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WILLIAM FRANCIS, Ph.D. F.L.S. F.R.A.S. F.C.S.

"Nec aranearum sane textus ideo melior quia ex se fila gignunt, nec noster vilior quia ex alienis libamus ut apes." Just. Lips. *Polit.* lib. i. cap. 1. Not.

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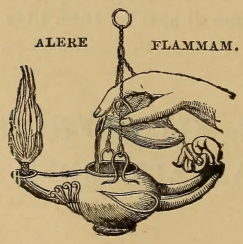
"Meditationis est perscrutari occulta; contemplationis est admirari perspicua .... Admiratio generat quæstionem, quæstio investigationem, investigatio inventionem."—*Hugo de S. Victore.*

——"Cur spirent venti, cur terra dehiscat,
Cur mare turgescat, pelago cur tantus amaror,
Cur caput obscura Phœbus ferrugine condat,
Quid toties diros cogat flagrare cometas,
Quid pariat nubes, veniant cur fulmina cœlo,
Quo micet igne Iris, superos quis conciat orbes
Tam vario motu."

J. B. Pinelli ad Mazonium.



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ERRATA.

Page 170, second line of table,
for Bell-metal at 0° vibrates during 55 secs.; at 100° it vibrates during 15 secs.
read „ „ „ 55 „ „ „ 40 „

Page 171, line 27 from top, *for* air of glass tubes of different diameters, renders
read air in glass tubes of different diameters, render

„ 173, line 11 from top, *dele* The frequency of the vibrations of

„ 195, „ 12 from bottom, *for* from the globular *read* having the globular

„ 367, Mr. W. H. Everett's paper, in first equation, for $rd^2\theta$ read $r^2d\theta$

THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[FIFTH SERIES.]

JANUARY 1896.

I. *The Latent Heat of Evaporation of Benzene.* By E. H. GRIFFITHS, *M.A., F.R.S., Sidney Sussex College, Cambridge, and* Miss DOROTHY MARSHALL, *B.Sc., University College, London**.

1. THE method of experiment and the nature of the apparatus employed have been fully described in previously published papers. Although it appears somewhat presumptuous to assume, on the part of the reader, a knowledge of such communications, it would on the other hand seem redundant to devote many pages to the mere repetition of what has already appeared in print. We therefore propose to give an explanatory statement of the theory and methods employed, but to omit all detailed description of the apparatus. Full references to former communications will be given when necessary, so that all particulars can be ascertained by any who desire them.

As reference has frequently to be made to the following papers (by E. H. Griffiths) they are, for convenience, denoted as follows :—

Paper J. "The Mechanical Equivalent of Heat," Phil. Trans. vol. clxxxiv. 1893, A, pp. 361–504.

Paper A. "The Influence of Temperature upon the Specific Heat of Aniline," Phil. Mag. Jan. 1895.

Paper W. "The Latent Heat of Evaporation of Water," Phil. Trans. vol. clxxxvi. 1895, A, pp. 261–341.

* Communicated by the Physical Society: read November 8, 1895.

2. *Description of the Method.*

In order to render our account of these experiments intelligible, we find it necessary to give the following somewhat lengthy extract from Paper W, pp. 270–273 :—

"The method adopted was of such a nature that the results would not be appreciably affected by

(1) errors in thermometry ;
(2) changes in the specific heat of water ;
(3) the capacity for heat of the calorimeter ;
(4) loss or gain of heat by radiation, &c. ;

and if these points are borne in mind, they may serve to explain some of the contrivances which might otherwise appear uncalled for.

"If the vessel in which the evaporation is taking place is kept at a constant temperature, we are independent of the capacity for heat of it and its contents ; we also dispense with the measurements of changes of temperature. Thus, if matters be so arranged that the loss and gain of heat throughout an experiment are balanced, many fruitful causes of error are avoided. Of course, the actual temperature of the calorimeter during evaporation must be determined, but a small error here is of little consequence. The change in the value of L is small as compared with the changes in θ. In fact, an accuracy of an order of $\frac{1}{10}$ of a degree would be sufficient when determining the actual elevation.

"The heat was supplied to the calorimeter by means of a wire whose ends were kept at a constant potential difference. The thermal balance could be maintained in one of two ways:—

(1) If the heat-supply was too great, the electric current could be temporarily stopped, or, the rate of evaporation of the water increased. (The latter was the method generally adopted.)
(2) If the cooling was too rapid, the only mode of maintaining the balance was (in the apparatus about to be described) to reduce the rate of evaporation.

"The liquid to be evaporated was contained in a small silver flask, connected with which was a spiral coil of silver tubing 18 feet in length. Both flask and spiral were within the calorimeter, and the water-vapour, after passing through the spiral, emerged from the apparatus at the temperature of the calorimeter. Surrounding the flask, and between it and the spiral, a coil of platinum-silver wire was arranged, and flask, spiral, and coil were entirely immersed in a certain singularly limpid oil consisting of hydrocarbons only.

"The calorimeter (which was filled to the roof with the oil, and the equality of temperature maintained by rapid stirring) was suspended by glass tubes within a steel chamber, whose walls were maintained at a constant temperature. So long, therefore, as the calorimeter and the surrounding walls were at equal temperatures, there was no loss or gain by radiation, &c. If during an experiment the temperature of the surrounding walls changed, the method of experiment involved a corresponding change in the temperature of the calorimeter, and, therefore, some loss or gain of heat would be experienced. The apparatus was so designed that any such change in temperature was extremely small (in no case amounting to $\frac{1}{100}°$), yet, in order to estimate the loss or gain, it was necessary to know approximately the capacity for heat of the calorimeter and contents.

"Small differences between the temperature of the calorimeter and the surrounding walls would, during an experiment, be of no consequence provided that the oscillations were of such a nature that the mean temperature of the calorimeter was that of the surrounding space, and it will be found that this condition was fulfilled.

"In addition to the heat supplied by the electric current, there is also a supply due to the work done by the stirrer, and it was in the estimation of this 'stirring supply' that the greatest difficulties were encountered. Fortunately the heat thus generated was only about $\frac{1}{100}$ of the heat supplied by the current, and thus any small error in that portion of the work becomes of little account.

"Of the accuracy with which the electrical supply could be measured there is no question; and even if the value of the E.M.F. of the Clark cells, or the absolute resistance of the box-coils given by the standardizations performed during the determinations of J, is in any way inaccurate, such errors would now eliminate, since the value of J was determined by means of the same standards as those by which the quantity of heat developed in these experiments was determined. Hence, by assuming the value of J obtained by the use of these standards, we get the comparison in terms of a thermal unit at 15° C., independently of the numerical value assumed in the reductions.

"One further correction remains to be noticed. The temperature of the calorimeter has been referred to as oscillating about the exterior temperature, and it might happen that at the close of an experiment this difference was not the same as that at the commencement—if any such difference existed. The magnitude of this correction depended, of course, on the

ability of the observer to maintain the thermal balance. In these experiments the correction was usually small, and in any case could be determined with great accuracy.

"Having indicated the nature of the observations, we proceed to state the relation between the various sources of loss or gain of heat.

"Let Q_E be the thermal units per second due to the electrical supply;

Q_S be the thermal units per second due to the mechanical supply;

Σq be the total heat-supply during an experiment from any other causes.

"Then, if M be the mass of water evaporated, L the latent heat of evaporation at temperature θ, and if the electrical supply is maintained for a time t_E, and the mechanical for a time t_S,

$$ML = Q_E t_E + Q_S t_S + \Sigma q. \quad . \quad . \quad . \quad . \quad (1)$$

"Now the D.P. at the ends of the coil was always some integral multiple of the D.P. of a Clark cell.

"Let e be the D.P. of a Clark cell, n the number of cells, and R_1 the resistance of the coil at the temperature θ_1, then

$$Q_E = \frac{e^2 n^2}{R_1 J}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

"If the calorimeter at the commencement and end of an experiment was at exactly the same temperature as the surrounding walls, then, if their temperature was unchanged, the term Σq would vanish; but although this term throughout these experiments was of small dimensions, it could not be entirely ignored.

"Let θ_0' and θ_0'' be the temperature of the surrounding walls at the beginning and end of an experiment; suppose the calorimeter temperature (θ_1) to exceed the surrounding temperature by d' at the commencement and d'' at the end of an experiment. Then fall in temperature of calorimeter

$$= (\theta_0' + d') - (\theta_0'' + d'').$$

Hence the heat given out by the calorimeter in consequence of this fall in temperature is

$$C_{\theta_1}\{(\theta_0' + d') - (\theta_0'' + d'')\},$$

where C_{θ_1} is the capacity for heat of calorimeter and contents at the temperature θ_1.

"If we neglect any small loss by radiation, &c., due to the

differences d' and d'' between the temperature of the calorimeter and the surrounding walls, we may conclude that the whole of the heat thus evolved by the calorimeter was expended in the evaporation of water, hence

$$\Sigma q = C_{\theta_1}\{(\theta_0' - \theta_0'') + (d' - d'')\}^* . \quad . \quad . \quad (3)$$

Hence

$$ML = \frac{e^2 n^2 \times t_E}{R_1 J} + Q_s \times t_s + C_{\theta_1}\{(\theta_0' - \theta_0'') + (d' - d'')\}. \quad (4)''$$

3. The following table shows the *comparative* mean values of the terms in equation (4) resulting from our experiments on benzene †.

TABLE I.

No. of Cells.	$Q_E t_E$.	$Q_S t_S$.	Σq.
5	1	·008	±·005
4	1	·011	±·009
3	1	·015	±·001

We found that the thermal balance was most easily maintained when the potential-difference was that of 4 cells, therefore the majority of our experiments were performed with that potential-difference, the experiments with 5 and 3 cells being used as a check upon the results. To secure an accuracy of (say) 1 in 1000 in the total heat-supply, the above table shows that it was necessary to measure

$Q_E t_E$ to (say) 1 part in 2000,
$Q_s t_s$ „ 1 „ 15,
Σq „ 1 „ 10;

but as the sign of the last term was in some experiments positive, in others negative, the above degree of accuracy was

* This apparently clumsy method of representing the quantity of heat evolved or taken up by the calorimeter was adopted because, as the method of experiment involved separate determinations of θ_0', θ_0'', d', and d'', the actual temperature of the calorimeter at any time could only be obtained in this manner.

† (Experiments Preliminary, XIX. *a*, XXXV., and XXXVI. were not included when calculating this table, as they are in several respects exceptional.)

not essential, since, when taking the mean of a number of observations, the effect of any error in Σq would be greatly diminished. We believe that the above order of accuracy was exceeded in the measurement of the respective terms. In the case of the experiments with 3 cells at 20° an order of accuracy of 1 in 50 would be required in the value of $Q_s t_s$ to secure the same standard; hence we do not regard our determinations at that temperature as possessing an equal authority with those at higher temperatures.

4. *The Method of maintaining the Space surrounding the Calorimeter at a Constant Temperature* θ_0.

A full description of the somewhat elaborate apparatus designed for this purpose will be found on pp. 374–378 of Paper J, and some improvements which were subsequently added are described on pp. 274–276 of Paper W. This portion of the apparatus did not, for some unknown reason, appear to be working with the same perfection this year as it did in the summer of 1894. Oscillations in θ_0 (the external temperature), amounting to $\frac{1}{100}$° C., were on several occasions observed, the greatest change during the whole series being ·0145°. This, however, was exceptional and, as an inspection of our final tables will show, the change in the course of an experiment (*i. e.* in about half an hour) was usually only about a few thousandths of a degree at temperatures varying from 20° to 50° C.; the constancy of θ_0 was therefore sufficient for our purpose.

Had it been possible, we should have preferred to determine θ_0 by means of a platinum-thermometer, as we could thus have detected smaller changes; but a third observer would, in that case, have been required, and circumstances did not permit of this addition*.

The mercury-thermometers used for indicating the temperature of the steel chamber in which the calorimeter was suspended were graduated in millimetres, and had been carefully calibrated and compared with two Tonnelot thermometers standardized at the Bureau International†. The stems of these thermometers (where they projected above the tank-lid) were surrounded by glass tubes up which a stream

* We believe that such irregularities as present themselves in our final results are partly due to the employment of mercury-thermometers for the determination of the changes in θ_0.

† See "The Measurement of Temperature," Science Progress, Sept. 1894.

of tank-water was forced by means of a pump driven by a water-motor : thus the stem-temperature was always that (or nearly that) of the tank, and the readings were not affected by changes of temperature in the room. The actual value of θ_0 was of small consequence, an accuracy of $\frac{1}{10}$° C. being sufficient ; but it was necessary to read *changes* of temperature during an experiment to nearly 0°·001 C., and this we believe we were able to do. A full description of the various precautions taken will be found in Paper A, pp. 55 and 56, and Paper W, pp. 275, 276.

The observations were taken by means of a reading-telescope fitted with a micrometer-eyepiece, which directly divided 1 millim. on the thermometer-stem into 10 parts : thus 0·1 of a millimetre could be directly read, and ·01 could be estimated. There is no doubt that the actual readings could be taken to ·025 millim., that is about 0°·001 C.

Thermometer A had a range of 16° to 26° C., and about 27 millim. were equivalent to 1° C.

Thermometer II. had a range of 28° to 53° C., and about 20 millim. were equivalent to 1° C.

All temperatures, both when obtained from these and from the platinum-thermometers, are throughout this Paper expressed in terms of the nitrogen-scale.

5. *The Calorimeter and its Connexions.*

A full description will be found on pp. 276–281 of Paper W, and drawings and sections on plates 5, 6 of the same paper. The brief account given in the description of the method *supra* is sufficient to indicate the nature of the arrangements to those who are not familiar with that paper.

The *differences* of temperature between the calorimeter and surrounding walls were determined by means of differential platinum-thermometers. A description of the method of standardization and observation of those thermometers is given on pp. 52–56 of Paper A, and some further details on pp. 285–290 of Paper W. Experimental evidence is there adduced in support of the following statement :—" It follows that differences of temperature could be determined to ·0004°, and differences of ·0001° could be detected."

The following Table shows the difference in temperature corresponding to a difference of 1 mean millim. of the bridge-wire whose opposite ends were connected with the thermometers :—

TABLE II.

Temp.	$\Delta\theta$ for difference of 1 mean millim. bridge-wire.
20°	0°·009046
30	0·009073
40	0·009101
50	0·009128

By experiments repeated this summer we found that no change had taken place in the value of the mean bridge-wire millimetre. A slight change (particulars of which are given in a subsequent section) had, however, occurred in the position of the null-point, *i. e.* in the reading on the bridge-wire when the temperature of the two thermometers was identical.

6. *Brief Description of the Method of obtaining the Value of the Terms in the Expression*

$$\text{Total heat} = Q_E t_E + Q_S t_S + \Sigma q.$$

$$Q_E t_E = \frac{(ne)^2 \times t_E}{R_I \times J}.$$

The ends of the platinum-silver coil (immersed in the oil surrounding the evaporating-flask and spiral) were kept at a constant potential-difference by means of the arrangements described in Paper J, pp. 382–388. This potential-difference was always some integral multiple (n) of the potential-difference of a Clark cell (e). During the spring of this year the Clark cells used were again compared with the Cavendish standard (R_I), which has shown no signs of change since its standardization by Lord Rayleigh in 1883 and by Messrs. Glazebrook and Skinner in 1891.

The mean value of the whole set of 30 cells differs from R_I by 0·00004 volt only; and although individual cells show larger discrepancies than in previous years, their mean potential-difference at 15° C. may be taken as 1·4342 volts (see Paper W, p. 297). A number of these cells were always placed in parallel arc: thus when n is given as 4, we were really using 12 cells as four files of 3 each. The arrangement for keeping the temperature of these cells at nearly 15° C. has been described in Paper J, p. 385. During the period covered by our experiments, some of the days were extremely hot, and

the tap-water became so warm that when turned on by the regulator it was unable to keep down the temperature sufficiently. The extreme range in the temperature of the cell-tank during these experiments was from 14°·8 to 16°·3 C. As, however, the movement was always extremely slow, it is probable that the tank-temperature closely corresponded to the effective temperature of the cells, and hence the correction

$$e = 1{\cdot}4342\{1 + (15 - \theta) \times {\cdot}00077\}$$

applied by us gave the value of e with sufficient accuracy.

The time t_E was determined by means of a chronograph controlled by an electric clock, whose gaining rate is now less than 1 in 20,000. Any movement of the keys by which the current was switched on to the calorimeter-coil was automatically recorded on the tape; and thus personal errors were eliminated. The times could have been read to $\frac{1}{100}$ of a second, but it was considered unnecessary to read to nearer than $\frac{1}{10}$, *i. e.* about 1 in 7000 of t_E.

The value of R_1 is expressed in terms of the "true ohm" as given by the B.A. standards, with which (by kind permission of Mr. Glazebrook) the coils used by us have been directly compared (Paper J, pp. 407–410).

The increase in R_1 due to the rise of temperature caused by the current was determined in the manner described in Paper J, pp. 404–407 (see also Paper W, p. 296).

The value of J assumed by us was 4·199 (Paper W, p. 314). We would emphasize the fact to which attention has been previously directed (section 2, p. 3), namely, that even if, in consequence of errors in the standards &c., this value of J is incorrect, it is still the right value to use for the reduction of these observations, for, provided that no change has taken place during the past three years in the standards used (and direct comparisons show no signs of any change which would affect the results), the values of L obtained by us are independent of the numerical values of J and R when expressed in terms of the same units as those assumed during the determination of the mechanical equivalent.

The value of $Q_E t_E$ is thus expressed in terms of a "thermal unit at 15° C."

7. $Q_s t_s$.

The method adopted for finding Q_s (the thermal units per second due to the work done by stirring) has been fully described in Paper W, pp. 290–293. It was there shown that the value of Q_s varied approximately as r^4 (where r was the

rate of revolution of the stirrer), and that this relation was sufficiently close for the reduction of experiments in which the value of r was somewhere between 5 and 6 per second.

The value of Q_s was found to increase rapidly as θ_1 (the temperature of the calorimeter) diminished. Our recent redeterminations of Q_s have explained a discrepancy which caused an uncertainty in the former determination of Q_s at 20° and 50°, and we now find that it was due to an arithmetical error in the reduction of the observations. Fortunately, the resulting correction in no way affects the conclusions arrived at in Paper W; and our present investigation confirms the accuracy of the experimental determinations of Q_s obtained in 1894. As the extreme differences between any of our determinations of Q_s at temperatures above 20° (after the arithmetical corrections at the 20° and 50° points) do not exceed 1 part in 70, even if we include the values found in 1894, and as the mean probable error at any temperature is much below 1 in 100, it is obvious that the values of Q_s have been ascertained with more than sufficient precision.

$$8.\ \Sigma q = C_{\theta_1}\{(\theta_0' - \theta_0'') + (d' - d'')\}.$$

We have previously indicated (sect. 4, p. 6) the manner in which θ_0' and θ_0'' (the initial and final temperature of the walls of the surrounding chamber) were determined by the direct observation of mercury-thermometers. The value of d (the difference between θ_1 and θ_0, *i. e.* the calorimeter-temperature and that of the surrounding walls) was ascertained by the differential platinum-thermometers previously referred to, and was always small—rarely greater than ·01° C. It was usually determined by the reading of the galvanometer-swings without altering the position of the contact-maker on the bridge-wire.

When evaporating water, the values of d were so small that an approximate value of the swing (in terms of a length of the bridge-wire) was sufficient. During these benzene experiments, however, we found it impossible to maintain the thermal balance with such perfection. The only values of d which were of consequence were d' and d'' (the initial and final values), and, especially in the earlier experiments, the internal temperature (θ_1) rose so rapidly as soon as the last drop of benzene had evaporated, that it was found impossible to switch off the electric current from the calorimeter-coil at the precise moment necessary to reduce d'' to negligible dimensions. As a consequence, the final swings (which were read by means of a micrometer-eyepiece) were often considerable, and therefore it was necessary to determine their value on each occasion; for, although throughout a series of experiments at

the same temperature the value varied but little, the changes were sufficient to affect the resulting value of Σq in exceptional cases. In the earlier experiments, especially from Preliminary to No. VII., when the difficulty of the final adjustment had not been fully realized, the values of d'' were unduly large, and their equivalent degree-measurements are therefore somewhat doubtful, owing to insufficient determinations of the value of the galvanometer-swings. We feel sure, however, that from No. VII. onwards any errors due to this cause must be very small—certainly not so great as 1 in 50 ; for the value of a swing of 100 (in terms of a millim. of the bridge-wire) would, throughout a group of experiments at the same temperature, vary (for example) from 1·31 to 1·34 millim.* This change in value would produce no appreciable effect on Σq when the difference between the initial and final swings did not exceed 50 or 60, as was the case in most of our later, and better, experiments.

The value of C_{θ_1} (capacity for heat of calorimeter and contents) remains the same as last year with the exception that 0·1 grm. of the oil was removed at the commencement of July when withdrawing the platinum thermometer for purposes of re-standardization. As the value of C_{θ_1} varied with θ_1 from 304 to 323, this loss was negligible.

The quantity Σq represents the heat absorbed by the calorimeter and contents, and it should be remembered that it is by the measurement of a similar quantity that the majority of the determinations of thermal constants have been made by previous observers. We were, however, anxious to diminish the importance of this term as much as possible, for we wished our values of L to be independent of any thermometric errors. As shown by Table II., the average value of Σq was only about $\frac{1}{200}$ of the total heat-supply. In the experiments in which the value of Σq is less than 1 (of which there are many examples), we may say that our final results are independent of such errors, for, as pointed out in Paper W, the value of Q_s is independent of temperature measurements, since it depends on the *ratio* of the rate of rise due to the mechanical supply to the rate of rise due to an electrical together with a mechanical supply (Paper W, p. 331).

The values of C_{θ_1} at 20° and 50° used in this Paper differ somewhat from those given in Paper W. This difference is due to the error, previously referred to, in the value of Q_s at those temperatures. Thus

the value of C_{θ_1} at 20° is reduced from 307·5 to 305·2,
and „ „ 50° „ „ 323·1 to 322·0.

* 1 millim. of the bridge-wire indicated a temperature-difference of about 0°·009 C.

This, however, in no way affects the values of L in Paper W, nor, except in the case of two experiments (viz., Preliminary and XIX. *a.*, *infra*), would it, if left uncorrected, have affected the values of L as given in this communication.

We believe that the values of Σq are correct to better than 1 in 100 ; and Table I. (p. 5) shows that an accuracy of 1 in 10 would have been sufficient.

9. *Measurement of Mass.*

The actual measurement of the mass of benzene evaporated presented several difficulties which were not encountered when working with water. A tube resembling a weight-thermometer was filled with benzene, placed in an air-tight case just large enough to contain it, and then weighed against a precisely similar case used as a tare. The weight-thermometer (termed by us a "dropper") narrowed at its open extremity to a capillary tube, which was doubled back on itself for rather over 1 cm., and again bent near the open end, so that the last 1 or 2 mm. were horizontal. These droppers varied in capacity from about 4 to 6½ c.c., and were filled in the following manner :—The dropper (point uppermost) was lowered by a fine wire to the bottom of a tube about 4½ ft. long, of which the lower 10 inches or so were filled with benzene, while the upper 3 ft. were surrounded by a "condenser-tube" through which tap-water was continually passing. The lower end, containing the benzene, was transferred at regular intervals from a vessel of water at about 86° C. to a vessel of cold water ; thus the benzene was alternately boiled and cooled without any escape of vapour into the room. Five or six such transferences were generally required to completely fill the dropper. The containing tube was then placed in a bath at about 65° C. until the temperature of the benzene was steady. The dropper was now removed and allowed to stand in the open air for some time, in order to get rid of any benzene adhering to its surface. Although simple and effective, these operations occupied a considerable time, and, as a rule, the whole of the morning had to be devoted to the filling of the droppers required for the experiments, which were usually performed at night.

Before an experiment the dropper was lowered, by means of a thread passed through a platinum wire sealed into the closed end, into the calorimeter, where it stood in a vertical position. In Paper W (p. 307) it was shown that, when filled with water, the evaporation through the capillary opening between the time of weighing and the commencement of an experiment might be neglected. In the case of benzene, however, it was found

that, for several reasons, loss of this kind could not be disregarded :—

(1) The vapour-pressure of benzene so greatly exceeds that of water at corresponding temperatures that the loss by diffusion through the capillary was appreciable.

(2) The surface-tension of benzene is so great that the liquid crept up the sides of the capillary to the opening, and the consequent loss by evaporation was increased.

(3) In order to supply the air necessary for starting the boiling when the exterior pressure was removed, a capillary tube, closed at one end, had been sealed within each dropper. It was found that this answered very well during some preliminary trials, when the dropper was placed within a glass tube connected with an exhaust pump so that its manner of discharging could be watched ; but on a second filling with our purest sample (the first filling having thoroughly cleaned the interior surface of the droppers) no action took place even when the surrounding pressure was reduced to a few millimetres. In order, therefore, that the expulsion of the benzene from the dropper should commence as soon as the external pressure was reduced to the right amount, it was found necessary to leave a very considerable air-bubble within the tube. Precautions had to be taken to prevent the expulsion of the liquid by the alternate contraction and expansion of the air-bubble when the temperature was changed from that of the balance-case to that of the tank.

These difficulties were surmounted in the following manner.

During the process of filling, as above described, the droppers stood in a vertical position, with the doubled over and open end uppermost, and were never inverted after their removal from the filling-tube, at a temperature of about 65° C., until their insertion into the calorimeter. The coefficient of expansion of benzene is very large, and, on cooling to the room-temperature, the whole of the upper bend together with a couple of centimetres of the neck between it and the body of the dropper was found to be free from benzene. The dropper was then placed in its case and left in a vertical position in the balance-case for, as a rule, some hours, when the small air-space in the enclosing-case no doubt became saturated with the vapour and thus further evaporation ceased. After being weighed, the dropper and case were lowered into a large tube placed within the tank and left until they acquired the tank temperature θ_0 ; they were then rapidly

withdrawn and for the first time inverted—the air-bubble rising to the closed end. The case was opened for a moment and the dropper at once lowered into place by means of the previously attached thread: this operation only occupied a couple of seconds. As the dropper left the case the latter was instantly closed by a second operator to prevent the escape of any benzene vapour left within it. Thus from the time of inversion no change took place in the temperature, and therefore in the volume of the air-bubble; also no appreciable change in the temperature of the calorimeter (θ_1) was caused by the introduction of the dropper and contents. However, the mere act of opening the tube leading down to the calorimeter caused a slight lowering of θ_1, and in order to re-establish the equality between θ_1 and θ_0 before commencing an experiment, it was necessary to switch on, for a second or two, the current from the exterior coil in the tank to the calorimeter coil. When the observer at the differential-thermometer galvanometer announced that $\theta_0 - \theta_1$ was small and steady, the time for commencing the experiment had arrived.

10. In our later experiments, when we had become more expert at the various operations, the time from the insertion of the dropper to the commencement of an experiment was from 3 to 5 minutes. In our earlier observations at 30° the time was, however, much longer—from about 8 to 15 minutes. During this interval there was generally apparent a slight lowering of θ_1, which made us fear that the evaporation through the capillary was appreciable, and it was not until our tenth experiment that a means of meeting this difficulty suggested itself. It was evidently necessary to wait until θ_1 became steady before commencing an experiment, and it was difficult to see how to shorten the time required to establish this condition.

From Experiment X. onwards the procedure was as follows. A glass rod whose upper end passed through a cork fitting the opening of the entrance-tube was lowered into the calorimeter. Round the lower three inches of this rod was strapped (by fine platinum wire) a thin roll of cotton-wool of which the upper two inches were saturated with benzene while the lower end was left dry, so that there was no danger of drops of benzene falling from it into the silver evaporating-flask.

The rod was withdrawn occasionally and more benzene added if the upper portion of the roll had become dry. Observation of the galvanometer showed when the cooling effect

caused by the evaporation had ceased, and we were thus able to determine when the flask and connecting tubes were saturated. The rod was not finally withdrawn until the dropper was introduced : thus the space being saturated, no further evaporation took place ; and we believe that from Experiment X. onwards any error arising from evaporation in the time preceding the experiment may be disregarded.

After we had completed our experiments, we made some observations with the object of ascertaining the probable magnitude of the error in Experiments I. to VII. due to evaporation when the flask had not been previously saturated. A dropper was placed within the evaporating flask for 20 minutes and kept at a temperature of 30° C. It was then withdrawn, and the loss determined. As might have been expected, it varied slightly according to the droppers used, probably owing to the different sizes of the capillary opening.

In 20 minutes,	dropper	II.	lost	10	milligrams,
„	„	„ III.	„	12	„
„	„	„ IV.	„	13	„

After Experiment VII. we adopted the plan of noting the time from the insertion of the dropper to the commencement of the experiment, but unfortunately we had not previously done so. We consider that the average time in these experiments must have been from 10 to 15 minutes ; the loss during this time would therefore appear to have been about 6 milligrams, or rather more than $\frac{M}{100}$. Thus the values of L resulting from Experiments I. to VII. are probably too low by about 0·12. Fortunately we have six independent experiments at 30° C. (Nos. XXIX. to XXXIV.) in which this cause of error was absent.

We have entered into this matter fully, as it shows the importance of extreme attention to details in work of this kind, and also it was necessary to explain why we practically neglected Experiments I. to VII. when drawing our final conclusions. The same cause of error would slightly affect Experiments VIII. and IX. at 40° C. At this time, however, we were attempting to minimize the evil by allowing as short a time as possible to elapse between the introduction of the dropper and the commencement of an experiment. We have also a note of the time, which in both cases was less than 4 minutes. The loss during this interval would probably not have affected the resulting values of L by more than ·05, and we therefore do not consider it necessary to reject these experiments.

At the close of an experiment, when observation of the

galvanometer showed that all evaporation had ceased, the dropper was extracted by means of a bent wire, immediately placed in its case, and weighed again after standing some hours in the balance-case.

Let m_1 and m_2 be the weights of the case and dropper before and after an experiment; the temperature of the balance-case was usually about 20° C., the specific gravity of benzene at that temperature may be taken as 0·88, hence M (true mass corrected to vacuo)

$$= m_1 - m_2 + (m_1 - m_2) \times \cdot 0012.$$

11. Before commencing the experiments it was necessary to ascertain if any alteration had taken place in the values of the various constants and variables since their determination in 1894.

Thermometers II. and A were re-standardized and it was found that a "zero-point rise" had taken place, as is customary with mercury thermometers.

Thermometer No. II. had risen 0·4 mm.=0°·02 C. since its standardization in August 1894.

Thermometer A had risen 1·4 mm.=0°·06 (nearly) since its standardization in July 1893, which is about the normal rise of thermometers of this description.

The platinum thermometer (AB) was removed from the calorimeter, strapped to its corresponding thermometer (CD), and placed in the tank whose temperature was raised from 18° C. to 40° C. Observations at different temperatures showed a rise of 0·45 mm. in the null-point whose position is now given by the formula $598 \cdot 8 + \cdot 03\,\theta$ in place of $598 \cdot 35 + \cdot 03\,\theta$. If, however, this change, which probably took place in the arms of the bridge rather than in the thermometers *, had not been detected the resulting error would have been negligible, for a difference of ·45 mm. in the setting of the null-point is equivalent (at 40°) to a temperature difference of about ·0041° C., and the total loss or gain by radiation, &c., corresponding to this difference between θ_1 and θ_0 would not exceed 0·2 thermal gram per half-hour, whereas the actual duration of the majority of these experiments was about 18 minutes.

The corrected formula was, however, used throughout these experiments for the adjustment of the contact-maker.

* [Note by E. H. G., August 12, 1895.—I have found by re-standardizing the bridge-arms, that the above supposition was correct.]

12. *Value of* R_1.

The values of R_1 were redetermined at temperatures 30°, 40°, and 50°. The corrected results showed falls of ·0012, ·0012, and ·0016 respectively from the values of 1894. These quantities have, therefore, to be subtracted from the values given in Paper W, table viii., but the correction is only 1 in 10,000 of R_1.

The following table gives the values of R_1 used during the reductions of the observations on benzene.

(The suffix to R denotes the potential-difference in terms of a Clark cell.)

TABLE III.

Temp.	R_{3e}.	R_{1e}.	R_{5e}.
20°	10·327	10·329	10·333
30	10·351	10·353	10·357
40	10·374	10·377	10·381
50	10·399	10·401	10·406

δR per 1° C. = ·0024.

The re-standardization of the Clark cells has already been referred to (section 6, p. 8).

13. Q_s.

No alteration in the values of Q_s at 30° and 40° C. appears to have taken place.

The values of Q_s at 50° and 20° as given in Paper W, pp. 332, 333, appeared to have undergone alteration, and we therefore made a careful redetermination at those temperatures. The method adopted was that described on p. 292, Paper W, viz., the rate of rise in temperature at null-point was determined

(1) when the heat-supply was that due to the stirring only

$$=\left(\frac{d\theta_1}{dt}\right)_s, \text{ and}$$

(2) when the heat-supply was that due to the stirring and a

potential-difference of 3 Clark cells

$$=\left(\frac{d\theta_1}{dt}\right)_{3es}.$$

Hence
$$\frac{\left(\frac{d\theta_1}{dt}\right)_s}{\left(\frac{d\theta_1}{dt}\right)_{3es}-\left(\frac{d\theta_1}{dt}\right)_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

gives the ratio of the heat supplied by the stirring to the heat-supply when the potential-difference was that of 3 Clark cells; and as the latter can be calculated if R_1 is known, the value of Q_s can be obtained without any assumptions as to the thermometric scale, the capacity for heat of the calorimeter, &c.

The individual experiments were in close agreement with each other and give the following results:—

At 50°, $\left(\frac{d\theta_1}{dt}\right)_{3es}$ $=\cdot 14533$ expressed in mm. of bridge-wire where rate of stirring $=5\cdot 380$,

$\left(\frac{d\theta_1}{dt}\right)_s$ $=\cdot 001155$, at rate $5\cdot 380$.

Hence Q_s (at rate $5\cdot 380$) $=\cdot 003404$;

but (see Paper W, p. 330) $\frac{r^4}{Q_s}=$ a constant $=\frac{837\cdot 8}{\cdot 003404}$,

hence $Q_s=\cdot 003206$ at rate $5\cdot 300$.

A second determination gave

$Q_s=\cdot 003202$ at rate $5\cdot 300$.

A third gave

$Q_s=\cdot 003205$ at rate $5\cdot 300$.

We therefore assume

$$Q_s=\cdot 00321+(r_1^4-789)\times\cdot 0000041,$$

as sufficiently accurate.

At 20°, $\left(\frac{d\theta_1}{dt}\right)_{3es} = \cdot 15941$ at rate 5·590,

$\left(\frac{d\theta_1}{dt}\right)_{s} = \cdot 00427$ at same rate.

Hence Q_s (at rate 5·590) $= \cdot 01174$;

but $\frac{r^4}{Q_s} =$ a constant $= \frac{976 \cdot 6}{\cdot 01174}$,

hence $Q_s = \cdot 00949$ at rate 5·300.

A second determination gave ·00951 at rate 5·300,
A third „ „ ·00991 „ „ 5·300.

In this last experiment, however, there is internal evidence of some error in the time over the second interval. If we omit this interval and calculate the value of Q_s from the remaining intervals of that experiment, we get ·00961 as the value.

It is evident that ·00950 is a sufficiently close approximation.

We therefore assume

$$Q_s = \cdot 00950 + (r^4 - 789) \times \cdot 0000120.$$

Our experiments at 30° and 40° show that the values of Q_s at those temperatures as given in Paper W are correct. The errors at 20° and 50° as given in that paper were due to an arithmetical mistake, a difference having been added, instead of subtracted, in each case.

The following Table gives the expressions by which the value of Q_s can be obtained at any of the temperatures or rates given in succeeding Tables :—

TABLE IV.

Temp.	Value of Q_s (in thermal grms.).
50°	$\cdot 00321 + (r_1^4 - 789) \times \cdot 0000041$
40	$\cdot 00466 + (r_1^4 - 789) \times \cdot 0000059$
30	$\cdot 00665 + (r_1^4 - 789) \times \cdot 0000084$
20	$\cdot 00950 + (r_1^4 - 789) \times \cdot 0000120$

These changes in Q_s probably indicate the changes in the viscosity of the oil.

14. *Alterations in the Apparatus.*

No alterations have been made except in exterior portions of the apparatus.

When working with water in 1894 there were, in the tubes leading to the air-pumps, one or two rubber joints which are now replaced by glass ones.

To prevent condensation, the benzene vapour after issuing from the tank passed over a row of small gas-jets and then down into a small Wolff's bottle, connected with the manometer and containing pumice-stone and sulphuric acid. It then passed through another tap * into a large globe (capacity about 35 litres) also containing pumice-stone and sulphuric acid. By means of a water-pump the pressure in this globe was reduced to that of the aqueous vapour at the temperature of the tap-water. Its capacity was so great that the pressure in the Wolff's bottle could, at any time, be brought below that required, at experiments above 20°, by simply opening the tap communicating with the globe. Thus the water-pump had not to be used during an experiment, and, consequently, the two motors worked with greater regularity. The Wolff's bottle was also in direct communication with a Geissler's mercury-pump, by means of which, when working at the lower temperatures, the pressure was greatly reduced near the close of an experiment in order to secure the boiling-off of the last drop of benzene. We found that the mercury-pump had to be kept in constant use during the experiments at 20°. With the exception of the above alterations and additions, the apparatus is the same in every respect as that figured in Plates 4, 5, and 6 of Paper W.

15. *On the Purity of the Benzene.*

The benzene, which was a sample of that used by Professor Ramsay and Miss Marshall for their comparative experiments, was supplied by Messrs. Kahlbaum of Berlin and guaranteed free from thiophene. It was redistilled twice from phosphoric

* The grease on the core of taps traversed by benzene was replaced by phosphoric acid.

anhydride, until it showed a perfectly constant boiling-point.

16. *Description of an Experiment.*

The dropper was placed in position (as described in section 9) and the contact-maker was then set, by means of a magnifying-glass, to the null-point corresponding to the tank temperature θ_0.

When the observer (Observer II.) at the thermometer-galvanometer (G_2) announced that θ_1 had become steady, three observations of the galvanometer-swing were taken, and the chronograph-key, being pressed at the second of those swings, recorded the time from which t_s was estimated. At the same moment Observer I. recorded the reading of thermometer II., which when reduced to the nitrogen scale gave the value of θ_0',—the initial tank temperature.

As a rule the initial swings were somewhere between 0 and +50 (a swing of +50 would correspond to about +·006° C.), and, for reasons which will appear later, we preferred to have this initial swing (which gave the value of d') positive.

The Wolff's bottle (condenser A) had previously been exhausted down to the pressure of the large globe (condenser B) already referred to, but the tap between the two condensers had been closed. A tap (immersed in the tank-water) between the silver flask and condenser A was then opened, and the air expanding into A caused a fall in the manometer attached to it. The tap connecting condensers A and B was then gradually opened, and the pressure in the flask fell until some benzene was expelled from the dropper. The instant this occurred the calorimeter temperature θ_1 commenced to fall, and Observer I. was acquainted with this fact by Observer II. who, throughout the whole experiment, was engaged in calling aloud the galvanometer-swings which resulted from the inversion of the battery connexions in the differential-thermometer circuit by means of a swinging key. The storage-cell current, which had been running for some time through a platinum-silver coil immersed in oil in the outer tank, was then switched on to the calorimeter-coil, the action recording itself on the chronograph-tape, and the potential balance then adjusted by means of the apparatus described in Paper J, p. 283. This balance had previously been approximately obtained while the current was running through the tank-coil; thus only a small additional adjust-

ment was required and, as the temperature of the calorimeter-wire remained constant, the electrical balance required little attention, for the potential-difference rarely altered by as much as $\frac{E}{100,000}$ throughout an experiment. As the benzene vapour passed into condenser A the pressure in the flask increased, and thus the loss of heat by evaporation diminished. By a rapid movement of the tap between condensers A and B, the pressure could again be diminished and the cooling effect increased.

It was found impossible to control the rate of evaporation with the same perfection when using benzene as was the case in the experiments with water. The taps had to be constantly manipulated, and a moment's inattention on the part of Observer I. was immediately followed by a sudden rise or fall in θ_1. This was more especially the case during the experiments at low temperatures. From about experiment V. or VI. onwards, however, the swings rarely amounted to as much as 200 or so, except during the first minute when the thermal balance was being obtained, at which time a swing of ± 400 or 500 was generally experienced. Throughout the whole of an experiment care was taken that any positive swing should be succeeded by a corresponding negative one. Although the announcement of swings of "$+200$" appeared alarming at the time, the extreme attention devoted to the keeping down of these oscillations was really unnecessary. A swing of $+200$ indicated that θ_0 was lower than θ_1 (at 40°) by about 0°·02 C., and the radiation &c. coefficient of the calorimeter being about ·00009 (in degrees per second per difference of 1°, Paper W, p. 289), this swing, even if maintained throughout the *whole time* that evaporation was proceeding (on an average less than about 7 minutes), would only have resulted in a *total* loss of about $\cdot 00009 \times 318 \times \cdot 0216 \times 7 \times 60 = \cdot 25$ thermal gram *. In no case, however, was a swing of such magnitude allowed to remain unaltered for more than a few seconds.

The chief difficulty was experienced near the close of an experiment. When working with water, there were always some indications that the end was near, for the pressure had to be diminished in order to maintain the thermal balance if only a drop or so remained. In the case of benzene, however, there were rarely any such indications; for the galvanometer-

* About $\frac{1}{2000}$ of the total "heat-supply."

swing might be announced as +8 or +10, and before the next announcement could be made it would be found that it had shot up to 300 or 400. After the preliminary experiment, it was of course possible to roughly estimate (knowing the weight of the dropper when filled) the probable duration of an experiment and, by adding it to the observed time of establishing the current, to predict approximately the time when evaporation would cease. Owing, however, to differences in the size of the air-bubbles necessarily left in the droppers, to the different rates of stirring, &c., and to errors in calculations made while all the attention of the observers was needed elsewhere, the current was very often switched off too late, thus increasing the value of the correction Σq unnecessarily. If any mistake was made, the final value of θ_1 was in consequence higher than the initial one, for in those cases where the current was switched off too soon, it was always possible to bring θ_1 up to its initial value by re-establishing the current for a second or two; if, however, θ_1 was too high there was no means of diminishing it, all the benzene having been exhausted. Observer I. endeavoured, if possible, to arrange so that the swing at the close of an experiment should be about −150, for the following reason. Evaporation having ceased and the current being switched off, the tap between the evaporating-flask and condenser A was closed, and a tap (also immersed in the tank-water) was slowly opened so that air (dried by having passed through sulphuric acid and phosphoric anhydride) was gradually admitted into the evaporating-flask through a 30 ft. copper coil immersed in the tank. Thus the air was at the temperature of the calorimeter. The heat liberated by its compression caused, however, a rise in θ_1 equivalent to a swing of about 120; thus no heat was, on the whole, gained by the calorimeter, for a corresponding loss had been experienced during the exhaustion. On this account we preferred to commence with a small positive swing, as it was not then necessary to reduce θ_1 so far at the close of an experiment to allow for this final increase. When Observer II. found that θ_1 had again become steady three final swings were taken, the chronograph-key being pressed at the middle one, thus giving the termination of the time t_s during which the stirring supply had to be estimated. Thermometer II. was read at the same time, and gave θ''_0, the final tank temperature. Throughout the experiment, every thousand revolutions of the stirrer had been automatically recorded on the chronograph-tape.

The method of removing and weighing the empty dropper has already been indicated.

17. *Remarks on the Experiments.*

We give particulars of all our experiments with two exceptions. In No. XXI. *a.* the chronograph ceased to work during a critical portion of the experiment, and we were thus unable to determine t_E. In XXIV. *a.* Observer I. omitted to close the entrance tap to the evaporating-flask before connecting with the exhaust, thus the attempt to diminish the vacuum was a failure and the experiment was relinquished after a minute or so. In these two cases we at once performed other experiments to replace the failures, but retained the numbers for convenience of reference.

We have, however, rejected, when drawing our final conclusions, several of the experiments whose details are given. Such experiments are marked by a † in the Tables. We have, in no case, rejected any experiment *except as a consequence of some note made during that experiment*; that is, before reducing the observations we had already decided as to those which should be regarded as of little value. Thus we have in no way been guided by the results. For example, XXV. differs from the mean of its group by a greater quantity than either of the rejected experiments at that temperature: however, as we have no note against it, we are compelled to give it equal importance with any of the others. A simpler plan would have been to reject entirely the † experiments, but we think that a fairer idea of the general accuracy of the work is given by including all those that we completed.

The reasons for the rejection of the † experiments are given (as they appeared in our original notes) at the end of each group. We have, however, in each case given the mean of all in a footnote, and it will be seen that (except at 30°) our results would not have been appreciably affected by the inclusion of those rejected ones.

As a rule, we found that the thermal balance was most easily maintained when the potential-difference was that of 4 cells, but this supply of heat would have been somewhat too great at 20°, when we worked with 3 cells only. At all temperatures above 20° we performed experiments with 3, 4,

and 5 cells, and we regard the agreement amongst the results as very satisfactory, and as establishing the validity of the various corrections. In thermal investigations it is as a rule difficult to alter all the conditions in so complete a manner as that caused by the changes above referred to. For example, if any "priming" had taken place (the usual cause of inaccuracy in determinations of the Latent Heat of Evaporation), its effect must have been greatly increased when the rate of evaporation was nearly trebled, which was the case when the potential-difference was altered from 3 to 5 cells. Again, the importance of the different terms undergoes such changes that any constant error in the determinations of $Q_s t_s$ and Σq would cause the values of L when $n=4$ to lie between the values when $n=3$ and $n=5$. An inspection of the Tables will show that no such effect is visible.

We also varied the "stirring supply" very considerably. The Tables show that (at the *same* temperature) the values of Q_s have been changed from ·00276 (Expt. XVII.) to ·00427 (Expt. XXIV.).

[*Note by* E. H. Griffiths.—My experience with apparatus of a similar nature to that used in these experiments has convinced me of the severity of the test above referred to. During my attempts at a determination of the mechanical equivalent during the years 1887–1891 the results invariably broke down when thus tested. The following quotation, referring to the experiments anterior to 1892, is from p. 364, Paper J :—

"The agreement amongst individual experiments taken under the *same conditions*, was, if anything, at times more marked than in the experiments of 1892 ; nevertheless, when the final reduction took place, fatal discrepancies invariably showed themselves. . . . Experiments conducted with a high electromotive force invariably gave too great a value for the time as compared with that obtained when a lower electromotive force was used."

It appears to me that it is this power of altering all the conditions which renders electrical methods of such great value when applied to thermal determinations.]

18. *Explanation of the Tables.*

Tables lettered A give the experimental *data*; the deductions from those data are given in Tables lettered B.

(We have arranged the experiments in order of temperature rather than historical order, as consecutive experiments were not always at the same temperatures.)

Tables A.

Col. I. *a* and *b* give the number and date of the experiment.

" II. *a* and *b* give the number of Clark cells in series, and the number of the dropper used.

" III. Gives the mass of benzene evaporated, after correction to *vacuo* (M).

" IV. The time during which the current was maintained (t_z).

" V. The time from commencement to end of experiment (t_s), *i. e.*, the time during which the "stirring supply" has to be estimated.

" VI. The number of revolutions per second (r_1) of the stirrer.

" VII. The difference between the initial (θ_0') and the final temperature (θ_0'') of the surrounding walls.

" VIII. Let θ_1' (initial calorimeter temperature) exceed θ_0' by d', and let θ_1'' (final calorimeter temperature) exceed θ_0'' by d''; then this column gives the value of $d'-d''$. Hence

$$\{(\theta_0'-\theta_0'')-(d'-d'')\}$$

gives the difference between the initial and final temperature of the calorimeter.

" IX. The capacity for heat of the calorimeter and its contents at the mean temperature of the tank (C_{θ_1}).

" X. The temperature of the Clark cells.

" XI. The value of R_{ne}, *i. e.*, the resistance of the calorimeter-coil at temperature θ_1, when the potential-difference of its end is *ne*. (From Table III. p. 17.)

19. Tables B.

(The numbers of the columns are similar to those in the corresponding tables of Paper W, where full particulars of the reductions are given.)

Col. XV. The value of $Q_E t_E = \frac{(ne)^2 \times t_E}{R_1 \times J}$, where e is the potential-difference of 1 Clark cell, at the temperature given in Col. X. (*supra*). The values of R_1 will be found in Table III. p. 17, and, as previously explained, the value of $J = 4{\cdot}199$ (see section 6, p. 8).

„ XVI. The value of Q_s deduced from Col. VI. by means of Table IV. p. 19.

„ XVII. The value of $Q_s t_s$, the "stirring supply," from Cols. V. and XIV.

„ XVIII. The term $\Sigma q = C_{\theta_1}\{(\theta_0' - \theta_0'') - (d' - d'')\}$ from Cols. VII., VIII., and IX.

„ XIX. The sum of Cols. XV., XVII., and XVIII., that is, the total thermal grams (Σ) required for the evaporation of M grms. of benzene.

„ XX. The mean tank temperature (θ_0) of the experiment, expressed in the nitrogen scale.

„ XXI. The value of $L = \frac{\Sigma}{M}$.

Remarks on Experiments at 50°.

Exp. XIX. *a*. "Mistake in switching off current, did not do so till θ_1 had risen so far that d'' could not be obtained by means of galvanometer-swing. Had to readjust contact on bridge at close of experiment. Decided to regard experiment as a failure and repeat it."

The experiment is included in the table to show that even the large resulting value of the term $\Sigma q(-18{\cdot}64)$ has but a small effect on the value of L, which differs from the mean at this temperature by less than 1 in 400. This value for L(98·90) should, however, certainly not influence our final conclusions.

Exp. XXIII.—"Mistake in connecting with vacuum, reduced pressure far below right amount, hence could not obtain proper thermal balance throughout experiment."

Conclusion.

When $\theta_0 = 50{\cdot}014$, $L = 99{\cdot}14$.

TABLE V.A.

Experiments at 50° C.

I.		II.		III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.
a.	*b.*	*a.*	*b.*									
Number.	Date. July	No. of Cells.	No. of Case.	M.	t_E.	t_S.	r_1.	$\theta_0' - \theta_0''$.	$d' - d''$.	C_{θ_1}.	Temp. of Cells.	R_{ne}.
XVII.	14	4	II.	5·0339	658·4	1319·2	5·107	0°	−·0010°	322	15·30°	10·402
XVIII.	14	4	III.	4·7071	622·4	1118·7	5·118	0	−·0154	,,	15·30	,,
†XIX. *a.*	14	4	IV.	4·9536	670·7	1204·3	5·126	−·0020	−·0560	,,	15·30	,,
XIX. *b.*	14	4	III.	4·7241	615·7	1093·2	5·241	+·0025	+·0046	,,	15·64	,,
XX.	16	4	V.	4·3654	569·1	997·0	5·171	+·0054	−·0030	,,	15·24	,,
XXI. *b.*	16	4	II.	4·9893	652·6	1314·9	5·219	+·0005	−·0030	,,	15·70	,,
XXII.	17	3	V.	4·3550	1010·1	1521·6	5·195	−·0024	−·0014	,,	15·58	10·399
†XXIII.	17	5	IV.	4·9726	416·2	1024·2	5·411	+·0024	−·0047	,,	15·77	10·406
XXIV. *b.*	18	5	IV.	5·0056	419·6	874·0	5·692	+·0017	−·0057	,,	15·50	10·406
XXV.	18	3	V.	4·3899	1011·4	1405·4	5·653	+·0054	−·0068	,,	15·50	10·399

Table V.B.

Reduction of Experiments at 50° C.

I. *a.*	XV.	XVI.	XVII.	XVIII.	XIX.	XX.	XXI.
Number.	$Q_E t_E$.	Q_S.	$Q_S t_S$.	Σq.	Σ.	θ_0.	L.
XVII.	496·00	·00276	3·64	− 0·32	499 32	49°·993	99·20
XVIII.	468·85	·00279	3·12	− 4·97	467·00	49·984	99·20
†XIX. *a.*	505·21	·00280	3·28	−18·64	489·95	49·983	98·90†
XIX. *b.*	463·55	·00307	3·35	+ 2·29	469·19	50·017	99·31
XX.	429·70	·00290	2·90	+ 0·78	433·37	50·026	99·27
XXI. *b.*	491·36	·00301	3·96	− 0·81	494·61	50·018	99·14
XXII.	427·93	·00296	4·50	− 1·22	431·21	50·027	99·02
†XXIII.	489·40	·00349	3·57	− 0·74	492·23	50·028	99·00†
XXIV. *b.*	493·52	·00427	3·74	− 1·29	495·97	50·021	99·08
XXV.	428·59	·00416	5·84	− 0·45	433·98	50·025	98·87
Mean (omitting those marked †)						50·014	99·14 ± ·034
If we include all experiments at this temperature we get ...						50·013	99·10 ± ·031

TABLE VI. A.

Experiments at 40° C.

I.		II.		III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.
a.	*b.*	*a.*	*b.*									
Number.	Date. July.	No. of Cells.	No. of Case.	M.	t_E.	t_S.	r_1.	$\theta_0' - \theta_0''$.	$d' - d''$.	C_{θ_1}.	Temp. of Cells.	R_{ne}.
VIII.	8	4	III.	4·7228	628·2	1092·0	5·274	0°	−·0049°	318	16·27°	10·377
IX.	9	4	IV.	5·0424	676·8	1155·0	5·220	+·0010	−·0192	,,	15·90	,,
X.	9	4	II.	5·0212	665·4	1169·3	5·171	−·0120	+·0023	,,	15·70	,,
†XI.	9	4	III.	4·7228	635·5	1074·6	5·176	−·0015	−·0202	,,	15·71	,,
XII.	9	4	V.	4·3846	577·8	1105·3	5·118	−·0020	−·0016	,,	15·72	,,
XIII.	10	4	IV.	4·9994	670·2	1143·7	5·097	−·0145	−·0085	,,	15·45	,,
XIV.	10	4	II.	5·0258	662·0	1194·2	4·859	+·0030	+·0054	,,	15·45	,,
XV.	11	4	IV.	5·0051	668·6	1084·8	5·169	−·0051	−·0116	,,	15·40	,,
XVI.	11	4	II.	5·0286	678·3	1281·4	5·162	+·0051	−·0430	,,	15·50	,,
XXVI.	19	3	V.	4·4191	1035·2	1471·5	5·242	−·0030	+·0027	,,	15·32	10·374
XXVII.	19	5	III.	4·7469	406·0	779·8	5·237	+·0015	−·0118	,,	15·32	10·381
XXVIII.	19	4	II.	5·0292	664·6	1066·6	5·238	+·0070	−·0070	,,	15·34	10·377

TABLE VI. B.

Reduction of Experiments at 40° C.

I. *a*.	XV.	XVI.	XVII.	XVIII.	XIX.	XX.	XXI.
Number.	$Q_E t_E$.	Q_S.	$Q_S t_S$.	Σq.	Σ.	θ_0.	L.
VIII.	473·86	·00457	4·99	− 1·56	477·29	39°·965	101·04
IX.	510·68	·00439	5·07	− 5·79	509·96	40·090	101·12
X.	502·22	·00423	4·95	− 3·08	504·09	40·068	100·39
†XI.	479·49	·00424	4·56	− 7·12	476·93	40·078	100·99†
XII.	436·08	·00405	4·48	− 1·14	439·42	40·071	100·22
XIII.	505·96	·00399	4·16	− 7·31	502·81	40·055	100·57
XIV.	499·81	·00330	3·94	+ 2·67	506·42	40·065	100·76
XV.	504·79	·00422	4·58	− 5·31	504·06	40·058	100·70
XVI.	512·05	·00419	5·37	−12·05	505·37	40·063	100·50
XXVI.	439·76	·00446	6·56	− 0·10	446·22	40·021	100·97
XXVII.	478·73	·00445	3·47	− 3·28	478·92	40·025	100·90
XXVIII.	501·77	·00445	4·75	0	506·52	40·025	100·70
Mean (omitting that marked †)						40·045	100·71 ± ·057
If we include all experiments, we get						40·049	100·74 ± ·054

TABLE VII. A.

Experiments at 30° C.

I.		II.		III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.
a.	*b.*	*a.*	*b.*									
Number.	Date. July	No. of Cells.	No. of Case.	M.	t_E.	t_S.	r_1.	$\theta_0' - \theta_0''$.	$d' - d''$.	C_{θ_1}.	Temp. of Cells.	R_{ne}.
† Prelim.	4	3	I.	3·5975	878·8	1617·0	5·691	0	−·0640	312	15·63	10·350
I.	5	4	III.	4·9107	661·6	1341·4	5·515	−·0108	−·0178	,,	15·44	10·353
† II.	5	4	IV.	5·0174	676·6	1210·9	5·177	−·0082	−·0212	,,	15·59	,,
III.	7	4	II.	5·1313	696·0	1102·3	5·473	0	−·0302	,,	15·57	,,
IV.	7	4	IV.	5·0071	674·7	1224·9	5·577	−·0088	−·0187	,,	15·74	,,
V.	7	4	V.	4·5336	607·9	1113·2	5·582	−·0096	−·0121	,,	15·70	,,
VI.	7	4	III.	4·8039	636·6	1144·5	5·120	+·0103	−·0063	,,	15·77	,,
VII.	7	4	II.	5·2472	692·4	1251·5	5·091	+·0061	+·0088	,,	15·89	,,
XXIX.	22	3	V.	4·4083	1045·0	1438·8	5·238	−·0088	−·0085	,,	14·85	10·351
XXX.	22	5	IV.	4·9873	431·1	939·5	5·252	+·0031	−·0178	,,	14·85	10·357
XXXI.	23	5	III.	4·7495	406·7	788·5	5·227	−·0077	+·0037	,,	14·82	10·357
XXXII.	23	3	V.	4·4459	1050·7	1434·8	5·156	0	−·0010	,,	14·90	10·351
XXXIII.	23	5	II.	5·0477	437·8	840·9	5·227	−·0198	−·0068	,,	14·78	10·357
XXXIV.	23	3	IV.	5·0226	1185·7	1645·3	5·241	0	−·0022	,,	14·85	10·351

TABLE VII. B.

Reduction of Experiments at 30° C.

I. *a*.	XV.	XVI.	XVII.	XVIII.	XIX.	XX.	XXI.
Number.	$Q_E t_E$.	Q_S.	$Q_S t_S$.	Σq.	Σ.	θ_0.	L.
† Preliminary.	373·98	·00883	14·3	−19·96	368·32	29°·820	102·36 †
I.	500·60	·00779	10·45	− 8·92	502·13	29·990	102·25
† II.	511·72	·00607	7·35	− 9·17	509·88	29·995	101·61 †
III.	526·68	·00756	8·33	− 9·42	525·59	29·962	102·44
IV.	510·41	·00815	9·98	− 8·58	511·81	30·000	102·24
V.	459·90	·00818	9·11	− 6·77	462·24	29·971	101·97
VI.	481·59	·00580	6·64	+ 1·25	489·48	29·969	101·90
VII.	523·62	·00567	7·10	+ 4·65	535·37	29·977	102·02
XXIX.	445·19	·00635	9·14	− 5·40	448·93	30·104	101·85
XXX.	509·70	·00641	6·02	− 4·59	511·13	30·117	102·46
XXXI.	480·95	·00629	4·96	− 1·25	484·66	30·034	102·05
XXXII.	447·50	·00596	8·55	− 0·31	455·74	30·047	102·50
XXXIII.	517·80	·00629	5·29	− 5·49	517·60	30·042	102·55
XXXIV.	504·26	·00636	10·46	− 0·69	514·03	30·052	102·34
Mean of Experiments XXIX. to XXXIV.						30·066	102·30 ± ·076
†Mean of all (omitting Preliminary and II.).....						30·006	102·22

Remarks on Experiments at 40° C.

Exp. XI.—"Reduced pressure far too much at commencement, was unable to obtain thermal balance until near end of experiment."

Exp. VIII. & IX.—As pointed out (section 10, p. 15), the values of L are here probably in error by about 0·05. This, however, does not seem sufficient reason for rejecting them.

Conclusion.

When $\theta_0 = 40{\cdot}045$, $L = 100{\cdot}71$.

Remarks on Experiments at 30° C.

Preliminary Exp.—The oscillations in θ_1 were very violent, as this was our first attempt at balancing when evaporating benzene; also, as we had no idea when to finish, the value of Σq is very great. We have, however, inserted it to show how little such matters affect the result.

Exp. II.—"After inserting the dropper, had to withdraw it again owing to its sticking in the tube; dropper exposed for some time." There is little doubt that, owing to the cooling when withdrawn, some air was sucked back, and hence there must have been some expulsion on re-introduction. We regret that we continued the experiment: the result is useless.

20. It will be noticed that the irregularities in this Table are greater than in preceding ones. These are due to two causes:—(1) The want of practice in the observers; (2) The value of the galvanometer-swings were not determined with sufficient accuracy until after Experiment VII., and, as the values of Σq are large, any error might affect L, but probably not by more than $\pm 0{\cdot}1$.

In Section 10, p. 15, we have given in full our reasons for considering that all these experiments (I. to VII.) are too low by about 0·12. The mean value of L given by them (omitting Preliminary and II.) is 102·14: thus it is probable that, had the flask been previously saturated with vapour, we should have obtained about 102·26, which is in fair agreement with the mean of the last six (102·30), where this cause of error was non-existent. In any case, taking *all* the values as they stand in Col. XXI. (rejecting the † experiments), we get 102·22. The number 102·30, however, is the most probable value.

Conclusion.

When $\theta_0 = 30{\cdot}066$, $L = 102{\cdot}30$.

Table VIII. A.

Experiments at 20° C.

I.		II.		III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.
a.	*b.*	*a.*	*b.*									
Number.	Date. July	No. of Cells.	No. of Case.	M.	t_E.	t_S.	r_1.	$\theta_0' - \theta_0''$.	$d' - d''$.	C_{θ_1}.	Temp. of Cells.	R_{ne}.
† XXXV.	24	3	III.	4·7593	1129·6	1575·8	5·562	+·0041	−·0194	305	15·30	10·327
XXXVI.	24	3	IV.	5·0083	1182·1	1833·2	5·586	−·0052	−·0143	,,	15·40	10·327

Table VIII. B.

Reduction of Experiments at 20° C.

I. *a.*	XV.	XVI.	XVII.	XVIII.	XIX.	XX.	XXI.
Number.	$Q_E t_E$.	Q_S.	$Q_S t_S$.	Σq.	Σ.	θ_0.	L.
† XXXV.	482·02	·01151	18·14	−4·67	495·49	19·947	104·10 †
XXXVI.	504·42	·01172	21·48	−5·95	519·95	19·947	103·82

Remarks on Experiments at 20° C.

Exp. XXXV.—This should be regarded as a preliminary experiment only. We had no idea of the pressure required to produce equilibrium, and had great difficulties in sufficiently reducing the pressure. "θ_1 was extremely irregular; the galvanometer-swing was often off the scale for a considerable period."

Exp. XXXVI.—"This was a very fair experiment; the thermal balance well under control."

We regret that we were unable to perform more experiments at 20° C. It was found necessary, however, to work the mercury-pump almost continuously, and this could not be done without further assistance*. As such assistance was but rarely available, and as our time was drawing to a close, we were compelled to content ourselves with the above experiments.

Conclusion.

When $\theta_0 = 19{\cdot}947$, $L = 103{\cdot}82$.

21. *Discussion of the Results.*

The values of L over the range 20° to 50° appear to be practically a linear function of the temperature.

If we assume

$$L = 107{\cdot}05 - 0{\cdot}1581\,\theta, \quad . \quad . \quad . \quad . \quad (\alpha)$$

we obtain a very close approximation to the experimental values, as shown by the following Table:—

TABLE IX.

Temperature. Nitrogen-scale.	L (expressed in terms of a thermal unit "at 15° C.").	
	By formula (α).	Experimental results.
50·014	99·14	99·14 ± ·034
40·045	100·72	100·71 ± ·057
30·066	102·30	102·30 ± ·076
19·947	103·90	103·82

It is evident, therefore, that the curvature over the above temperature-range must be very slight.

The value of L cannot continue to be a linear function of θ at very high temperatures, either in the case of benzene or

* We return our sincere thanks to Mr. C. T. Heycock, F.R.S., for his help during these experiments at 20° C.

water ; for the critical point of benzene would fall at about 677° C., and that of water at about 990° C. It has, however, been shown that in the case of water the curvature from 0° to 100° (that is, over a pressure-range of about 4 mm. to 760 mm.) may be neglected (Paper W, pp. 316–321) ; and the above Table would indicate that we cannot be far wrong in making a similar assumption with regard to benzene, especially when we consider that both the temperature-range (0° to 80°·2 C.) and the pressure-range (26 mm. to 760 mm.) are in the latter case diminished.

We originally intended to perform a group of experiments at 60°, which is about the highest temperature to which it is advisable to expose the apparatus ; but before doing so it would have been necessary to standardize a third mercury-thermometer—a work which would have occupied at least a week of the short time at our disposal.

Assuming formula (α) to hold to 760 mm., and taking the boiling-point at that pressure as 80°·20 C. (Ramsay and Young, Phil. Mag. 1887), we get

When $\theta = 80{\cdot}2$, $L = 94{\cdot}37$ ("thermal units at 15° C.").

The results of the experimental work described in this paper may be summarized as follows :—

The Latent Heat of Evaporation of Benzene over the temperature-range 20° to 50° C. (nitrogen-scale) is represented by the equation

$$L = 107{\cdot}05 - 0{\cdot}158\,\theta,$$

where L is expressed in terms of a "thermal unit at 15° C."

Assuming this expression to hold over the range 50° to 80°, the resulting value of L at 80°·2 C. (760 mm. pressure) is **94·37**.

22. *Historical.*

We have been unable to find records of any determinations in addition to the following :—

REGNAULT. *Mémoires de l'Académie*, 1862, vol. xxvi. p. 761 : L = 109·0.

R. SCHIFF. Liebig's *Annalen*, 1886, vol. ccxxxiv. p. 338 : L = 93·4 at 80°·35.

K. WIRTZ. Wiedemann's *Annalen*, 1890, vol. xl. p. 438 : L = 92·9 at 80°·1.

JAHN. *Zeitschrift für Physikalische Chemie*, 1893, vol. xi. p. 79 : L = 107·6.

II. *A Method of Comparing directly the Heats of Evaporation of different Liquids at their Boiling-points.* *By* Miss Dorothy Marshall, *B.Sc.*, *and* Prof. W. Ramsay, *Ph.D.*, *F.R.S.*, *University College, London* *.

1. THE heat of evaporation of a liquid is usually determined by measuring with the aid of a calorimeter the amount of heat liberated during the condensation of a known mass of the vapour ; or conversely, by measuring the heat absorbed during the vaporization of a known mass of the liquid. The former method is the one more frequently adopted, *e. g.* by Berthelot and by Schiff.

Apart from the direct experimental difficulties of the method (and these are by no means inconsiderable), it is open to the objection that it measures, not simply the heat set free during the passage of a given mass of the substance from the gaseous to the liquid condition, but the sum of this quantity of heat with that liberated during the cooling of the liquid from the temperature of its boiling-point to the final temperature of the calorimeter. It is therefore necessary to make a separate determination of the thermal capacity of the liquid between these temperatures.

Whatever the cause may be, the fact remains that the data regarding heats of vaporization are very scanty. The subject has been worked at by few observers, and the results obtained show very considerable discrepancies. Taking, for instance, the case of ethyl alcohol, the heat of vaporization of a gramme is given as 208 calories by Despretz, 227 by Brix, 202·4 by Andrews, and 208·9 by Favre and Silbermann. It is clear that no method can be really satisfactory that leads to such discordant results.

2. Professor Ramsay suggested some time ago that it should be possible to compare directly the heats of evaporation of two liquids, by raising the temperature of each to its boiling-point (surrounding each vessel with a jacket of its own vapour), and then determining the loss of weight sustained by each vessel when a current of electricity was passed through a carbon filament immersed in the liquid. All other conditions being made the same for the two vessels, the ratio of their losses of weight should give the inverse ratio of the heats of evaporation of the liquids. This method of comparison we have since been engaged in working out ; and having once overcome the initial difficulties of apparatus &c., we have been able to obtain some very satisfactory results,

* Communicated by the Physical Society : read November 8, 1895.

and intend to apply the method to a number of other liquids that we have not yet had time to investigate.

In such an arrangement as this there is no direct heat measurement, and therefore a fruitful source of error is eliminated. As the current passes, heat is developed in each carbon filament; and this heat is entirely expended in converting a portion of the liquid into vapour. The thermal capacity of the liquid no longer needs to be known, since the experiment is conducted entirely at the temperature of its boiling-point.

Each vessel is weighed at the temperature of the atmosphere at the beginning and end of the experiment; and the loss of weight of the vessel during the experiment gives the amount of liquid vaporized. If the same current be passed for the same time through two filaments of equal resistance, equal amounts of heat will be developed: hence the ratio of the loss of weight of the first vessel to the loss of weight of the second is the inverse ratio of the heats of vaporization of the liquids employed.

If the filaments have not equal resistances, the amounts of heat developed will be directly as the resistances. If the heat of evaporation of any one liquid be known, the absolute value of the heat of evaporation of any other liquid can be calculated from the ratio directly found.

3. [*Note, July* 1895, *by D. Marshall.*—At the time when this work was begun, the only liquid for which L was at all certainly known was water; but it was not considered advisable to compare directly with water in all the experiments: (i.) because the heat of evaporation of water is so very much greater than that of any other liquid; (ii.) because it is so exceedingly difficult to get it sufficiently pure; and (iii.) because the continued use of water seemed to make the apparatus more liable to break.

Ethyl alcohol was in every way more suitable for a standard, and I therefore tried to make a particularly careful comparison between alcohol and water. But both in those first attempts, and again in several other attempts that I made later on, I had the greatest difficulty in getting even moderately concordant results; and I have never, when working with water, been able to approach the degree of accuracy that I know from my other results I am justified in expecting.

Thus I was finally obliged to conclude that water is not a liquid to use in these experiments at all, much less is it fitted to form our ultimate standard. It became necessary therefore to find some other liquid for which L was accurately known; and as we did not feel sufficient confidence in any

of the published values with which we were acquainted, Mr. E. H. Griffiths, F.R.S., kindly consented to make a determination of the Heat of Evaporation of Benzene by the same method that he used with such conspicuous success in the case of water.

His experiments (in which I also took part) measured the heat of evaporation over the range 30° to 50° C.; and gave by extrapolation the value at 80°·2, the boiling-point. This number is therefore the one finally adopted by us in the calculation of our results; and we take this opportunity of recording our sincere thanks to Mr. Griffiths for furnishing us with such a trustworthy standard.

It should be observed that this number finally obtained for the heat of evaporation of benzene at its boiling-point does not in any way affect our work regarded as a method of comparison; but it does enter as the most important feature into the calculations by which we pass from our ratios to the absolute values of the heats of evaporation, and bring our results into line with those of other observers.]

4. *Description of the Apparatus.*

In the first experiments vessels were used closely resembling the ordinary incandescence-lamps, with carbon filaments; but a good deal of trouble was caused by the extreme difficulty of making satisfactory connexions. After the preliminary experiments, these lamps were therefore rejected in favour of others which had a spiral of fine platinum wire in place of the carbon filament: the ends of the spiral were attached to stout platinum terminals sealed into the glass; the terminals were gilt and amalgamated, and thus a safe and good contact was secured. The upper part of each lamp was drawn out into a rather narrow open tube, through which the liquid could be introduced, and which could be securely closed to prevent loss during weighing. The lamp was set up in an ordinary vapour-jacketing arrangement provided with side-bulb and condenser. The jacket was closed at the bottom by an indiarubber cork, through which passed two U-tubes containing mercury; the terminals of the lamp rested on the inner ends, into the outer ends dipped the wires carrying the current. The cork was protected by a layer of mercury so that it could not come into contact with the liquid. Each lamp was jacketed with the vapour of its own liquid, so that the temperature of its contents would be raised to the boiling-point without the possibility of ebullition taking place until the current was started; the liquid then lost by boiling from

the lamp was condensed in the jacket, and there was no loss of material.

The two lamps were connected in series.

In the earlier experiments the current used was that supplied at 110 volts to run two incandescence-lamps of 32-candle power (in parallel): it was further cut down to a convenient strength by the introduction of an appropriate resistance. At first this resistance consisted of a solution of copper sulphate —therefore both indefinite and variable: in the later experiments a coil of wire of about 20-ohms resistance was substituted for the copper-sulphate solution.

In performing an experiment, the liquid in the side-bulb of the jacket was first caused to boil; and the current was not started until the whole contents of the lamp were judged to have reached the temperature of the condensing vapour: if this were so, the liquid would pass into tranquil ebullition the moment the circuit was completed. It was generally found advisable to drop into the lamp a little glass capillary tube to provide a starting-point for boiling, as most of the liquids showed a great tendency to become superheated and bump. A number of experiments were rejected for this reason.

5. *Preliminary Experiments.* February 1894.

In the experiments with carbon lamps, the resistance of each filament was determined by the Wheatstone-bridge method, using a post-office box of coils for bridge, at five different temperatures; and the results were plotted on squared paper. The observations for each lamp lie approximately on a straight line, the resistance falling off uniformly as the temperature rises.

The resistance of each filament at 0° and 180°, the decrease per degree, and the coefficient of change of resistance between 0° and 180° were found from the curves.

Table I.

	I.	II.	III.
Resistance at 0°	42·5 ohms.	43·17	42·60
Resistance at 180°	36·69 „	36·58	36·10
Decrease per degree	0·0323	0·0366	0·0361
Coefficient of change of resistance	0·00076	0·00085	0·00085

These experiments gave no satisfactory results.

6. *First Series of Experiments.* March 1894.

Two lamps were now made with spirals of fine platinum wire in place of carbon filaments. These platinums were each 18 inches long, and coiled into spirals with all possible care to prevent neighbouring turns from touching and short-circuiting. The resistances of the platinums were determined by the Wheatstone-bridge method. The coefficient of increase of resistance of platinum with rise of temperature being (approximately) known, the resistance of each spiral for any given temperature could be calculated. It was assumed that the temperature of the spiral was the same as that of the liquid boiling round it.

A comparison of chlorobenzene and alcohol may serve for an example :—

Date.	B.	Weight evaporated.		R_1.	R_2.	Corrected Weights.		Ratio $\frac{L_1}{L_2}$.	L_1.
		C_6H_5Cl.	C_2H_5OH.			C_6H_5Cl.	C_2H_5OH.		
March 15.	746·3	14·5830	4·7975	4·89	4·27	14·5830	5·5252	0·364	76·1
April 20.	763·0	16·6645	5·4195	4·87	4·31	16·6645	6·1236	0·367	

TABLE II.

Substance.	Date.	B.	Ratio.	L.
Alcohol referred to water*	May 1	767	0·387	208·1
	„ 4	755	0·388	
CS_2	Mar. 11	741·9	0·405	85·0
Acetic Acid	May 10	754·4	0·475	98·6
„ „	„ 15	756·8	0·472	
„ „	„ 16	757·7	0·472	
Ether	„ 17	752·7	0·396	82·5
„	„ 30	752·3	0·397	
„	June 4	753·9	0·396	
Ethyl Acetate	„ 5	754·4	0·409	85·1
„ „	„ 6	753·8	0·409	
Methyl Formate	„ 7	755·2	0·514	106·7
„ „	„ 11	754·1	0·511	
Methyl Butyrate	„ 12	758·3	0·389	81·0

* The other substances are referred to Alcohol.

7. *Second Series of Experiments.* June 1894.

Up to this time the resistance of the platinum spiral during the passage of the current had never been directly measured, but was calculated from the cold resistance and the coefficient of increase with rise of temperature, on the assumption that the temperature of the spiral was the same as that of the surrounding liquid. The truth of this assumption was now called in question, and it was found to be unjustifiable. The resistance of the spiral was directly measured, first when surrounded by a liquid at its boiling-point, then when the current was passed so that the liquid boiled.

The resistance was always found to be greater when the current was passing; that is to say, the temperature of the spiral is always higher than that of the liquid boiling round it*. One does not exactly know what conditions govern the difference of temperature between spiral and liquid. The probability is that the ratio of the resistances when two liquids are compared will be altered, as well as the absolute values of the resistances themselves; and this would tend to alter all the results hitherto obtained.

It should also be observed that the coefficient of increase of resistance of platinum with rise of temperature is that given by Callendar (1·338; Phil. Trans. clxxxii. A, p. 136) for a particular sample of wire employed by him; but as this coefficient is found to be different for different samples of wire, the value adopted may very probably not hold for the particular wire here used.

It was therefore necessary to determine directly the resistance of the platinum spirals in each experiment. The resistance of the wire during the passage of the current was now determined by Poggendorff's method:—viz., comparison with the resistance of a standard wire carrying the same current. [For diagram of arrangements *vide infra*, p. 45.]

The standard resistance was a piece of uninsulated german-silver wire, immersed in cold water to keep its temperature constant.

Table III. shows the results obtained.

* Griffiths, Phil. Trans. clxxxiv.A (1893), pp. 400–407.

TABLE III.

Substance.	Date.	Ratio $\frac{L_1}{L_2}$.	L_1.
Water	July 2.	0·391	
,, ...	,, 6.	0·390	209·74
,, ...	,, 11.	0·391	
Chlorobenzene	,, 18.	0·355	
,, ...	,, 19.	0·363	75·2
,, ...	,, 19.	0·358	
Ether	,, 12.	0·398	83·5
,, ...	,, 13.	0·398	
Ethyl Acetate	,, 20.	0·392	82·2
,, ,, ...	,, 23.	0·392	
Acetic Acid	,, 24.	0·471	100·2
,, ,, ...	,, 25.	0·484	

8. *Third Series of Experiments.* February 1895.

In the early part of this year the apparatus was further modified with a view to increasing the accuracy of the resistance measurements.

The readings of resistance had been found very troublesome and very uncertain. In many experiments it was almost impossible to obtain anything like a point of balance on the bridge-wire: it was useless to attempt to read to within 1 centim. length of the wire; and during any one experiment the points of balance did not remain steady at all, but seemed to shift indefinitely. This might be the consequence either of variations in the strength of the current in the main circuit, or of change in the resistance of the bridge-wire from unequal heating. [The bridge consisted of a coil of uncovered platinoid wire immersed in cold water and connected in series with a platinum wire stretched against a scale.] Probably both causes were working together.

Two important changes were made:—

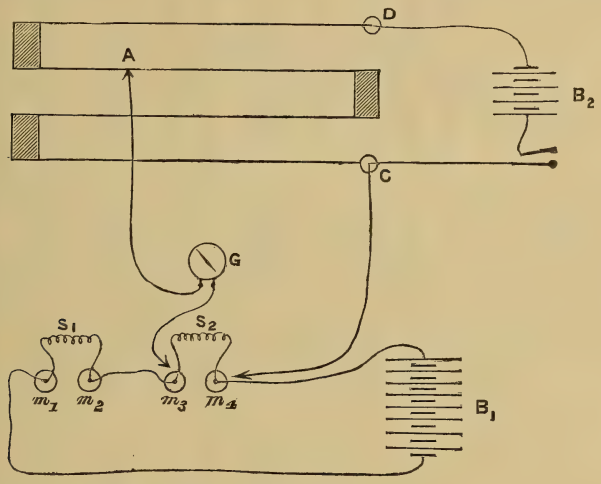
(i.) The current in the primary or lamp-circuit was taken from 8 secondary cells, instead of being taken from the mains through a (probably) variable resistance.

(ii.) A new bridge was made, consisting of four parallel german-silver wires of approximately equal resistance stretched against four metre-scales.

The whole of the resistance in the secondary circuit now falls into the bridge itself; whatever heating-effect there may be is distributed all along the bridge, and its effect is much lessened if not altogether removed. The resistance of the bridge-wire does not need to be known, provided it is tolerably

uniform ; because we are only concerned with comparative measurements, which are simply proportional to the lengths measured.

For the same reason the standard wire of known resistance is really unnecessary, and was omitted in all the remaining experiments.



S_1 S_2 are the platinum spirals of the lamps.
$m_1 \ldots\ldots m_4$ are mercury-cups by which connexion is made with S_1 and S_2.
B_1, B_2 are the batteries supplying the current in the primary and secondary circuits respectively.
C, D are the terminals of the fourfold potentiometer wire.
A is a sliding contact, and has to be moved until the fall of potential on the bridge between C and A is equal to the fall of potential between m_4 and m_3, or m_2 and m_1.
G is a simple reflecting-galvanometer.

It became evident at once—working in these improved conditions—that the whole method had been made very much more sensitive and accurate than before ; hence it was of the utmost importance to repeat the fundamental comparison between alcohol and water from which all the other results were calculated. These experiments are entered in Table IV.

It was at this point that we gave up the idea of taking water for our standard, and looked round for some other liquid to use in its place. It is quite clear, from the numbers obtained, that there is something in the water itself that interferes with the experiments. Probably there is some electrolysis during the passage of the current along the uninsulated wire ; it appears impossible by any ordinary means to get water so pure that its conductivity may be disregarded.

TABLE IV.

No.	Date.	B.	Weights evaporated.		V_1.	V_2.	Ratio $\frac{V_1}{V_2}$.	Corrected weights.		Ratio $\frac{L_1}{L_2}$.
			Alcohol.	Water.						
1	March 1.	754·1	15·2470	5·4645	274·7	260	1·057	15·2470	5·7740	0·379
2	„ 2.	754·7	14·7365	5·2370	326·2	308·8	1·060	14·7365	5·5510	0·377
3	„ 5.	757·8	14·1615	5·1310	227·2	210·3	1·057	14·1615	5·4215	0·383
4	„ 6.	756·0	13·5130	4·7255	208·0 248·5	195·7 233·1	1·063 1·065	13·5130	5·028	0·372
5	„ 7.	755·6	15·3565	5·5735	207·4 247·4	194·9 233·1	1·064 1·061	15·3565	5·9255	0·385
6	April 9.	759·5	14·0570	5·6460	327·2 275·1	307·8 258·4	1·063 1·064	14·0570	6·0015	0·427
7	„ 9.	759·5	13·7710	5·5770	327·9 275·7	307·4 258·5	1·066 1·066	13·7710	5·9450	0·432
8	„ 10.	761·3	13·7425	5·5420	328·3 276·2	307·6 258·7	1·067 1·067	13·7425	5·9130	0·430

Note.—Experiments 1 to 5 were made with the ordinary distilled water of the laboratory; experiments 6 to 8 were made with some water that had been specially purified for use in water-analysis.

In the later tables the numbers given for the heats of evaporation are calculated from the absolute value for benzene, viz.:—L = 94·4 at 80°·2.

Table V. records the results obtained by us for benzene, toluene, metaxylene, acetic acid, and ten of the lower esters prepared by Prof. Sydney Young. Each liquid is compared with alcohol.

In Table VI. we give for each liquid :—

(A) the ratio compared with benzene;
(B) the value of L in thermal units at 15°;
(C) the value of L obtained from the thermodynamic equation (see p. 51, *infra*);
(D) values obtained by other observers.

In Table VII. we give for each,

L, the heat of evaporation;
t, the temperature of the boiling-point;
M, the molecular weight;
$\frac{ML}{T}$, the quotient of the molecular heat of vaporization by the absolute temperature; according to Trouton's Law this should be a constant.

9. Table V.

No.	Date.	B.	Weights evaporated.		Ratio $\frac{V}{V^1}$.	Corrected weights.		Ratio $\frac{L_1}{L_2}$.	Mean.
				Alcohol.					
Benzene.									
1 ...	May 21.	752·8	16·976	8·193	0·907	16·976	7·431	0·438	
2 ...	„ 22.	755·7	18·662	8·950	0·906	18·662	8·1085	0·434	
3 ...	„ 23.	757·9	19·010	9·150	0·901	19·010	8·244	0·434	
4 ...	„ 24.	756·7	17·098	8·286	0·905	17·098	7·499	0·438	0·436
Toluene.									
5 ...	May 28.	...	17·359	7·136	0·972	17·359	6·936	0·400	
6 ...	„ 29.	763·2	17·046	7·040	0·972	17·046	6·843	0·401	0·401
Metaxylene.									
7 ...	May 30.	755·7	16·309	6·063	1·031	16·309	6·251	0·386	
8 ...	June 6.	767·6	16·755	6·150	1·036	16·755	6·3715	0·380	0·383
Acetic Acid.									
9 ..	June 11.	758·6	19·517	8·956	0·980	19·517	8·777	0·450	
10 ...	„ 12.	762·2	17·033	7·754	0·984	17·033	7·630	0·448	
11 ...	„ 19.	752·4	19·330	8·613	0·984	19·330	8·475	0·447	0·448

Table V. (*continued*).

No.	Date.	B.	Weights evaporated.		Ratio $\frac{V_1}{V_2}$.	Corrected weights.		Ratio $\frac{L_1}{L_2}$.	Mean.
	Methyl Formate.								
12 ...	March 25.	740·8	15·896	10·1455	0·800	15·896	8·1165	0·511	
13 ...	„ 26.	742·3	14·5725	9·289	0·795	14·5725	7·385	0·507	
14 ...	„ 27.	736·8	10·1955	6·5265	0·794	10·1955	5·181	0·508	0·509
	Ethyl Formate.								
15 ...	April 3.	758·8	17·208	8·883	0·843	17·208	7·4885	0·435	
16 ...	„ 3.	758·8	17·8375	9·2205	0·846	17·8375	7·8105	0·437	
17 ...	„ 5.	754·3	12·9885	6·712	0·844	12·9885	5·665	0·437	0·436
	Methyl Acetate.								
18 ...	March 27.	735·7	17·158	9·0365	0·851	17·158	7·690	0·448	
19 ...	„ 28.	731·6	18·1105	9·547	0·853	18·1105	8·1435	0·449	
20 ...	„ 28.	731·6	16·220	8·5165	0·853	16·220	7·2645	0·448	0·448
	Propyl Formate.								
21 ...	Dec. 20.	...	16·896	7·7695	0·916	16·896	7·117	0·421	
22 ...	Feb. 6.	754·5	17·6585	8·108	0·908	17·6585	7·362	0·417	
23 ...	„ 20.	768·1	17·675	8·2215	0·910	17·675	7·4815	0·423	0·420
	Ethyl Acetate (referred to Benzene).								
24 ...	Oct. 24.	748·5	16·137	15·451	1·121	16·137	17·321	0·932	
25 ...	„ 25.	748·2	15·001	14·275	1·129	15·001	16·110	0·931	
26 ...	„ 26.	748·9	15·279	14·449	1·131	15·279	16·342	0·935	0·933
	Methyl Propionate.								
27 ...	March 19.	760·5	17·6085	8·012	0·903	17·6085	7·235	0·411	
28 ...	„ 22.	758·8	16·204	7·369	0·904	16·204	6·6615	0·411	
29 ...	„ 23.	758·0	16·158	7·3375	0·903	16·158	6·626	0·410	0·411
	Propyl Acetate.								
30 ...	March 18.	764·9	17·8735	7·206	0·951	17·8735	6·855	0·384	
31 ...	„ 28.	731·6	17·508	7·0885	0·950	17·508	6·734	0·385	
32 ...	„ 29.	736·8	14·545	5·9035	0·948	14·545	5·5965	0·385	0·385
	Ethyl Propionate.								
33 ...	April 4.	762·9	18·5595	7·3745	0·942	18·5595	6·947	0·374	
34 ...	„ 4.	762·9	14·111	5·676	0·943	14·111	5·3625	0·380	
35 ...	„ 5.	754·3	16·2815	6·5735	0·944	16·2815	6·2055	0·381	0·378
	Methyl Butyrate.								
36 ...	March 29.	736·8	14·836	5·738	0·952	14·836	5·4625	0·368	
37 ...	April 2.	755·7	15·207	5·8455	0·950	15·207	5·553	0·365	
38 ...	„ 2.	755·7	15·064	5·853	0·951	15·064	5·466	0·372	0·368
	Methyl Isobutyrate.								
39 ...	April 8.	759·9	17·0765	6·4145	0·926	17·0765	5·940	0·348	
40 ...	„ 8.	759·9	15·7095	5·814	0·926	15·7095	5·384	0·343	0·346

TABLE VI.

	A.	B.	C.	D.
Benzene	1·000	94·4	.....	109·0 Regnault II.
				93·4 Schiff X.
				92·9 Wirtz XI.
				107·6 Jahn XII.
Toluene	0·920	86·8	.....	83·6 Schiff X.
Metaxylene	0·877	82·8	.....	78·3 Schiff X.
Alcohol	2·293	216·5	.....	202·4 Andrews I.
				227 Brix VI.
				208 Despretz VII.
				208·9 Favre & Silbermann VIII.
				206·4 Schall IX.
				205·0 Wirtz XI.
				201·4 Longuinine XIII.
Acetic Acid	1·028	97 0	92·7	120·8 Berthelot IV.
				84·9 Berthelot & Ogier V.
				101·9 Favre & Silbermann VIII.
Methyl Formate ...	1·167	110·1	119·1	117·1 Andrews I.
				115·2 Berthelot III.
Ethyl Formate ...	1·000	94·4	93·4	105·3 Andrews I.
				100·4 Berthelot III.
				92·2 Schiff X.
				113·25 Jahn XII.
Methyl Acetate ...	1·028	97·0	96·8	110·2 Andrews I.
				94·0 Schiff X.
				113·86 Jahn XII.
Propyl Formate ...	0·956	90·2	87·3	85·3 Schiff X.
Ethyl Acetate	0·933	88 1	84·3	92·7 Andrews I.
				83·1 Schiff X.
				84·3 Wirtz XI.
				102·14 Jahn XII.
Methyl Propionate	0·943	89·0	87·7	84·2 Schiff X.
Propyl Acetate ...	0·881	83·2	79·8	77·3 „
Ethyl Propionate ..	0·867	81·8	79·2	77·1 „
Methyl Butyrate...	0 844	79·7	72·5	77·3 „
Methyl Isobutyrate	0·794	75·0	76·3	75·5 „

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TABLE VII.

	Ratio to benzene.	L.	*t*.	M.*	$\frac{M.L}{T}$.
Benzene	1·000	94·4	80°·2	77·40	20·65
Toluene	0·920	86·8	110·8	91·30	20·61
Metaxylene	0·877	82·8	138·5	105·20	21·03
Water	.....	536·6	100°	17·86	25·64
Alcohol	2·293	216·5	78·2	45·66	28·09
Acetic Acid	1·028	97·0	118·5	59·52	14·72
Methyl Formate	1·167	110·1	31·8	59·52	21·45
Ethyl Formate	1·000	94·4	54·3	73·42	21·13
Methyl Acetate	1·028	97·0	57·1	73·42	21·53
Propyl Formate	0·956	90·2	80·9	87·32	22·38
Ethyl Acetate	0·933	88·1	77·15	87·32	21·93
Methyl Propionate	0·943	89·0	79·7	87·32	21·99
Propyl Acetate	0·881	83·2	101·25	101·22	22·45
Ethyl Propionate	0·867	81·8	99·2	101·22	22·22
Methyl Butyrate	0·844	79·7	102·7	101·22	21·43
Methyl Isobutyrate	0·794	75·0	92·3	101·22	20·74

* The molecular weights are calculated by taking

H=1.
O=15·86 (Rayleigh & Scott).
C=11·90 (Stas :—C=12·005 when O=16).

The absolute zero is taken as −273°·7.

10. The results of these experiments may, we think, be regarded as satisfactory so far as they go.

A good many difficulties have been encountered, but most of them have been overcome; and with a few more improvements in matters of detail we believe that the method will become almost as simple in practice as in theory, and will give results as good as can be required for ordinary purposes.

We do not profess or desire to aim at such accuracy as Mr. Griffiths has reached in his work on water; our method does not and cannot lend itself to such refinements.

We rather hope to obtain for a large number of liquids values that are certainly within one per cent. of the truth, and often much nearer.

Up to the present our experiments have mostly been limited to Dr. Young's esters, which were ready to hand; but we propose to extend them now, as far as possible, to any other liquids that we can obtain pure, and whose boiling-points fall between 30° and 150°.

We desire, in conclusion, to express our thanks to Prof. Carey Foster for the use of his laboratory in the later part of the work, and to Mr. A. W. Porter for his kind assistance and advice in the matter of the resistance measurements.

Appendix.

I. Calculation of L from the thermodynamic equation; worked out for the case of *Methyl Acetate* from experimental data given by Prof. Young (J. C. S. lxiii. p. 1191):—

$$L=(S'-S)\cdot\frac{dp}{dt}\cdot\frac{T}{J}.$$

(i.) To find $\frac{dp}{dt}$, by means of Biot's formula

$$\log p=a+b\alpha^t.$$

t.	P.
50°	588·2 millim.
60°	837·5 ,,
70°	1167·0 ,,

hence $\log p=5{\cdot}2852200-2{\cdot}515695\times({\cdot}9937262)^{t-50}$

at 57°·0 . . . $p=755{\cdot}0043$ millim.
at 57°·2 . . . $p=760{\cdot}2895$,,

$$\therefore\ \frac{dp}{dt}=26{\cdot}4260 \text{ millim. per } 1^\circ \text{ at } 57^\circ{\cdot}1.$$

$$\frac{dp}{dt}=2{\cdot}64260\times 13{\cdot}596\times 980{\cdot}94 \text{ dynes per sq. cm. per } 1^\circ.$$

(ii.) S'=vol. of 1 gr. vapour at 57°·1=350·6 c.c.
S=vol. of 1 gr. liquid at 57°·1=1·1 c.c.
$\therefore\ (S'-S)=349{\cdot}5$ c.c.
$T=273{\cdot}7+57{\cdot}1=330{\cdot}8.$
*$J=4{\cdot}199\times 10^7.$

Hence **L=96·81 thermal units.**

II. *Regarding the Purity of the Liquids employed.*—The benzene was bought from Messrs. Kahlbaum, of Berlin, and guaranteed free from thiophene. It was distilled twice from phosphoric anhydride until its boiling-point remained constant.

* The value of J given here is that determined by Griffiths (Phil. Trans. vol. clxxxiv. 1893 A), and used in working out the experiments on benzene; even if not correct, it is still the right value to use here, because it was determined by means of the same standards as those by which the quantity of heat developed in the benzene experiments was determined; so that any errors would eliminate.

This assumption as to the value of J is the only one that enters into the calculation; all the other data were obtained by direct experiment.

The toluene had been used in some of Prof. Ramsay's previous work; it was distilled once and showed a constant boiling-point.

The metaxylene was fractionated three times, and the portion distilling between 138·1 and 138·7 was used; it was probably not quite pure.

The alcohol was obtained from Messrs. Baird and Tatlock, and distilled from quicklime, and finally with a little sodium, until its boiling-point remained constant.

The acetic acid was bought from Messrs. Kahlbaum; it was fractionated by freezing, and the portion melting at 16°·2 was separated and used.

The esters were prepared by Prof. S. Young, F.R.S., and all details may be found in his paper (J. C. S. lxiii. p. 1191).

III. *The Radial Cursor: a new addition to the Slide-Rule.* *By* F. W. Lanchester*.

THE slide-rule in its ordinary form is chiefly useful in working simple problems involving multiplication and division, also involution and evolution where the indices occurring are integers. When an expression involves fractional indices its solution is only possible, on a slide-rule of ordinary construction, by employing the scale of logarithms on the back of the slide in the same manner as a table of logarithms, when the process becomes so clumsy as to practically destroy its utility.

The object of the present improvement is to enable the operator to solve problems of the latter class with the same ease and degree of accuracy as that only previously attainable in connexion with the simpler cases.

In thermodynamic problems such as arise in dealing theoretically with gas- and other heat-engines, the need of some method of computation more rapid than that of ordinary arithmetic involving many references to tables becomes evident, and it was in connexion with work of this description that the author was led to devise the instrument which is the subject of the present paper.

In the case of the isothermal curve of a perfect gas, the ordinary slide-rule is all that can be required, for the curve is represented by the simple expression

$$pv = \text{constant}.$$

* Communicated by the Physical Society: read October 25, 1895.

For the sake of brevity let us name the scales in a rule of the ordinary "Gravet" type thus

A

B

C

D

becoming when the slide is inverted,

A

C

B

D

Then with the slide in the inverted position the corresponding pressures and volumes of an isothermal can be read off directly on the scales A and B.

In the case of an adiabatic curve, however, we have an expression of the form

$$pv^{\gamma} = \text{const.},$$

where γ has a value 1·4 more or less according to the physical properties of the gas or mixture of gases we are dealing with. In order to deal with an expression of this form, it is necessary to provide some ready means of dividing the scales A and B proportionately to the value of γ which corresponds to the division and multiplication of the respective logarithms of the quantities dealt with, in the proportion of the indices of p and v, that is $1 : \gamma$.

This proportionate division of the scales is effected in the new Cursor by a radial index-arm, indicated in fig. 1 by lines f and g: this is arranged to swing about a stud which is itself carried on a sliding bar running in guides at right angles to the rule. All readings are taken at the points of intersection of the line with the edges of the slide. In the actual instrument the radial arm is made of a transparent slip of celluloid on which is scratched a straight line true to the centre of the stud-hole. The back view shows (fig. 2) the transverse bar graduated to give the indices to which it is desired to work. In fig. 2 this transverse bar is shown set to 1·408, the γ of a perfect gas.

A few words may be added with regard to the method of

Fig. 1.

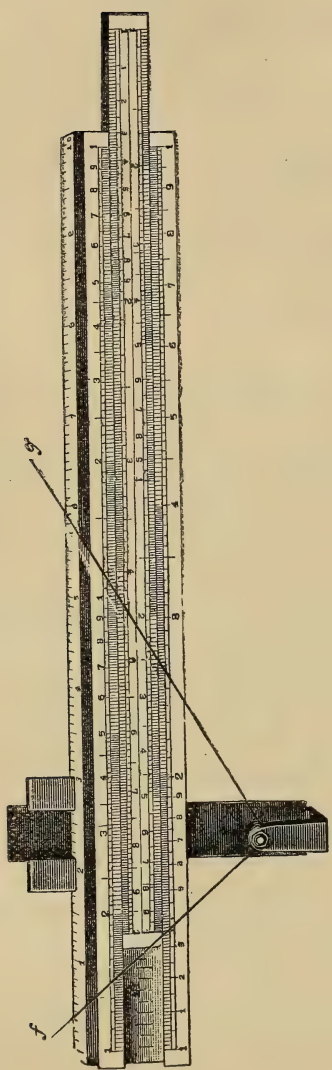


Fig. 2.

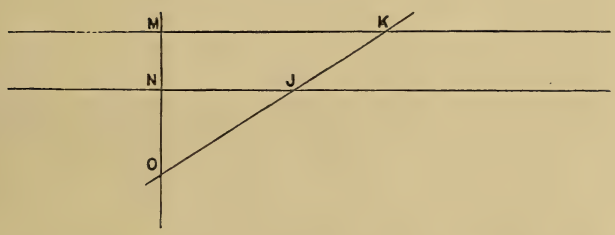
graduation of the transverse bar. Referring to fig. 3, in which M N is the width of the slide, and O is the centre of the radial arm stud-hole, for any index γ, we have

$$\gamma = \frac{MK}{NJ} = \frac{MN+NO}{NO} = \frac{MN}{NO} + 1,$$

where OK is any position of the index line, or

$$NO = \frac{MN}{\gamma - 1}.$$

Fig. 3.



The following examples will serve as an illustration of the class of calculation for which the instrument is designed.

1. *To determine the maximum pressure of an adiabatic compression.*

Set the slide on back of cursor to 1·408 (γ). Suppose, for example, that the initial pressure be 15 lbs. per square inch, and that 100 vols. be compressed to 26 vols.

Invert slide. Set index line of cursor to 15 on A, then set 1 (100) on $\mathfrak{g}$ to index line of cursor. Now set index line to 26 on $\mathfrak{g}$ and read on scale A the corresponding pressure $=100$ lbs. absolute, 85 above atmosphere (figs. 1 and 2).

2. The work done in any adiabatic compression can easily be calculated from the above by the following formula :—

$$W = \frac{1}{\gamma - 1} p_1 V;$$

where W = work done in ft.-lbs., V = final vol. of the gas (after compression) in units of 12 cub. in. (to harmonize with ft.-lbs. and pressure in pounds per square inch), and $p_1 =$ difference of pressure between adiabatic and isothermal maximum pressure for the given ratio of compression (fig. 4).

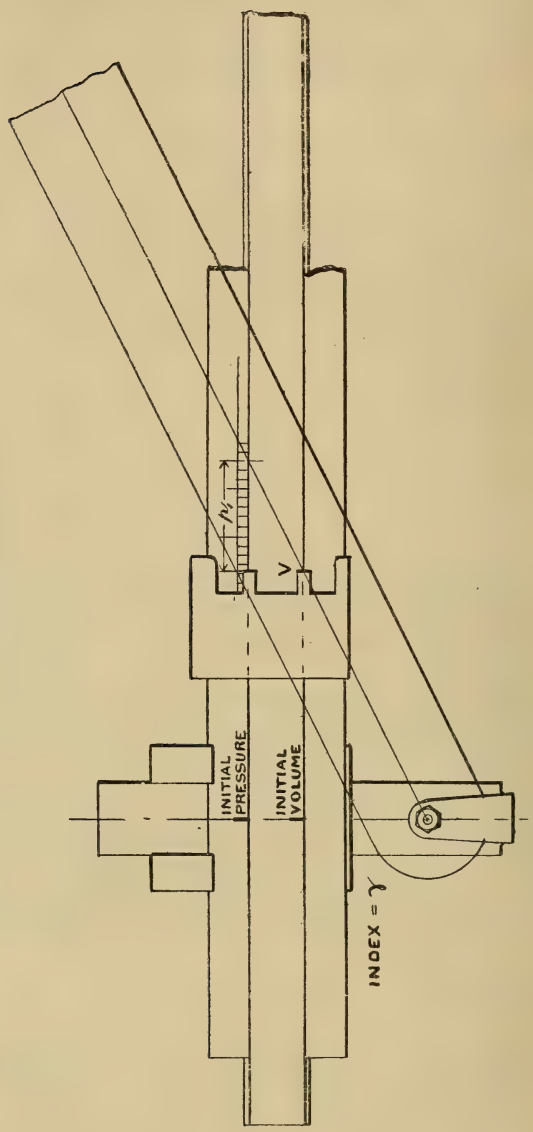


Fig. 4.

Work done $= \frac{1}{\gamma - 1} p_1 V.$

3. The temperature of an adiabatic at any point can be determined from either pressure or volume by using suitable indices for

$$v \,.\, T^{\frac{1}{\gamma-1}} = \text{constant};$$

also

$$p \,.\, T^{\frac{\gamma}{1-\gamma}} = \text{constant}.$$

In determining temperatures from volumes or *vice versâ* it is only necessary to proceed as before (1). In dealing, however, with pressures and temperatures the index is negative (where γ is $>$ unity), the interpretation of this being that an increase of pressure is accompanied by an increase in the temperature. The slide therefore does not require to be reversed as in pv and vT calculations; but with a rule of the Gravet type, owing to the scales C and D being the square roots of A and B, it is necessary to use a fictitious index obtained by dividing $\frac{\gamma}{1-\gamma}$ by the logarithmic ratio of the scales, giving $\frac{\gamma}{2(1-\gamma)}$. It would also be quite possible to graduate one edge of the index slide with fictitious values to compensate for the ratio of the scale-readings.

4. In the process described in (1) we have the means of plotting a series of adiabatics or isentropics, and each position of the radial cursor on the rule gives in its readings on the index line a corresponding isentropic, and a movement of the cursor to the left corresponds to a gain in entropy and *vice versâ*, and it can be shown that equal increments of linear movement of the cursor correspond to equal increments of entropy. In other words, the scale of inches or centimetres on the edge of the rule may be regarded as a scale of entropy in arbitrary units.

5. The following is a very good example of the use of the radial cursor in practice.

The theoretical efficiency of an Otto cycle gas-engine is independent of the working temperatures and is given by the expression $1-\left(\frac{v_2}{v_1}\right)^{\gamma-1}$, where v_1 is the total effective volume of the cylinder and compression-space, and v_2 = volume of compression-space alone.

Taking $\gamma=1{\cdot}408$, $\gamma-1={\cdot}408$. This is beyond the range of the instrument, so we will employ $\frac{1}{{\cdot}408}=2{\cdot}45$.

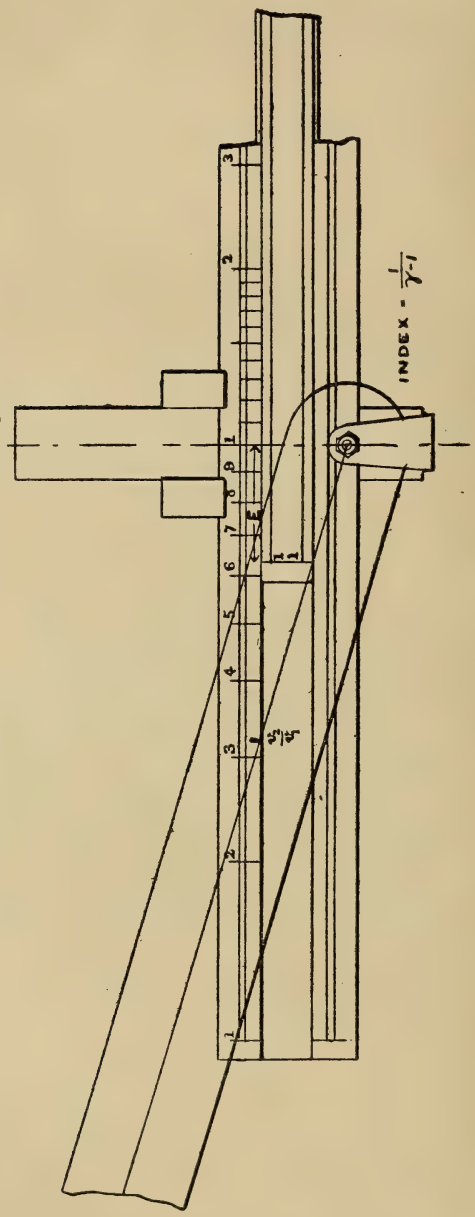


Fig. 5.

Efficiency of Otto Cycle Gas Engine $= 1 - \left(\frac{v_2}{v_1}\right)^{\gamma-1}$.

Let us suppose that in the case we are dealing with $\frac{v_2}{v_1}$ comes out at ·32 (an actual example, see 'The Engineer,' April 6th, 1894).

Set the index slide to 2·45 and place cursor accurately opposite middle 1 on scale A (use one of end lines on slide to centre with), and set index line to ·32 $\left(\frac{v_2}{v_1}\right)$ on scale A (over to the left, fig. 5).

Now using one of the end lines on slide read on scale A opposite to intersection of index line on scale D, *but read backwards from the centre* 1. This will be found to give ·37, or the theoretical efficiency of an Otto engine having ratio of expansion ·32 is ·37.

6. To determine x in such an expression as

$$7^x = 3.$$

Here x will be a fractional index, and we shall first find $\frac{1}{x}$.

Set cursor to 1 in centre of scale A.

Set index line to 7 on scale A, and keeping it there move slide of cursor till index line intersects scale D opposite figure 3 on scale A (use end line of slide).

Now read result on index-scale

$$= 1{\cdot}76,$$

Or $$x = \frac{1}{1{\cdot}76} = {\cdot}568.$$

IV. *Graphical Method for finding the Focal Lengths of Mirrors and Lenses. By* Edwin H. Barton, *D.Sc., F.R.S.E., Senior Lecturer and Demonstrator in Physics at University College, Nottingham*.*

In his 'Geometrical Optics' Aldis gives a graphical method for exhibiting simultaneously the focal length of a concave mirror and the distances from it of any two conjugate foci (page 30 in third edition). The present note consists of the extension of this principle to the cases of a convex mirror and thin lenses and its application to the practical problem of finding focal lengths. This experiment is usually a little perplexing to junior students in a physical laboratory. It is therefore hoped that by giving publicity to this method (which, simple and obvious though it is, does not seem to

* Communicated by the Author.

have found its way into any of the ordinary text-books) the labours both of students and demonstrators may be somewhat lightened.

Let the following notation and convention of signs be adopted :—

Focal length,	f.
Distance of object,	u.
Distance of image,	v.

All distances to be measured from the *mirror* or *lens*, and to be reckoned *positive* when measured *against* the direction of the incident light, and *negative* if *with* that direction.

Then we have :—

For mirrors, $$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}; \quad \ldots \ldots \ldots \quad (1)$$

and for lenses, $$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}. \quad \ldots \ldots \ldots \quad (2)$$

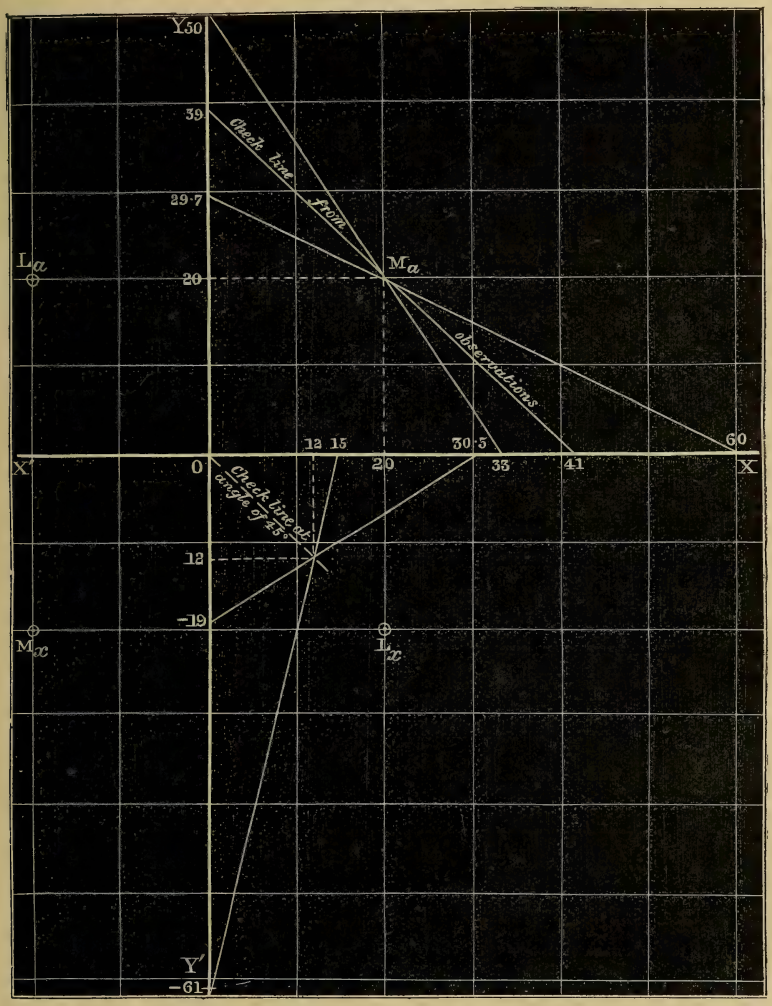
For the graphical method take two rectangular axes of coordinates OX and OY, and let a straight line pass in any direction through the point M_a, whose coordinates are (f, f); then it follows at once from (1) and elementary geometry that its intercepts on the axes of x and y give a pair of corresponding values of u and v respectively for a concave mirror of focal length f (see fig. 1). This is the case already referred to as given in Aldis's 'Geometrical Optics.' Now if the rotating line pass through the point M_x, whose coordinates are $(-f, -f)$, we have, as seen from (1) also, the case for a convex mirror. Similarly, by consideration of (2), we see that the points L_a and L_x, whose coordinates are $(-f, f)$ and $(f, -f)$, give the completing cases for a concave and a convex lens respectively.

The first uses of this graphical method are to afford a picture of the corresponding values of u and v for any mirror or lens for which f is known, and to enable the student to trace the series of values through which v passes as u changes continuously from infinity to zero. But in this view of the matter no very great advantage is gained, as it is almost as troublesome to the student to remember the positions of the fixed points through which the revolving line must pass for the various cases as to recall and rightly use the corresponding formulæ.

The second and more important use of this method is its application to the determination of f for a given mirror or lens when the optical bank is available with which to observe

corresponding values of u and v. This supersedes the use of the formulæ, or the recollection of the positions of the four

Fig. 1.



fixed points mentioned above, avoids the cumbrous calculation of reciprocals, and at the same time exhibits a criterion of the accuracy of the experiment. The student has simply to (1) mark off along the axes of x and y with due regard to sign the corresponding values of u and v as observed on the

bank; (2) join the two points thus obtained; and (3) repeat these operations for another pair of values of u and v. Then the coordinates of the intersection of the two lines give the focal length required. An additional check is obtained by taking a third pair of values of u and v, or by drawing through the origin of coordinates a line inclined at an angle of 45° with the axes. In either case the three lines obtained should intersect in a single point. In any actual experiment this is not likely to be precisely the case; but the value of f can readily be inferred, or, in case of a great discrepancy, the results rejected and the experiment repeated. Below are the data for determining the focal lengths of a concave mirror and a convex lens, which are utilized in the figure.

Example I.—Concave Mirror.

	Observed values of u.	Observed values of v.
	cm.	cm.
First pair	60	29·7
Second pair	33	50
Check pair	41	39
Result:—f=20 centim. nearly.		

Example II.—Convex Lens.

	Observed values of u.	Observed values of v.
	cm.	cm.
First pair	30·5	−19
Second pair	15	−61
Result:—f=−12 centim. nearly.		

V. *A "Direct-reading" Platinum Thermometer.* *By* Rollo Appleyard*.

Messrs. Callendar and Griffiths and their colleagues at Cambridge have proved that the platinum thermometer is capable of measuring temperature with an accuracy of *one thousandth* of a Centigrade degree. For general

* Communicated by the Physical Society: read November 22, 1895.

purposes, temperature is never required to be measured with a precision greater than *one tenth* of a degree. It is my object to consider how the reserve of precision may be utilized in facilitating the processes of testing. The results seem to indicate that it is possible, with a platinum thermometer, to determine ordinary temperatures by a single operation, of such a kind that the readings are "direct"—that is to say, no calculation requires to be applied to them.

In the course of some experiments upon the variation of the electrical resistance of dielectrics with temperature, an apparatus was needed which, by a quick and simple operation, would measure the temperature within the substance of the dielectric. As dielectrics, generally, are bad conductors of heat, it is desirable to take simultaneous readings of temperature at many points within their mass. A platinum thermometer may be regarded as having an *extensible bulb*, for the wire may be distributed to various points within the dielectric; the temperature deduced from the platinum thermometer, at any instant, is therefore equivalent to the mean of the simultaneous readings of a large number of distributed mercurial thermometers. Further, the dielectrics, in their protecting covers, are submerged in a tank of water; the stems of ordinary mercurial thermometers placed, vertically, beneath a water-surface would be difficult to observe; the platinum instrument is far more convenient for this work, for it has the advantage of a *portable stem and scale.*

The resistance of a dielectric* is by no means such a definite quantity as the resistance of a metallic conductor. The difference in quality between samples of the same name, the age of the substance, the nature of the contact between the dielectric and its electrodes, and the so-called "electrification," are such variable factors that the term "dielectric resistance" has only an empirical significance. To arrange for the measurement of temperature with extreme precision is therefore unnecessary.

Range.

For certain practical reasons, the temperatures mentioned in this paper are referred to the Fahrenheit scale. The required range was 32°–120° F.; and, within the present limits of accuracy, it is safe to assume that the resistance, R_t, of the platinum coils, at any temperature t, within this range, is represented by the expression

$$R_t = R_{32}[1+\alpha(t-32)] \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

* See "Dielectrics," Proc. Physical Soc. xiii. p. 155, 1895; Phil. Mag. 1894, xxxviii. p. 396.

The Platinum Coils.

There were six platinum coils, forming six thermometers, each having a resistance of about 7 ohms. Each consisted of about 2 metres of pure platinum wire, of 8 mils diameter, wound in double spirals upon an ebonite rod. Thick copper leads were soldered to the ends of the coils. *To determine* α *and* R_{32} the six coils were separately tested in melting ice, and then at various temperatures in heated water, side by side with a mercurial thermometer. These measurements were made with the coils submerged in a large tank, the temperatures being controlled by a gas-regulator. The resistances were measured on an ordinary "dial" bridge having german-silver coils. This part of the work occupied much time : α was derived from the mean of the best of 30 determinations ; it was somewhat tedious, but the values R_{32} and α proved, in the result, very accurate. The six coils of the six thermometers were afterwards connected in series, and regarded as one thermometer. The mean values were :—

$$R_{32} = 40{\cdot}05 \text{ ohms (six coils, in series).}$$
$$\alpha = {\cdot}002097.$$

The six platinum coils, still connected in series, were then imbedded within their respective dielectrics. *Hollow metal tubes*, for mercurial thermometers, were inserted at the same time within the dielectrics, for check-readings. The dielectrics were then immersed together into one tank of water.

Measurement of R_t *by "Dial" bridge.*

The apparatus was now complete, and tentative tests were made for R_t at different temperatures. The results, when compared with the mean of the readings of the mercurial thermometers in the tubes, were rather discouraging. The temperatures could not be depended upon within about 2° Fahr. These errors were due either to the german-silver coils, or the plugs, of the dial-bridge. Slightly better results were obtained by reducing the battery to a single cell. It was impossible to determine the bridge-temperature, at any time, with accuracy; and, moreover, even if that temperature could have been found *, the temperature-coefficient for different samples of german-silver varies so widely with different samples, that no dependence whatever could have been placed upon values "corrected" by text-book coefficients. The

* A platinum coil interwound with one of the resistance-bobbins would probably be the best method under these circumstances.

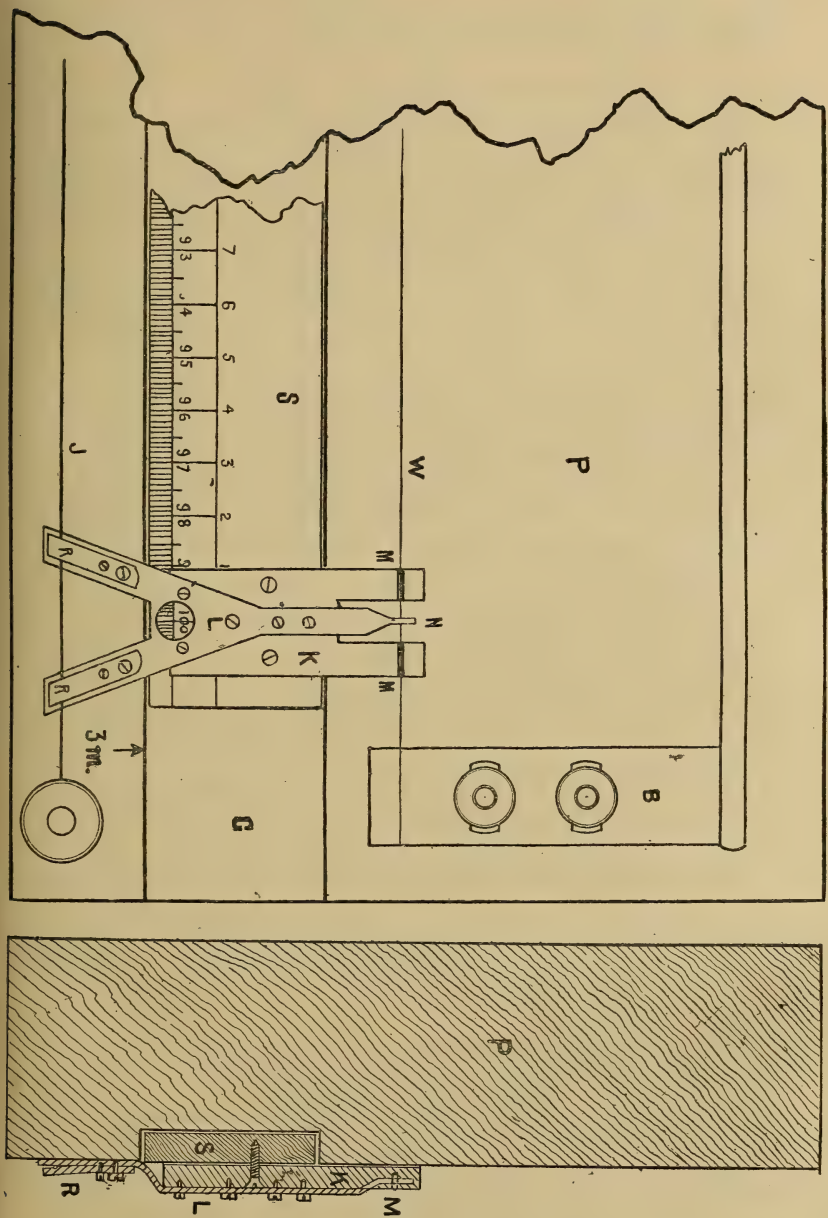


Fig. 1.

"dial" bridge was therefore abandoned, and a slide-bridge of special form was designed for the purpose.

Measurement of R_t *by Slide-bridge.*

The slide-bridge, fig. 1, consists of a 12-ft. plank, P, with a groove G cut along its whole length. A metre-scale, S, can slide in the groove, flush with the upper surface of the plank. Datum-marks are made at successive metres along the plank; subdivisions being read upon the sliding metre-scale. The contact-piece, L, is attached to the metre-scale at its zero-point. This saves the cost of an elaborate graduation of the plank; and keeps the sliding-contact very accurately in position. When necessary, the metre-scale with its contact-piece is easily transferred to other such slide-bridges. There is no distortion of the slide-wire, and indentation is almost impossible*. The slide-wire, W, is of platinoid † stretched between two adjustable blocks of copper, B. Its ends are soldered into saw-cuts made in the copper blocks. On the other side of the scale-groove there is a copper wire, J, for the galvanometer-contact. The contact-piece consists of an ebonite block K, with a gap at one end. The platinoid wire slides through a saw-cut M transversely to this gap in the ebonite. A Y-shaped piece of spring-brass, L, is screwed to the ebonite block. One end of the Y, projecting over the gap, is provided at N with a platinum knife-edge. The other two arms of the Y project beyond the other end of the ebonite block, and carry clips, R, through which the wire for the galvanometer-contact slides.

The bridge-coils, fig. 2, consist of five silk-covered platinoid resistance-bobbins, immersed in a double vessel of copper, the inner vessel containing $1\frac{1}{2}$ gallons of paraffin-oil.

A glow-lamp, submerged in the oil, is used for maintaining the temperature constant at 80° F.; this temperature was chosen as being just above ordinary air-temperature.

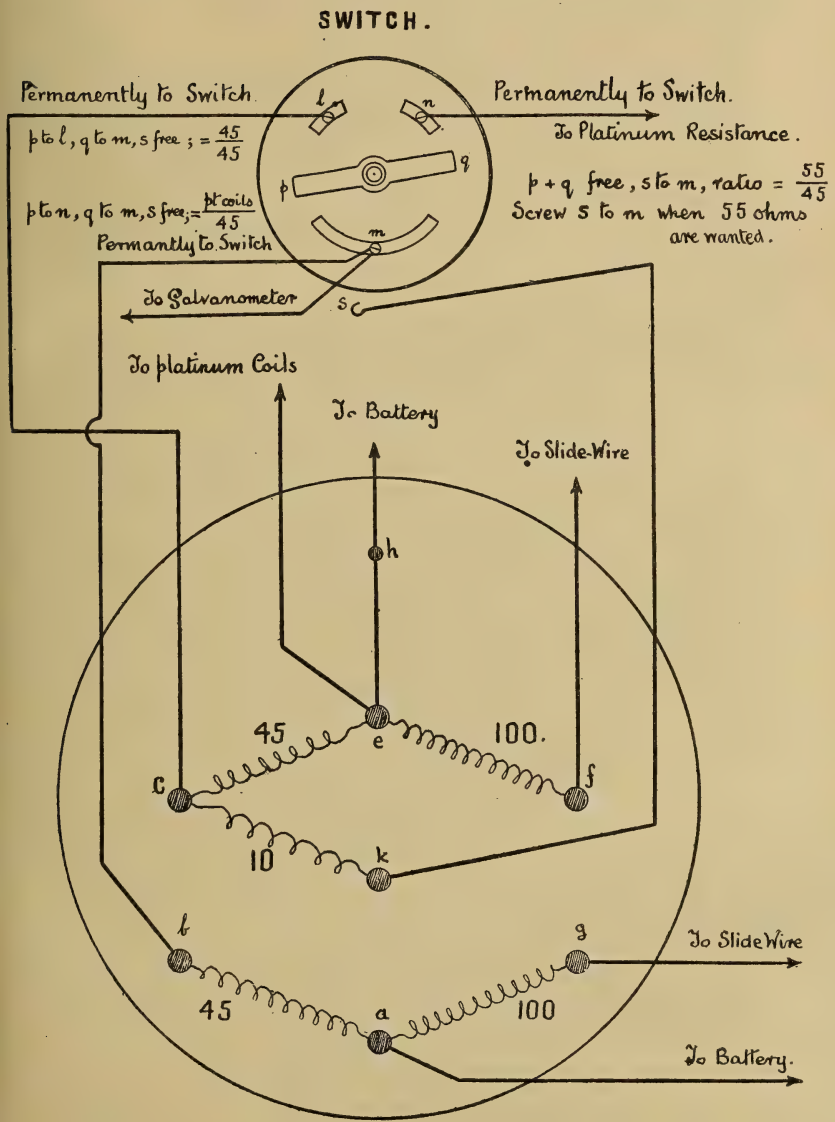
Copper rods (a, b, c, e, f, g, k) connect the coils; their ends pass upward, through the wooden cover of the calorimeter, to the switch.

Fig. 2 shows the arrangement of these coils. The two ends of the slide-wire go to f and g respectively; to which points the 100-ohm coils (f, e) and (g, a) are connected. By switching p to l, the fixed arm (45 ohms) of the bridge,

* A very convenient *optical bench* could be made by such a grooved plank.

† Platinoid is not good for slide-bridge wires. After about two months' exposure this wire has become blackened, and I am replacing it by a wire of platinum-silver.

Fig. 2.


SWITCH.
Permanently to Switch.
Permanently to Switch.
To Platinum Resistance.
p to l, q to m, s free; = 45/45
p to n, q to m, s free; = pt coils/45
Permantly to Switch
p + q free, s to m, ratio = 55/45
Screw s to m when 55 ohms are wanted.
l
n
p
q
m
s
To Galvanometer
To platinum Coils
To Battery
To Slide-Wire
h
45
e
100.
c
f
10
k
b
g
To Slide Wire
45
a
100
To Battery.

connected between a and b, is balanced against the 45 ohms between b and e; the slider has then a position at the middle of the slide-wire. When p is switched to n, the platinum thermometer is substituted for the (b, e) coil.

Calibration.

To calibrate the bridge, the switch (fig. 2) is set with p, q horizontal, and s screwed firmly to m. This puts $45+10=55$ ohms between b and e, and the slider has to be moved to a new position along the plank. A simple calculation enables us to find an expression for the temperature, t, of the platinum coils in terms of n, the distance in cms. of the sliding-contact from the extreme left of the plank-scale. The actual figures for this particular bridge are here given.

Corrected values of the bridge-coils, at 80° *F.*

Between a and b = 45·138 ohms.
" c " e = 45·150 "
" k " c = 10·032 "

The coils 100, 100 require no correction; they were adjusted at 64°·5 F., and we are only concerned with their *equality* at the slightly higher temperature.

Value of r, the resistance of 1 cm. of the slide-wire.—Balancing (45·15 + 10·032) ohms against 45·138 and noting the corresponding value* of n, we have

$$45{\cdot}138[100+241{\cdot}95\,r]=55{\cdot}182[100+58{\cdot}05\,r].$$

Whence $$r={\cdot}1301 \text{ ohm}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

R_t *in terms of* t.—Putting the values for R_{32} and α in (1), the equation becomes

$$R_t=40{\cdot}05[1+{\cdot}002097(t-32)], \quad . \quad . \quad . \quad (3)$$

that is, within the required range of temperature.

To this may be added, without sensible error, the resistance of the thick copper leads going to the platinum coils; this was ·025 ohm, so that

$$R_t=37{\cdot}39+0{\cdot}084\,t. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

R_t *in terms of* n.—The general equation of the bridge will be

$$R_t[100+(300-n)r]=45{\cdot}138(100+nr).$$

Or $$R_t=\frac{4513{\cdot}8+5{\cdot}8745\,n}{139{\cdot}04-{\cdot}1301\,n}. \quad . \quad . \quad . \quad . \quad (5)$$

* The full length of the slide-wire is $n=300$ cms.

t in terms of n.—From (3) and (5),

$$t = \frac{10{\cdot}7406\,n - 684{\cdot}9}{11{\cdot}677 - 0{\cdot}0109\,n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

That is,

$$n = \frac{11{\cdot}677\,t + 684{\cdot}9}{10{\cdot}7406 + 0{\cdot}01093\,t} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

From (7) the plank may be calibrated "directly," in temperature degrees. It will, however, be more convenient to plot a curve co-ordinating n and t. Such a curve does not greatly differ from a straight line ; it is therefore sufficiently accurate to calculate n for every successive five degrees within the range.

Accuracy attained.

Mr. E. H. Griffiths has been good enough to examine my figures, and he agrees that, if the original standardization of the platinum coils is carefully attended to, this apparatus may be relied upon to measure temperature with an accuracy of one tenth of a Fahrenheit degree. Unfortunately, the platinum coils were sealed hard and fast within their respective dielectrics before this slide-bridge method was adopted ; so that it has not been possible to check the readings in a liquid, as against a mercurial thermometer. This is very much to be regretted. By taking the mean of the readings of mercurial thermometers in the tank, and the readings of the mercurial thermometers in the tubes within the dielectrics, and comparing this mean with the value of t derived from (6), it is possible to make a very rough comparison. It should be carefully noticed, however, that "tube" temperatures include all the errors which it is the very object of the present method to avoid. The following table must be regarded with that limitation.

TABLE I.

Date.	Tube-temperature.	n.	t, from (6).	Air-temperature.
Aug. 22	76°·0 F.	135·8	75°·9 F.	81° F.
„ 23	72·5	132·9	72·6	77
„ 28	68·3	128·9	68·1	73
„ 29	80·5	139·75	80·4	76
„ 30	90·6	147·7	89·6	76
Sept. 2	101·0	156·64	100·1	78
„ 3	110·1	164·8	109·9	79

It may be remarked here that, with the same apparatus, by taking the mean of the two values of n, corresponding to reversal of the battery, a second approximation could have been made. Also the slide-wire is thinner than is required by the temperature range. A thicker wire would have given greater possibilities of accuracy, and it would have been less liable to mechanical injury and temperature fluctuations.

The Action of Sulphur Vapour upon Copper.

While standardizing platinum thermometers in the vapour of sulphur (444° C.) some mica plates, which formed part of the apparatus within the vapour, were bound together with copper wires which passed through holes in the mica. After five hours' exposure to the vapour the apparatus was dismantled, and, as had been expected, the copper wire was completely demetallized. An examination of the resulting wire, a brittle compound of cuprous or cupric sulphide, showed that it was, in some cases, perforated by a small axial hole; the wire had, in fact, become a tube of very small bore.

I am able to show this to you by projecting upon the screen one of the pieces of mica through which the original wire was threaded; the ends of the sulphide wire were broken off flush with each face of the mica, thus forming a section of the sulphide wire.

Prof. Ramsay tells me that, as a rule, it is possible to draw an inner metallic core* from a copper wire which has been treated with sulphur vapour. It therefore seemed probable that my specimen was a piece of sulphide wire out of which a core had fallen. This, however, does not appear to be the explanation, for I have since succeeded in threading an iron wire, of 8 mils diameter, completely through a sulphide wire, which was more than half an inch long and not quite straight. It is impossible that a core of 8 mils could have fallen out of this specimen.

An examination, with the microscope, of the original copper wire does not reveal the slightest perforation; although the end was carefully removed with nitric acid so as to avoid the closing of the hole, if one existed, by any mechanical cutting of the wire.

It was suggested that an occasional hole might occur in the process of drawing the wire. Prof. Boys asked me to try a piece of copper cut from a block. I therefore had a piece, of square section, sawn from a large commutator-bar. After

* I have obtained lately several specimens which show this core.

six hours' boiling in sulphur vapour, this copper rod became circular in section; it had a central perforation, as could be proved by passing a wire into it. The colour of the surface was a deep "beetle" blue.

In order to show the amount of expansion which the copper experiences during the change to sulphide, a second piece of copper, of square section, was sawn out of the commutator-bar, half the length being turned down to a circular section. This was put into sulphur vapour for six hours, with the result that the diameter of what had been the square part had increased from 215 to 386 mils; the circular part had increased from 120 to 232 mils; and a third piece, of ordinary drawn copper wire, increased from 31 to 77 mils. During the transformation from copper to sulphide the diameter in all cases is approximately doubled. The length is only slightly increased; all sharp edges disappear; and there is a general tendency towards the circular section. The fracture is crystalline, resembling that of an aerolite.

A piece of Delta metal submitted to the same test came out of the sulphur vapour without appreciable change. It was discoloured but otherwise unaltered, and I propose to adopt this metal in future for the electrodes of platinum thermometers.

Summary.

The foregoing results indicate that platinum-coil thermometry may be reduced to a simple operation, and that the simplified method is sufficiently precise for general work. The exact conditions which determine the numerical limits of accuracy, generally, cannot be very concisely stated. The possible causes of error are well known, and with a little care may be nearly eliminated. The first precaution is to maintain the bridge-coils at some constant temperature just above the maximum air-temperature. For this purpose the oil-bath and glow-lamp will be found satisfactory.

If the temperature of the slide-wire is likely to vary considerably, r may be redetermined; or a correction-factor may be applied. This will seldom be necessary; no such correction was used in the above tests.

There must be some easy arrangement for calibrating the slide-wire. The auxiliary-coil test is expeditious. The wire now supplied by good makers is sufficiently uniform; and, except where great precision is essential, it will not require to be corrected for differences of diameter; it should be of platinum-silver.

The resistance of the leads can generally be added as in equation (4). In other cases one of the "dummy-lead" methods of Siemens or Callendar must be adopted.

In conclusion I have to thank Mr. E. H. Griffiths, Mr. W. A. Price, and Prof. W. N. Stocker for their help in preparing this paper; and the Silvertown Telegraph Company, who have kindly allowed me to carry out the experiments and publish the results.

VI. *On a Method of Drawing Hyperbolas.* *By* George J. Burch, *M.A. Oxon.**

THE ordinary methods of drawing hyperbolas fail when the portion of the curve required lies some distance from the vertex, small errors of measurement being then so much magnified as to render the results practically useless. Cunynghame's hyperbolagraph, an admirable instrument for describing the parts near the vertex with a single movement, is also, for the same reason, inapplicable to the cases dealt with in the present communication.

In using graphic methods for the investigation of a problem in Optics, the author had occasion, in 1885, to draw a number of hyperbolas all passing through a fixed point far away from the vertices of most of them, the asymptotes and the vertex of each being given. After vainly endeavouring to draw the curves in the usual way, he devised the following method which proved entirely successful, and which is, so far as he has been able to ascertain, a new one.

Given the asymptotes Ox *and* Oy, *and the vertex* A, *to construct an hyperbola.*

The equation of an hyperbola, when referred to its asymptotes as axes of coordinates, is

$$4xy = a^2 + b^2.$$

In the simplest case, that of the rectangular hyperbola, $a = b$, and the equation may be conveniently written

$$xy = c^2 = \text{a constant.}$$

To any point C on Ox draw AC, and from A draw a line parallel to Oy, cutting Ox in B.

Make CE upon the axis of x equal to BO, and from E draw a line parallel to Oy, cutting AC in D.

Then yOx being a right angle, and $\triangle$ABC and $\triangle$DEC

* Communicated by F. J. Smith, F.R.S.

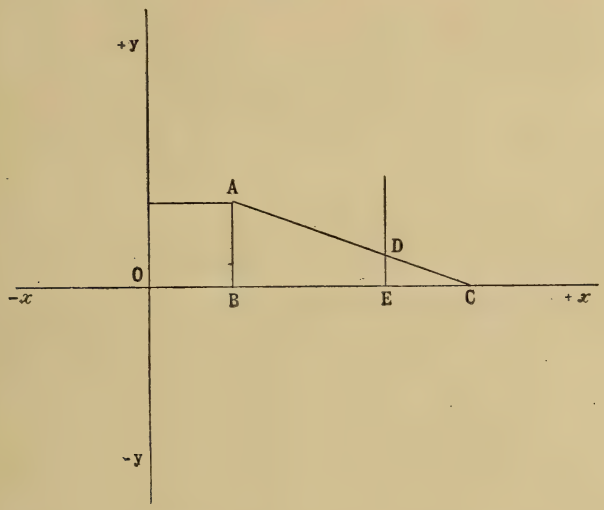
being similar triangles,

$$BC : AB :: EC : DE,$$

or

$$BC . DE = AB . EC.$$

Fig. 1.



But by the construction, AB and EC are constant;

and also $BC = OE = x$;

and $EC = OB$.

Therefore if $ED = y$,

$$xy = OB . BA = \text{a constant},$$

and D is a point on the hyperbola.

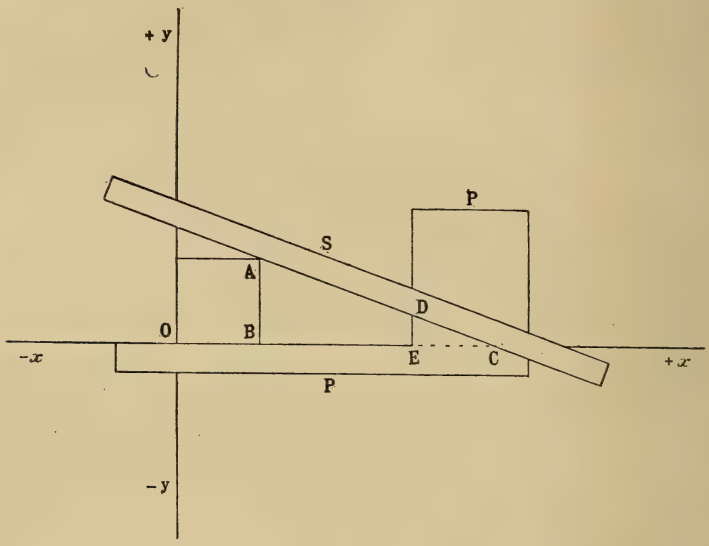
It is scarcely necessary to point out that this construction applies also to hyperbolas other than rectangular, since the lines AB and ED are drawn parallel to Oy, the other asymptote.

In practice, when it is required to plot several curves, the simplest plan is to draw the asymptotes on a piece of stout paper and then cut it into the shape of a modified T-square, as shown in P, fig. 2, where the edge OBE forms part of the asymptote Ox, the continuation of which is ruled on the paper and represented by the dotted line, and the edge DE is parallel to the asymptote Oy.

A distance EC equal to OP is measured along the line Ex, and a fine pin-hole made at C.

Drawing-pins are fixed in the board at O the origin and A the vertex of the proposed hyperbola, and the asymptotes

Fig. 2.



drawn. Then the paper square, P, is laid against one asymptote and a third pin inserted at C. This can be done with greater accuracy if the line EC is continued to the end of the paper. It remains to place a straight-edge S against the pins A and C, and to mark the intersection of it with the line ED at D. Then D is a point on the hyperbola, and by shifting the pin C together with the paper square P to a fresh position on the line Ox, another point can be determined in like manner. One great advantage of the method is that the value of y can be found directly for any given value of x. To do so, it is only necessary to place the paper P so that OE $= x$, and proceed as before.

Obviously, too, the instrument might be constructed in metal and arranged so as to slide along the asymptote Ox, drawing a continuous curve. It might consist of two brass bars hinged at C, with a cross-bar clamped to one of them in such a way that its distance from C and the angle between it and the bar P could be adjusted. This cross-bar might carry a pencil or writing-point free to slide along it, and pressed by a light spring against the edge of S. A rough model of such an instrument was made at the time by the author, and

was found to work very well. It should be noted that all the long lines being given by the straight edge, great accuracy is easily obtainable, whether with the simple paper square or the more complex instrument.

21 Norham Road, Oxford,
September 28, 1895.

VII. *Notices respecting New Books.*

Elements of the Mathematical Theory of Electricity and Magnetism. By J. J. THOMSON, *M.A., F.R.S.* Cambridge University Press, 1895.

STUDENTS of Electricity who desire to read the more mathematical portions of the subject, and particularly those who wish to follow the development of the æther theory of electricity and the electromagnetic theory of light, usually find some difficulty in the choice of a text-book. From a first-year experimental course to Maxwell's treatise is too great a step, in mathematics as well as in physics; some text-book of an intermediate character is therefore required. Maxwell appears to have realized this, and he attempted to remove the difficulty by his Elementary Treatise, which, unfortunately, he did not live to complete. In the present volume Prof. Thomson has a similar aim: he retains nearly the same order of subject-matter as in Maxwell's treatise, but (with few exceptions) only such problems are considered as can be solved by the aid of the differential calculus. By this treatment the mathematical difficulties are greatly diminished, while the physics of the subject is satisfactorily developed and illustrated by a sufficiently large number of examples.

The author has made frequent use of Faraday tubes of force in explanations of phenomena occurring in the electric field: he shows very simply that such tubes will be in equilibrium if the tension along their axes is accompanied by an equal pressure at right angles to them. In discussing the case of an insulated sphere in a uniform field, the idea of an electric doublet is introduced, and an expression is found for the moment of a doublet representing the external effect of the charge on the sphere. Among other new modes of treatment we may mention the use of the dissipation function in determining the distribution of currents in any network of conductors. Kirchhoff's laws are shown to be equivalent to the statement that the currents distribute themselves so as to make the total rate of development of heat-energy a minimum; by writing down the expression for this heat-energy and making it a minimum, the values of the currents in each branch may be obtained. Prof. Thomson has found a companion for κ, the specific inductive capacity, and μ, the magnetic permeability, in dimensional formulæ. The work done by a unit magnetic pole in threading a closed circuit is $4\pi/p$ times the current flowing in the circuit; as the

definition connects magnetism with electricity clearly p may have dimensions, but the electrostatic and electromagnetic theories each assume it to be a mere number. The result of retaining p is to make $p^2/\mu\kappa$ the square of a velocity.

Chemists will probably regret that the subjects of migration of ions, ionic velocity, and electrolytic conductivity find no place in the volume, electrolysis being treated in a very elementary fashion. The chapter devoted to dielectric currents and the electromagnetic theory of light is very readable, even by those who are not familiar with the ordinary equations of wave propagation; it contains a concise though somewhat brief account of the recent confirmations of Maxwell's theory by Hertz and others. We notice that the discovery of the action of electric waves in diminishing the resistance of a tube of metal turnings is erroneously attributed to Prof. Lodge: the experiment was originally described in this form by M. Branly.

The volume is nicely printed, with bold headings at the commencement of each section, and is of a handy size for students' use. There is a subject-index at the end of the book, but a list of titles of the various chapters inserted at the commencement of the volume would be of great service to the student.

JAMES L. HOWARD.

An Exercise Book of Elementary Physics for Organised Science Schools etc.; arranged according to the Headmasters' Association's Syllabus of Practical Physics. By RICHARD A. GREGORY, *F.R.A.S.* London: Macmillan & Co., 1895.

THE Government having at last recognized the suitability of elementary practical physics and mechanics as school subjects, this series of exercises suggests a method of making such subjects effective in training students to acquire habits of observation, accuracy, and carefulness. The exercises include the general operations of weighing and measuring, mechanics and heat; they are arranged according to subject and in order of difficulty. Spaces are provided in which the results of experiments may be recorded, and in each case the student is left to make his own deductions from his experiments.

Some of the directions for experiment are not very explicit; for example, in order to measure the rise of water in capillary tubes the student is told to take two very narrow tubes, one having twice the internal diameter of the other, and to wash them by drawing distilled water through them. A good student, and not a few teachers, will want to know how the diameters of the tubes are to be measured; and the results of the experiment will not be very satisfactory unless the tubes have been previously washed with some cleansing agent, such as potash or alcohol. A few exercises on glass-blowing, soldering, and similar manipulation might be added with advantage. The construction of the model lift or force-pump, used in one of the experiments, would be excellent practice in this direction.

The author suggests the use of balances weighing to 0·01 grm., and recommends supplying one balance to every six students. Taking into account the number of experiments requiring the use of a balance this seems hardly sufficient; we incline to the opinion that a larger number of cheaper balances weighing to 0·1 grm., say one balance to each pair of students, would be preferable for elementary physical work.

Mr. Gregory has rendered good service by suggesting in so practical a fashion a programme of physical work, which indicates the nature of the instruction to be given, without binding a teacher to follow it out entirely or even in consecutive order.

JAMES L. HOWARD.

VIII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from vol. xl. p. 547.]

November 20th, 1895.—Dr. Henry Woodward, F.R.S., President, in the Chair.

THE following communications were read:—

1. 'Additional Notes on the Tarns of Lakeland.' By J. E. Marr, Esq., M.A., F.R.S., Sec.G.S.

This paper is supplementary to one by the author published in the Q. J. G. S. vol. li. (1895). He gives additional notes on Wateredbath Tarn, describes Hard Tarn on Helvellyn, a pond whose outlet has gradually been diverted from a course over screes to one over solid rock; Hayeswater, a lakelet referred to by Dr. H. R. Mill as in some respects intermediate between the mountain-tarns and the valley-lakes; and Angle Tarn, Patterdale, a good example of a plateau-tarn. The results of his fresh observations tend to confirm the views expressed in his former paper.

2. 'Notes on the Glacial Geology of Arctic Europe and its Islands.—Part I. Kolguev Island.' By Col. H. W. Feilden, F.G.S.; with a 'Report on the Erratic Boulders from the Kolguev Beds,' by Prof. T. G. Bonney, D.Sc., LL.D., F.R.S., F.G.S.

Kolguev Island, about the size of Norfolk, lies about 50 miles from Arctic Russia and about 130 miles south-west of the nearest part of Novaya Zemlya, with soundings not exceeding 30 fathoms between it and Russia, and probably not more than 75 fathoms between it and Novaya Zemlya. It is entirely composed of a vast accumulation of glacio-marine beds. The northern two-thirds of the island consists of an elevated ridged area with a maximum height of 250 feet. The author has been furnished with notes by Mr. Trevor-Battye concerning the geology of this region. It is inferred from his observations that this elevated region is composed of beds of sand

with erratic boulders not less than 80 feet deep, resting on clays—the 'Kolguev Clays.' Mount Bolvana rises as a symmetrical cone above the tundra, detached from the northern plateau, pointing, in the opinion of the author, to the occurrence of marine erosion.

The southern portion of the island is tundra, a dead flat of grass, bog and peat-levels reaching to the sea; good sections of the Kolguev Clays are exposed in the gullies traversing it near the sea on the western coast. In the vicinity of the Gobista river the Kolguev Beds consist of clays merging here and there into sands. They are charged with boulders often ice-scratched, indicating continuous deposition in a comparatively deep sea. The beds yielded many shells of Arctic mollusca, such as *Saxicava arctica*, *Mya*, etc., apparently dispersed from top to bottom. The ice-pack has forced many fragments of semi-fossil wood on to the shore, no doubt worked up from a bed immediately below sea-level. No deposit was met with in Kolguev Island precisely similar to what is called 'Till' in Scotland, though there are many Boulder Clays in Britain which are in no measure superior in toughness to those of Kolguev, for instance, those of the Yorkshire coast and the Chalky Boulder Clays of Norfolk.

It is suggestive that all the glacial deposits met with by the author in Arctic and Polar lands (except the terminal moraines now forming above sea-level) should be glacio-marine beds.

Prof. Bonney in his report describes the rocks brought home by the author. They include granite-gneiss (very like Archæan rocks), grit, chert, limestone with *Favosites* (Silurian or Devonian), limestone with *Amphipora ramosa* (Devonian), limestone with *Lithostrotion irregulare* (Carboniferous), and a fragment of a Jurassic belemnite. The fossils have been examined by Mr. E. T. Newton, F.R.S.

IX. *Intelligence and Miscellaneous Articles.*

ON UNDERGROUND TEMPERATURES AT GREAT DEPTHS. BY ALEXANDER AGASSIZ.

FOR several years past I have, with the assistance of our engineer, Mr. Preston C. F. West, been making rock-temperature observations as we increased the depth at which the mining operations of the Calumet and Hecla Mining Co. were carried on. We have now attained at our deepest point a vertical depth of 4712 feet, and have taken temperatures of the rock at 105 feet, at the depth of the level of Lake Superior, 655 feet, at that of the level of the sea, 1257 feet, at that of the deepest part of Lake Superior, 1663 feet, and at four additional stations, each respectively 550, 550, 561, and 1256 feet below the preceding one, the deepest point at which temperatures have been taken being 4580 feet. We propose, when we have reached our final depth, 4900

feet, to take an additional rock-temperature and to then publish in full the details of our observations.

In the meantime it may be interesting to give the results as they stand. The highest rock-temperature obtained at the depth of 4580 feet was only 79° F., the rock-temperature at the depth of 105 feet was 59° F. Taking that as the depth unaffected by local temperature variations, we have a column of 4475 feet of rock with a difference of temperature of 20° F., or an average increase of 1° F. for 223·7 feet. This is very different from any recorded observations; Lord Kelvin, if I am not mistaken, giving as the increase for 1° F., fifty-one (51) feet, while the observations based on the temperature-observations of the St. Gothard Tunnel gave for an increase of 1° F., sixty (60) feet. The calculations based upon the latter observations gave an approximate thickness of the crust of the earth, in one case of about 20 miles, the other of 26. Taking our observations, the crust would be over 80 miles and the thickness of the crust at the critical temperature of water would be over 31 miles, instead of about 7 and 8·5 miles as by the other and older ratios. With the ratio observed here, the temperature at a depth of 19 miles would only be about 470°, a very different temperature from that obtained by the older ratios of over 2000° F.

The holes in which we placed slow-registering Negretti and Zambra thermometers were drilled, slightly inclined upward, to a depth of ten feet from the face of the rock and plugged with wood and clay. In these holes the thermometers were left from one to three months. The average annual temperature of the air is 48° F.; the temperature of the air in the bottom of the shaft was 72° F.—*American Journal of Science*, December 1895.

ON THE INFLUENCE OF ELECTRICAL WAVES ON THE GALVANIC RESISTANCE OF METALLIC CONDUCTORS. BY H. HAGA.

At the meeting of the Berlin Physical Society on Nov. 30, 1894, M. E. Aschkinass* communicated observations according to which the resistance of a grating of tinfoil was found to be about two per cent. less in consequence of electrical radiation; and this small resistance lasted even after the cessation of radiation, until heating or mechanical agitation restored the former value.

Since radiation will produce electrical vibrations in a tinfoil resistance, it appeared worth while to investigate whether in general the resistance of a metallic conductor is also altered by electrical vibrations passing over its surface.

Using the apparatus described and recommended by Ebert†, by which long continuous electrical vibrations could be obtained, such vibrations were transmitted through different specimens of copper

* *Verhandl. der Phys. Gesellschaft zu Berlin*, vol. xiii. p. 103.
† Wied. *Ann.* vol. lii. p. 144 (1894).

and iron wire and through tinfoil. No influence could be observed, although in the way the experiments were arranged an alteration in the resistance of $\frac{1}{20}$ per cent. could easily have been observed.

After this negative result I returned to radiation, and allowed primary sparks of very various duration to strike across, while the resistances to be investigated were at a distance of less than 50 cm. To produce the sparks an induction-coil 25 cm. in length was used worked by three accumulators, but more frequently an induction-coil 60 cm. in length worked by ten accumulators (24 amperes).

By means of a spark-resonator it was always ascertained that the primary spark was an oscillating one. The following resistances were used:—Various gratings of tinfoil fastened on ebonite, the thickness of the tinfoil being 0·05 mm. to 0·01 mm., and the resistances 0·478 Ω, 3·155 Ω, 11·31 Ω, 36·31 Ω; a grating of thin iron wire on an ebonite frame (5·504 Ω); a german-silver wire (0·578 Ω), and a strip of tinfoil (1·068 Ω): both these last were resonant with the primary spark; secondary sparks could be obtained between the wire or the strip. In none of these resistances was any trace of a variation observed (to $\frac{1}{20}$ per cent.).

When the gratings were not fastened on the ebonite, but were fixed in an ebonite frame, I observed the phenomena described by Aschkinass, but only when the strips were very close to each other. The variations in the resistances were often very considerable. The resistance of a grating of 6·7 Ω was 3·6 Ω after radiation by the primary spark; by an agitation again 5·1 Ω, after radiation 3·2 Ω. This grating was dipped in solidifying paraffin; it then showed a resistance of 7·85 Ω, which became 4·75 Ω by radiation, and by shaking increased to 7·6 Ω. It was necessary to agitate violently to get back to the original high resistance. No change in resistance could be observed with gratings having great spaces between the strips. In one grating the 22 strips were 0·75 mm. in breadth and likewise 0·75 mm. apart. The resistance 11·85 Ω was the same after radiation as before.

Hence, from these experiments, the amount of the variation depends on the position of the strips in reference to each other. If these are close the primary spark may give rise to the formation of one or more bridges between adjacent strips, as Branly* assumes with metal filings, and Lodge† in what he calls "microphonic detectors." Although much remains to be explained in the latter investigation, it appears to me indubitable that, as Aschkinass also thinks possible, the phenomena with metal gratings belong to the same category, and are not to be ascribed to some unknown action.—Wiedemann's *Annalen*, No. 11, 1895.

* *Journal de Physique*, p. 459 (1892); p. 273 (1895).
† The Work of Hertz, pp. 20–26.

THE

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PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[FIFTH SERIES.]

FEBRUARY 1896.

X. *The Filar Anemometer.* *By* CARL BARUS, *Hazard Professor of Physics, Brown University, Providence, U.S.A.**

1. IN a remarkable paper, published some sixteen years ago, Prof. V. Strouhal† investigated certain laws subject to which a sound is produced when the wind passes transversely across a slender solid body. Placing his chief reliance on the results obtained with metallic wires, Strouhal found for thicknesses ranging from 0·018 cm. to 0·325 cm., and for speeds from 2·1 met./sec. to nearly 12 met./sec., that the whole group of phenomena could be expressed by an equation of striking simplicity,

$$n = Cv/d\,;$$

where n is the frequency of vibration, v the speed of the wind, d the diameter of the wire, and where C is constant except as to temperature. When all data are expressed relatively to metres and seconds, this constant has at atmospheric temperatures a mean value

$$C = 0{\cdot}200\,;$$

and it is thus at once possible to compute the speed of the air

* Communicated by the Author.

It gives me pleasure to acknowledge my indebtedness to the Secretary of the Smithsonian Institution, for materially promoting the present series of researches.

† *Ueber eine besondere Art der Tonerregung,* Würzburg, Stahel, 1878. The subject was suggested by Prof. F. Kohlrausch.

normally to the axis of a given wire, when the note made by the whistling wind is located, at a given temperature, in pitch. Other data *, such as the material, or the tension, or the length of the wire, &c., are without marked effect, except as regards the intensity of the sound produced.

Strouhal further found that whenever the air-tone n, varying continuously with speed v, approaches the fundamental or any of the overtones of the transversely vibrating wire, the sound bursts forth with accentuated intensity. In virtue of this discovery, Strouhal was able to give a degree of precision to his results which for the case of the unassisted air-tone would have been unattainable; for it was merely necessary to work out such speeds as kept the wire in a state of *permanent resonance*. Indeed it was now possible to obtain sounds from the wire even after the actuating air-tone alone had ceased to be audible.

Commenting on the application of his results, Strouhal notes its immediate bearing on anemometry. The device is peculiarly adapted to the measurement of variable gusts and high winds, and is thus supplementary to the ordinary anemometer.

2. It is from this point of view, *i. e.* in relation to what may be called micro-anemometry, that the filamentary anemometer interested me during my connexion with meteorological research †. Here is an instrument virtually without mass, which therefore does not state the case summarily, but represents the wind as it actually is. In consideration of the actual complexity of aerodynamic phenomena, the simplicity of Strouhal's law is an ulterior advantage. Whoever believes that much is to be learned from a more searching investigation of the nature and origin of gusts of wind ‡ will be in sympathy with the development of what is in many respects an ideal instrument of research.

There is another important problem which lies within the scope of the filar anemometer. I refer to the class of researches recently accentuated by the paper of S. P. Langley § on the work of the wind.

* The law applies more accurately in proportion as thickness increases, and may be considered exact for diameters above 0·1 cm.

† *Cf.* my letter to Prof. M. W. Harrington, in Rep. Am. Assoc. of State Weather Services, Oct. 1892; Bull. U. S. Weather Bureau, No. 7, pp. 44, 45.

‡ Barus, American Meteor. Journal, March 1893, pp. 488–489; *cf.* Bulletin U.S. Weather Bureau, No. 12, 1895.

§ "The Internal Work of the Wind," Smithsonian Contributions, No. 884, Washington, 1893.

If the velocity of air normally to the wire is registered, three wires at right angles to each other would register the respective velocities in three cardinal directions. From this the actual direction of the gust is at once obtainable as the resultant of the three components.

It is necessary, however, before a like deduction can be accepted with confidence, to further elucidate Strouhal's law with data referring to the oblique passage of a wire through the air. For just what will occur under these circumstances cannot certainly be foreseen. It is also desirable to obtain accurate data for the effect of temperature on the phenomenon, an inquiry which Strouhal only carried far enough to apprehend that the pitch of a note, other things being equal, is depressed with increasing temperature.

From a theoretical side, the subject has recently been elucidated by Lord Kelvin*.

In constructing my whirling table I aimed at quantitative decision on these questions. I therefore made a whirling machine one end of the axis of which was pivoted to the ceiling of a high room, and the other end rigidly attached to the spindle of an ordinary whirling table about a metre above the floor. The crank-wheel was placed at some distance, so as to allow two horizontal arms, about 1·4 metre long and 1·9 metre apart in the same vertical plane, to rotate freely. Clamp-screws at the end of these arms carried the wire to be tested, vertically stretched and 1·9 metre long.

The whole framework was made of gas-pipe and snugly screwed together. It was therefore possible to move the horizontal arms so that the vertical planes through them would subtend a given (small) angle, and the (elongated) wire be rotated through the air obliquely to the line of motion. Due care had to be taken not to carry the obliquity too far.

The customary electric brushes were added at the axis of the whirling arm, and thus the rotation could be timed by a chronograph pendulum of the simple kind sketched in a preceding paper†. Contact brushes for other electric apparatus (*cf.* § 3) were also supplied.

On trial, however, the whirling arm at the higher speeds developed a weakness. It was found to yield seriously under the strain of rotation, so that I did not feel sure of its constants. I therefore abandoned further attempts at solving the subsidiary questions just specified, for the present.

3. A wire singing on the housetops, however, is as yet no

* 'Nature,' l. pp. 524, 549, 573, 597 (1894); see particularly pp. 524 and 525, where the whistling of a strong wind is discussed.

† Barus, American Journal, xlviii. p. 396 (1894).

anemometer, least of all when its own song is drowned for the ear by the noises round about. I therefore proposed to myself in the first place to devise some means by which this air-tone could be conveyed to any distance without change of quality, and thereafter to endeavour to register this indication automatically. It is only with the first of these problems that the present paper is concerned. All attempts made at the second, which is seriously difficult, have thus far been failures*.

The first condition appeared easy of accomplishment. It seemed merely to be necessary to attach the wire to the plate of a telephone in order to catch and transmit the sounds from the roof to the office. However, telephonic devices over which I laboured a long time failed to the last to give me any result whatever.

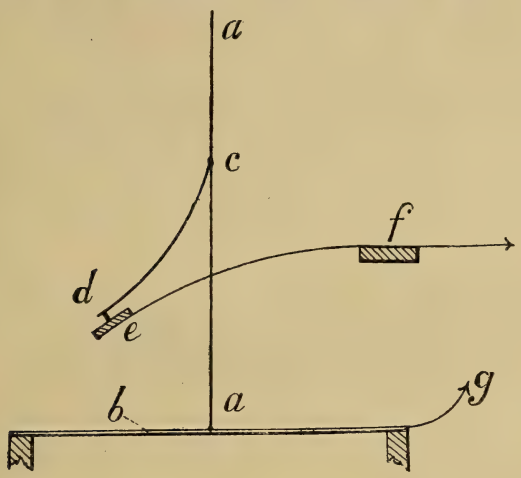
I then turned to the microphone, at first equally without success, though finally an arrangement was developed which behaved satisfactorily. I may in passing refer to certain of the earlier forms of microphonic contact, some of which were at times useful. The difficulty encountered arises from the fact that a delicate microphonic contact is to be maintained without breakage, at the end of a rapidly revolving and necessarily vibrating whirling arm. Most devices give sounds interrupted by a terrific racket at the receiver, which is very trying to the ears. Apart from this the effect of centrifugal force in varying the contact is to be counteracted, and the air-resistance or the noises made by moving parts of the microphone eliminated.

In figure 1 *aa* is the revolving wire, attached at its upper end to one arm of the whirling machine (not shown), and at the lower end to the metallic plate or sounding-board *b*, at the end of the lower arm of the machine. Near the bottom of the wire a very delicate brass spring *cd* is soldered on at *c*, carrying a short platinum stylus at *d*. A similar spring, *ef*, attached at *f* to the sounding-board, carries at *e* a light pellicle of carbon. The latter has been electroplated on one side with copper, so that it can be soldered. The two springs are so adjusted that the microphonic contact is just made at *d*, *e*, and kept intact when *e* is slightly moved up and down, *d* sliding on the flat face of *e* in this case. Moreover, as *e* is relatively heavier than *d*, centrifugal force will tend to straighten the spring *fe* more than *dc*, and the contact is not

* Unfortunately the ingenious devices of Prof. C. R. Cross and his pupils (Proc. Am. Acad. xxv. p. 233 (1890); May 1892; Jan. 1893) are not available here.

interrupted at high speeds if both springs lie in a plane through the axis and are of proper lengths and curvature.

Fig. 1.—Vertical Spring Microphonic Contact on Air-Wire.



As *f* is insulated, the current of four Leclanché cells passes through *f*, *e*, *d*, *c*, *a*, *g*, and the Bell telephone included in the circuit. The sounds are very much increased (and unfortunately the thunder due to breaks of contact also) when the primary of a transformer is placed in the circuit with the telephone in the secondary, in the now well-known way*. A great advantage is gained by filing the flat surface of *e* with a smooth file, probably owing to the carbon powder which lodges in the pores. Polishing the surface on a stone is harmful. High tones seem to require firmer contacts than low tones, and this state of things is brought about by the action of centrifugal force as stated. The position of the spring *cd* on the wire *aa* seems to be immaterial. I made tests throughout 30 cm. without marked results. Nor is much advantage gained from the shape or material of the sounding-board. Curiously enough, the latching of *d* and *e* often favoured the intonation of low sounds. No rules can be given for the bending of the springs, and the best contact is usually the result of chance, and is obtained only after many trials.

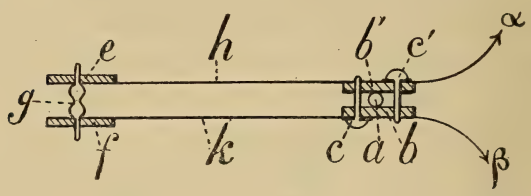
It is with the above arrangement that I once noted a peculiar phenomenon for which I have failed to find an

* The reader is referred to either Preece's or Du Moncel's Treatise on the Telephone.

explanation. After having made a delicate contact I noticed a low sonorous sound in the telephone, which continued with undiminished intensity for fully five minutes. No spring could have vibrated for this length of time, so that the acoustic apparatus must have contained its own motor. The sound ceased only when contact was broken, but on succeeding days I could not reproduce it.

4. A second microphonic contact with which there is usually less racket than in the preceding case is shown in figure 2 in plan. The sounding wire is shown in cross

Fig. 2.—Horizontal Spring Microphonic Contact on Air-Wire.



section at a; h and k are two very delicate flat springs attached to the wire a at its lower end by two insulating plates of hard rubber b, b', and the insulated clamp screws c, c'. The free ends of these springs carry two small carbon pellicles, e and f, centrally perforated so as to hold the shouldered rod of graphite g loosely between them. The current of four Leclanché cells passes through α, h, e, g, f, k, β (α and β being the terminals) to the enclosed primary of the transformer, and thence back to the battery. The rod g may also fit into cavities in e and f, and be held in place by the very gentle clutch of the springs h and k; or g may be replaced by a short wire of platinum. There is a groove in the middle of g to attach small weights to vary the contact; h and k are set radially, so as not to be influenced by centrifugal force.

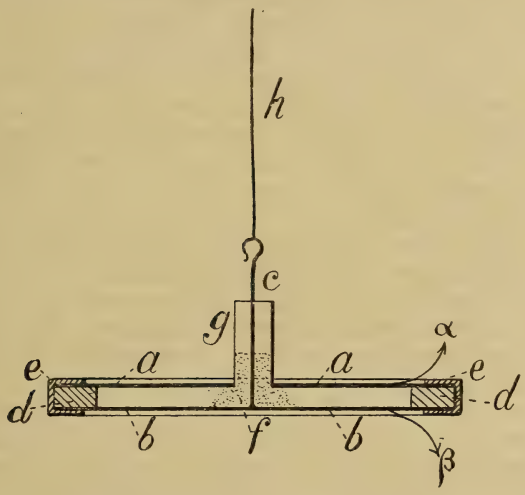
No effect was obtained when either of the microphonic contacts were placed on the sounding-board in which the lower end of the air-wire terminates. It seems essential, therefore, to tap the sound from the wire itself. A weighted sounding-board gives no novel results. A gridiron of wires produced air-tones which the ear appreciated with an intensity proportional to the number of wires, but the effect in the telephone did not exceed that of a single wire. Strips of metal in place of the wires were also ineffectual, and it was difficult to keep the sharp edge turned in the direction of motion. In this case, too, the bulging of the central parts of the revolving filament proved to be a serious annoyance.

A regular speed of the wire favours the occurrence of tones, particularly those which are awakened by resonance.

5. The beneficial effects obtained in § 3, as the result of filing, induced me to try a form of microphone in which the sensitive contacts are produced by a pulverulent conductor*. After many attempts I finally succeeded in devising an arrangement which is far superior to the forms just described, inasmuch as the thunder is altogether absent, and other extraneous noises are excluded. The sounds heard in the telephone are not loud, but clear and flute-like, and the range of pitch obtained is enormous. I passed almost from utter cessation of motion to the highest speed which I could give the whirling arm, and heard sounds quite inaudible to the unaided ear.

The annexed figure (3) shows the form which seemed to give the best results. Here *aa* and *bb* are two tin-plates about

Fig. 3.—Granular Microphonic Contact for Air-Wire.



0·016 cm. thick and 8 cm. in diameter, held apart by an interposed flat ring of hard rubber *dd* about 0·1 cm. thick or less. An elastic band of soft rubber *ee*, is stretched quite around the circumference and secures the plates by pressure against the hard rubber ring. The upper plate is centrally perforated, and a tube, *g*, about 1 cm. wide and 2 cm. long communicates with the perforation. A stylus *c*, is soldered to the middle of the lower plate and passing axially through

* Originally devised by Hunning, I believe. See Preece or Du Moncel, *l. c.*

the tube g ends in a hook to which the air-wire h is attached. The whole arrangement is held down horizontally on the sounding-board by two flat clamps (not shown) or by a ring insulated from the plates a and b. Carbon powder f, is finally introduced into the tube g, not in too large quantity, and not packed tightly. The current therefore passes into the upper plate at α, then through f into the lower plate and out at β, α and β being the terminals.

My first trials with finely pulverized graphite failed. I obtained the best results with gas-carbon ground in a mortar and sifted in a way to keep the particles granular. The plates must be clean and bright. The sounds usually start with a creak, as if the powder must first be shaken loose; and tapping frequently improves it. The plates must not be pressed upon the powder, and the part of it around the stylus is chiefly effective. In other forms I have quite filled up the discoid cavity, but it was then frequently necessary to pull up the upper plate with a spring.

6. The following results were obtained with the microphone of § 5. A series of wires of different diameters were tested, and those larger than 0·05 cm. and smaller than 0·15 cm. were found best adapted for the purpose. Thicker and thinner wires behaved peculiarly, as will presently be seen. It has been stated that Strouhal's law requires a correction for thin wires ($<0·1$ cm.). The following is a record of some experiments:—

(A) Copper wire, 0·072 cm. in diameter, 190 cm. long.—Sounds were heard from g' (faint), a' (faint), b', c'' chromatically to c''', to c^{IV}, to e^{IV}, which corresponded to the highest speeds safely attainable in my whirling arm. The range of frequency is thus from $n=391$ to above $n=2610$ indefinitely; interpreted by Strouhal's law (since $d=0·00072$ met.), from 1·41 met./sec. to above 9·40 met./sec., or from about 3 miles per hour to 21 miles per hour, and above this indefinitely. The sound therefore begins with a calm or the zero of Beaufort's scale. Strouhal's range began with frequency 840, showing advantages in the microphonic method in detecting low notes, some of which were indeed quite inaudible in the air.

I counted the revolutions of the whirling arm at the pitches c'', d'', e'', g'', c''', e''', g''', and compared the data so found with Strouhal's formula, obtaining values which agreed well enough to evidence the correctness of pitch in both cases.

If the notes in question could be registered automatically, the limit of accuracy would be subject only to the production of beats, for the sounds vary continuously; but the ear can

hardly detect a smaller interval than a half tone, so that speeds are not definable closer than about 6 per cent. This, however, in view of the excessively variable character of a gust of wind, is no serious disadvantage.

(B) Copper wire, 0·126 cm. diameter, 190 cm. long.—I heard all the chromatic intervals from c' to c'' to c''' to e''', which corresponded to the limiting speed used. The frequency is thus from $n=261$ to above $n=1305$, indefinitely; and the speeds are from 1·7 met./sec. to above 8·5 met./sec., or from 3·8 miles per hour to above 19 miles per hour. Comparisons with Strouhal's formula were made for g', b', d'', g'', c''', checking the pitch in the two cases. All the notes were clear and strong. Strouhal's lower limit was here about $n=600$, so that the microphone again detects inaudibly low notes.

(C) Copper wire, 0·240 cm. diameter, 190 cm. long.—Faint cello-like notes were heard, ranging from c to c' to c''. The pitch was checked for g, c', and e'. The limit of speeds used corresponded to c'', above which the notes would, doubtless, have come out more ringing. The lower limit of frequency was thus actually $n=130$, whereas Strouhal failed to hear tones below $n=500$ about. I did not enter into this case at much length, because of the confusion produced by overtones. Very regular rotation was needed to bring out the low notes.

(D) Brass wire, 0·020 cm. diameter, 190 cm. long.—The behaviour of this wire was very peculiar. No sounds were obtained except for tensely drawn wire, and I then heard the exceedingly shrill notes b^{IV}, c^{V}, $c\sharp^{\text{V}}$, d^{V}, $d\sharp^{\text{V}}$, e^{V}, f^{V}, $f\sharp^{\text{V}}$, ranging therefore in frequency from $n=3915$ to $n=6000$ nearly. A comparison of pitch with Strouhal's data was made for b^{IV} and d^{V}. Curiously enough, Strouhal in the case of these thin wires actually heard a whole range of low sounds from less than $n=2000$ to about $n=3000$, and none above. The fact that I had to draw the wire tensely to affect the microphonic plates may be adduced in explanation. Apart from the range there is no anomaly. It is probable that these wires are not massive enough to actuate the microphone.

I have stated that my reasons for not going to higher speeds than about 21 miles per hour were merely the imperfections of my whirling arm. The last example shows forcibly enough that excessively high pitches are by no means beyond the scope of the microphone. If the note $n=6000$ had been heard from the first wire ($d=0·072$ cm.), it would actually mean a speed of about 22 met./sec. or 49 miles per hour. This is fairly a gale. If heard from the second wire ($d=0·126$ cm.), it would mean a speed of 87 miles per hour

or a hurricane; if heard from the third wire ($d=0{\cdot}240$ cm.), it would mean 160 miles per hour, a speed* which fortunately transcends our storm nomenclature.

7. At the close of these experiments I put up a couple of these anemometers in my yard, but I have not yet obtained sufficient material for discussion. The means of registry is to bring the sounding wire on the whirling arm into unison with the exposed anemometer, and to let the former wire make its registry on the chronograph (§ 2).

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XI. *The Asymmetrical Probability-Curve.* *By* Professor F. Y. Edgeworth, *M.A.*, *D.C.L.*†

THE Probability-Curve may be described as an approximation to the law of frequency which governs the set of values assumed by a function of numerous independently varying small quantities; the function and the limits within which the variables range being such that the function may be regarded as approximately *linear*; so that we have nearly

$$Q=Q_0+Q_1'q_1+Q_2'q_2+\&c.+Q_n'\,q_n\ddagger;$$

where Q is the compound quantity under consideration; q_1, q_2, &c. are the elementary quantities; Q_1' is what Q becomes when we differentiate with respect to q_1, and substitute zero for each of the variables $q_1, q_2, \ldots$; Q_2', Q_3', ... are similarly defined; Q_0 is what Q becomes when zero is substituted in Q for each of the q's—an absolute term which may usually be omitted.

The *symmetrical* probability-curve is a *first* approximation which is commonly written

$$y=\frac{1}{\sqrt{\pi}c}e^{-\frac{x^2}{c^2}};$$

where x is the abscissa along which the values of Q are

* There is a possible optical analogy to which I may allude in passing. If in relation to speeds of the order of molecular velocities, the luminiferous æther may be considered as evidencing viscosity (following in the line of a well-known hypothesis of Lord Kelvin), we might then expect a molecule in its passage through æther to "sing" optically; in other words, we might expect the æther to awake resonant vibrations in the molecule, in the way in which the transverse harmonic vibrations of a wire are evoked (§ 1 *et seq.*) when the air-tone is the fundamental.

† Read before the Royal Society, June 1, 1894. Communicated by the Author; in a revised and abridged form.

‡ On this condition see the present writer's paper in the Philosophical Magazine, Nov. 1892.

measured from an origin such that the average value of each of the q's, and therefore also the average value of Q, vanishes (Q_0 being omitted); $y\,\Delta x$ is the proportional number of values of Q occurring between x and $x+\Delta x$; $\frac{c^2}{2}$ is the sum of the n quantities each of which is the mean square of error (measured from the average value or centre of gravity) for one of the elements $Q'q$.

The *asymmetrical* probability-curve is a second approximation which may be written

$$y=\frac{1}{\sqrt{\pi}c}\,e^{\frac{-x^2}{c^2}}\left(1-\frac{2j}{c^3}\left[\frac{x}{c}-\frac{2}{3}\frac{x^3}{c^3}\right]\right);$$

where x, y, and c have each the same meaning as before; j is the sum of the n quantities each of which is the mean cube of error for one of the elements $Q'q$. I propose to give a new proof of this formula; after first adverting to one or two old proofs.

I. The formula may be derived from an analysis which Todhunter, following Poisson, has indicated. Todhunter inquires what is the probability, P, of the value of a certain quantity E occurring between $c+\eta$ and $c-\eta$; E being $=\gamma_1\epsilon_1+\gamma_2\epsilon_2+\ldots$, where $\gamma_1, \gamma_2, \ldots$ are constants, and each of the quantities ϵ_1, ϵ_2, &c. fluctuates between given limits, a and b, according to a law of frequency which is of the form $y=f_i(x)$; the value of $\int_{b_i}^{a_i} xf_i(x)\,dx$ being k_i; the corresponding value of mean *second* powers being k_i', of mean *third* powers k_i''. For a first approximation to P, Todhunter finds

$$P=\frac{1}{2\kappa\sqrt{\pi}}\int_{-\eta}^{\eta} e^{\frac{-(l-c+v)^2}{4\kappa^2}}\,dv;$$

where $l=\Sigma\gamma_i k_i$, and $2\kappa^2=\Sigma\gamma_i^2(k_i'-k_i^2)$. This coincides with the first approximation given above* when k_1, k_2 each $=0$, and when the interval 2η is indefinitely small. For then P reduces to

$$\frac{1}{2\kappa\sqrt{\pi}}\,e^{\frac{-c^2}{4\kappa^2}}\times 2\eta;$$

where c^2 may be replaced by our x^2, 2η by our Δx, and $4\kappa^2=$ our c^2.

For a second approximation Todhunter indicates as the

* P. 90.

correction of, or addendum to, the above expression an expression which is described as the third differential with respect to l of the value of P above written (as a first approximation) multiplied by a constant l_1 which

$$=\tfrac{1}{6}\Sigma\gamma_i^3(k_i''-3k_ik_i'+2k_i^3).$$

Performing this operation* and putting (as before) each of the k's$=0$, and therefore $l=0$,

$$l_1=\tfrac{1}{6}\Sigma\gamma_i^3k_i''=\tfrac{1}{6}\text{ our } j;$$

and (as before) substituting x for c, Δx for 2η, c^2 for $4\kappa^2$, we find the addendum (to the formula for the symmetrical probability-curve), in our notation,

$$\frac{2j}{c}e^{\frac{-x^2}{c^2}}\left[\frac{x}{c^2}-\frac{2}{3}\frac{x^3}{c^4}\right]\Delta x.$$

Adding the correction to the first approximation, we have our formula for the asymmetrical probability-curve†.

II. A second proof is derivable from the reasoning by which Mr. E. L. De Forest in the 'Analyst' (Iowa) (vol. ix. p. 163) obtains for the asymmetrical probability-curve a certain formula which has been independently discovered by Prof. Karl Pearson‡. The expansion of this formula in ascending powers of x will be found to coincide with the expansion of our formula in ascending powers of x; *provided that* the second and higher powers of $\frac{j}{c^3}$ may be neglected—a condition which is employed by Mr. De Forest in his proof§, and may readily be established‖.

* This and subsequent operations are performed more fully in the MS. deposited in the Archives of the Royal Society.

† When writing this paragraph I had not adverted to the similar work in Galloway's treatise on *Probability* (forming the article on that subject in the seventh and eighth editions of the *Encyclopædia Britannica*), art. 136.

‡ Philosophical Transactions, 1894. Proceedings of the Royal Society, 1893, p. 331. *Cf.* 'Nature,' 1895, p. 317.

§ E. g. *loc. cit.* p. 138, regard being had to the definition of his symbols.

‖ For example, let the elemental frequency-locus consist of two points, at a distance b, assumed with respective frequencies p and q. Then the mean square of error for a single element is pqb^2, and the sum of these means for all the elements, our k, is $npqb^2$. Also the mean cube of error for an element is $\pm pq(p-q)b^3$ (*cf.* Phil. Mag. vol. xxi. (1886) p. 320); whence $j\div k^{\frac{3}{2}}=(p-q)\div\sqrt{n}\sqrt{pq}$. Which is small when $\sqrt{n}$ is large relatively to $(p-q)\div\sqrt{pq}$, a quantity which vanishes when the element is symmetrical, and is finite for all but infinite degrees of asymmetry. By parity it will be found that $j\div k^{\frac{3}{2}}$ in general $=$ a finite quantity $\div\sqrt{n}$; so that it becomes small when n is sufficiently large.

III. The new proof of the formula for the asymmetric probability-curve, which is offered here, is analogous to that which has been given by Mr. Morgan Crofton for the symmetrical probability-curve*. The proof consists in determining y, the required error-function, as the solution of a system of partial differential equations which must be satisfied by such a function. Put $y=\mathrm{F}(x, k, j)$ where x and j have the same signification as before and k now $=$ our $c^2 \div 2$, ($=$Todhunter's $\kappa^2 \times 2$). A first equation is obtained from the condition that, if each of the constituent elements $q_1, q_2 \ldots$ be multiplied by a constant γ†,

$$\frac{y}{\gamma}=\mathrm{F}(\gamma x, \gamma^2 k, \gamma^3 j).$$

Putting $\gamma=(1+\omega)$, where ω is indefinitely small, expanding and neglecting powers of ω above the first, we have

$$y+x\frac{dy}{dx}+2k\frac{dy}{dk}+3j\frac{dy}{dj}=0. \quad . \quad . \quad . \quad (1)$$

Two more equations are given by the conditions that, if $y=\mathrm{F}(x, k, j)$ is the law of frequency for the sum of the n elements $\mathrm{Q}_1' q_1+\mathrm{Q}_2' q_2+\ldots$, then the superposition of a new element of the form $\mathrm{Q}'_{n+1} q_{n+1}$ for which the mean-square-of-error (measured from its centre of gravity) is Δk, and the mean-cube-of-error (similarly measured) is Δj, must obey the law of frequency

$$y+\Delta y=\mathrm{F}(x, k+\Delta k, j+\Delta j).$$

Let $\eta=f(\xi)$ be the law of frequency for the new element. Then the law of frequency for the compound (of $n+1$ elements) ‡ is

$$\int_b^a f(\xi)\mathrm{F}(x-\xi)\, d\xi,$$

where a and b denote the extreme limits of the range of $f(\xi)$—limits which are by hypothesis finite §. Expanding $\mathrm{F}(x-\xi)$ in terms of ξ, and neglecting powers of ξ above the third (upon the hypothesis that the range of ξ is comparatively

* In the article on *Probability*, 'Encyclopædia Britannica,' 9th edit., vol. xix. p. 781.

† *Cf.* Mr. Morgan Crofton, *loc. cit.*

‡ According to the rule for compounding laws of error indicated by Mr. Morgan Crofton in the article referred to. Compare the present writer, Camb. Phil. Trans. vol. xiv. p. 141.

§ Above, p. 90, and Phil. Mag. vol. xxxiv. (Nov. 1892).

small); and observing that

$$\int_b^a f(\xi)\xi\, d\xi = 0$$

(since ξ is measured from the centre of gravity of the corresponding curve),

$$\int_b^a f(\xi) d\xi\, \xi^2 = \Delta k, \quad \int_b^a f(\xi) d\xi\, \xi^3 = \Delta j\,;$$

we have

$$y + \Delta y = \mathrm{F} + \tfrac{1}{2}\Delta k \frac{d_2\mathrm{F}}{dx^2} - \tfrac{1}{6}\Delta j \frac{d_3\mathrm{F}}{dx^3}*.$$

This expression ought to be identical with

$$\mathrm{F} + \Delta k \frac{d\mathrm{F}}{dk} + \Delta j \frac{d\mathrm{F}}{dj}.$$

Here Δk and Δj may be regarded as independent observations; it is therefore proper to equate the coefficients of Δk in the two expressions for $y + \Delta y$; and similarly the coefficients of Δj. Thus we obtain two additional partial differential

* The reasoning requires that the expansion of $y + \Delta y$ should form a descending series. This condition is fulfilled by our solution. For let the mean square of error for each element be of the order $\frac{1}{n}$, then k, the sum of these mean squares, will be of the order unity. Accordingly if y, as proposed, $= \frac{1}{\sqrt{\pi}\sqrt{2k}} e^{\frac{-x^2}{2k}}$ approximately, then

$$\frac{d_2y}{dx^2} \div y = -\frac{1}{k} + \frac{x^2}{k^2},$$

$$\frac{d_3y}{dx^3} \div y = \frac{3x}{k^2} - \frac{x^3}{k^3}.$$

These and higher differentials may be regarded as being of the order unity for values of x between limits ± 1. Whence it follows that the terms of the expansion in the text form a descending series; since $\frac{d_2\mathrm{F}}{dx^2}$, $\frac{d_3\mathrm{F}}{dx^2}$, &c. are of the order unity, and Δk, Δj, &c., being integrals of $\xi^2 f(\xi)$, $\xi^3 f(\xi)$, &c. between limits separated by a very small interval, will in general form a descending series. The reasoning is not affected if we change the unit: *e. g.* suppose the range of the elements to be of the order unity; in which case $\frac{d_2\mathrm{F}}{dx^2}$, $\frac{d_3\mathrm{F}}{dx^3}$, &c. will form a descending series, while Δk, Δj, &c. will be of the same order. Also the order of $\frac{d_2y}{dx^2} \div y$, $\frac{d_3y}{dx^3} \div y$, &c. is not affected by taking into account the second term of approximation to the value of y given in the text; it being observed that $j \div c^3$ is small, and that the formula only professes to be applicable for values of x which are of the same order as c.

equations; the whole system being

$$y+x\frac{dy}{dx}+2k\frac{dy}{dk}+3j\frac{dy}{dj}=0, \quad . \quad . \quad . \quad (1)$$

$$\frac{dy}{dk}=\frac{1}{2}\frac{d_2y}{dx^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{dy}{dj}=-\frac{1}{6}\frac{d_3y}{dx^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

To these data are to be added the conditions (*a*) that $j \div k^{\frac{3}{2}}$ is small*, and (*b*) that

$$\int_{-\infty}^{+\infty} dyx=1\dagger.$$

Of the problem thus stated the following two solutions are offered:—

First Method.—From equation (1) we have by a familiar method

$$y=\frac{1}{\sqrt{k}}\phi\left(\frac{x}{\sqrt{k}}, \frac{j}{k^{\frac{3}{2}}}\right), \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where ϕ is an arbitrary function.

From equation (2) by a known method‡ we have

$$y=\left(1+\frac{1}{2}x^2\left(2\frac{d}{dk}\right)+\frac{1}{\underline{|4}}x^4\left(2\frac{d}{dk}\right)^2+\ldots\right)\psi_1$$
$$+\left(x+\frac{1}{\underline{|3}}x^3\left(2\frac{d}{dk}\right)+\ldots\right)\psi_2; \quad . \quad . \quad (5)$$

where ψ_1 and ψ_2 are arbitrary functions of k (and j). These functions are restricted by equation (4) to the form

$$\frac{1}{\sqrt{k}}\phi\left(\frac{j}{k^{\frac{3}{2}}}\right).$$

To further determine the forms of ψ_1 and ψ_2 we must utilize equation (3); from which, by combination with equation (2), we have

$$\frac{dy}{dj}=-\frac{1}{3}\frac{d_2y}{dx\,dk}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This condition may be fulfilled by assuming ψ_1 and likewise ψ_2 to consist of a series of ascending powers of $\frac{j}{k^{\frac{3}{2}}}$; which is permissible by condition (*a*).

* As shown above, p. 92 note. † A condition of a probability-curve.
‡ Forsyth's 'Differential Equations,' Art. 256.

To determine the absolute term in the expansion of y, it may be observed that in the case of symmetry ψ_2 vanishes (since y cannot involve odd powers of x); whence it appears, since j also vanishes in this case, that ψ_2 has no absolute term, ψ_1 reduces to $\frac{A_0}{\sqrt{k}}$. A_0 is found by condition (b) to be $\frac{1}{\sqrt{2\pi}}$, while the value of y is (the expansion in powers of x of) the well-known (symmetrical) probability-curve.

To proceed another step, let the first term of ψ_2 be $B_1 \frac{j}{k^{\frac{5}{2}}}$, a form which is prescribed by equation (1); and the second term of ψ_1,

$$\frac{A_1}{\sqrt{k}} \frac{j}{k^{\frac{3}{2}}}.$$

By equation (6) combined with (5) we have

$$\left(x+\frac{1}{\underline{|3}}x^3\left(2\frac{d}{dk}\right)+\ldots\right)\frac{B_1}{k^{\frac{5}{2}}}+\left(1+\frac{1}{2}x^2\left(2\frac{d}{dk}\right)+\ldots\right)\frac{A_1}{k^2}$$

$$\equiv -\frac{1}{3}\left(x\left(2\frac{d}{dk}\right)+\frac{1}{\underline{|3}}x^3\left(2\frac{d}{dk}\right)^2+\ldots\right)\frac{d}{dk}\frac{1}{\sqrt{2\pi k}}.$$

This identity evidently requires that $A_1=0$. The identity is then satisfied by $B_1=-\frac{1}{2\sqrt{2\pi}}$.

It is unnecessary to proceed further with the determination of the coefficients, since the higher powers of the expansion may be neglected. We have therefore for the solution

$$y=\left(1+\frac{1}{2}x^2\left(2\frac{d}{dk}\right)+\ldots\right)\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{k}}$$

$$+\left(x+\frac{1}{\underline{|3}}x^3\left(2\frac{d}{dk}\right)+\ldots\right)\frac{-j}{2\sqrt{2\pi}\,k^{\frac{5}{2}}};$$

an expression of which the expansion in powers of x proves to be identical with the expansion of the formula given above, when k is replaced by $\frac{c^2}{2}$.

Second Method.—The following is perhaps a simpler solution. Put as the correction of the first approximation

$$\left(\text{viz. } y=\frac{1}{\sqrt{\pi}\,\sqrt{2k}}e^{\frac{-x^2}{2k}}\right)$$

the expression $\theta(x,k,j)$. Then if we put F_1 for the first approximation,

$$y=F_1+\theta\,;\quad \frac{dy}{dx}=F_1'+\theta'\,;\quad \&c.;$$

where F_1', θ' ... denote partial differentials with respect to x. Now, if θ is small with respect to $F_1(x)$, then the functions being continuous, θ' will be small with respect to $F_1'(x)$. And we may likewise assume that θ'' and θ''' are small in comparison with F_1'' and F_1''' respectively. Therefore in the expression for $\frac{d_3y}{dx^3}$ it is allowable to neglect θ'''. But it is not equally allowable to neglect θ''. For considering the expansion of $y+\Delta y$, viz.

$$F+\frac{1}{2}\Delta k\frac{d_2F}{dx^2}-\frac{1}{6}\Delta j\frac{d_3F}{dx^3}$$

(above, p. 94), we could not be sure that the neglected quantity $\Delta k\theta''$ is not of the same order as the retained quantity $\Delta j\frac{d_3F}{dx^3}$, the terms of the expansion forming a descending series. Rejecting therefore only θ''', we have approximately

$$\frac{d_3y}{dx^3}=\frac{d_3F_1}{dx^3}=\frac{1}{\sqrt{2\pi k}}e^{\frac{-x^3}{2k}}\left(\frac{3x}{k^2}-\frac{x^3}{k^3}\right);$$

whence by equation (3),

$$\frac{dy}{dj}=-\frac{1}{6}\frac{1}{\sqrt{2\pi k}}e^{\frac{-x^2}{2k}}\left(\frac{3x}{k^2}-\frac{x^3}{k^3}\right).$$

Integrating, we have

$$y=-j\frac{1}{\sqrt{2\pi k}}e^{\frac{-x^2}{2k}}\left(\frac{x}{2k^2}-\frac{x^3}{6k^3}\right)+\chi,$$

χ being a "constant" with regard to j; which by equation (1) must be of the form

$$\frac{1}{\sqrt{k}}\phi\left(\frac{x}{\sqrt{k}}\right).$$

Put $j=0$; then the first term of the value for y vanishing while the curve becomes symmetrical, the second term, the "constant" χ, must be the expression for the ordinary probability-curve, viz.

$$\frac{1}{\sqrt{\pi}\sqrt{2k}}e^{\frac{-x^2}{2k}}.$$

Thus the required expression for y is

$$\frac{1}{\sqrt{2\pi k}}\, e^{\frac{-x^2}{2k}}\left(1-j\left(\frac{x}{2k^2}-\frac{x^3}{6k^3}\right)\right);$$

which, when c^2 is substituted for $2k$, coincides with the expression given above.

This solution may be completed by observing that it satisfies the fundamental equations (1) and (2) unconditionally, as well as (3) when account is taken of condition (a).

A further verification of the theory is afforded by showing that if the sum of m independent elements obeys the law of frequency $y=\mathrm{F}(x, k_1, j_1)$, F having the form which has been found, and k_1 and j_1 being the sum of the respective mean squares of error and mean cubes of error for the m elements; and likewise the sum of another set of m independent elements, n in number, obeys the similarly defined law of frequency, $y=\mathrm{F}(x, k_2, j_2)$; then the sum of $(m+n)$ elements of which m are of the first class and n of the second obeys, as it should, the law

$$y=\mathrm{F}(x, k_1+k_2, j_1+j_2)^*.$$

A particularly interesting case of the asymmetrical probability-curve is that in which an element has only two possible values, say zero and unity, occurring with the respective probabilities p and q—the case considered in a former number of the Philosophical Magazine (vol. xxi. p. 318, 1886). Observing that the mean square of error for this elementary locus is $pq^2+qp^2=pq(p+q)=pq$, and the mean cube of error $=pq(p-q)$, we have by the general formula for the curve representing the law of frequency for the sum of n such elements, an expression in terms of those constants which, *mutatis mutandis*, proves to be identical with the expression which Todhunter, after Laplace, has obtained by a method peculiar to the Binomial†.

The general or multinomial probability-curve, involving (in addition to the centre of gravity) only two constants k and j, may always be replaced by a binomial; through the equations

$$npqi^2=k, \quad npq(p-q)i^3=j,$$

where i is the length of each element‡. There are thus only two equations for three quantities, n, i, and $p \div q$ $(p+q=1)$.

When it is proposed to construct a binomial from a given

* The work is given in the original paper.

† History of Probabilities, Art. 993.

‡ *Cf.* above, note to p. 92.

set of observations, there is given a third condition, namely, that ni must be greater than the distance between the greatest and least observations. But this inequation (coupled with the other equations) is not sufficient to determine n, i, and $p \div q$ with any precision.

Of course, whether a binomial or multinomial probability-curve is to be adapted to a given set of observations, the set must fulfil the condition that $j \div k^{\frac{3}{2}}$ should be a small fraction*. In fact the condition is frequently unfulfilled: for instance, in the statistics of the duration of American marriages†, where the observed $j \div k^{\frac{3}{2}}$ forms a large integer‡. In such cases it may be inferred that the number of independent elements is too small (or their asymmetry too great) to generate a probability-curve.

XII. *On the Existence of Vertical Earth-Air Electric Currents in the United Kingdom.* *By* A. W. Rücker, *M.A., F.R.S.*§

IN a paper by Dr. Adolph Schmidt, read before Section A of the British Association at Oxford (Report Brit. Assoc. 1894, p. 570), the author stated that he had expanded the components of the earth's magnetic force in series, and had deduced expressions, two of which give the magnetic potential on the surface of the earth in so far as it depends on (1) internal, and (2) external forces. "The third series represents that part of the magnetic forces which cannot be expressed in terms of a potential, but must be due to electric currents traversing the earth's surface." The author concludes that such currents amount on the average to about 0·1 ampere per square kilometre.

It appeared therefore desirable that this conclusion, drawn from the magnetic state of the earth as a whole, should be tested by means of those portions which have been most fully studied.

* Above, p. 92.

† Given by Dr. W. F. Wilcox in "The Divorce Problem" (Studies in History &c., Columbia College, vol. i.).

‡ Many other instances in which the condition fails are given by Prof. Karl Pearson in his masterly "Contributions to the Mathematical Theory of Evolution," No. II. (Philosophical Transactions, 1895). For some criticism of Prof. Pearson's theory of asymmetric frequency-curves see the present writer's paper on "Recent Contributions to the Theory of Statistics," in the Journal of the Royal Statistical Society, Sept. 1895.

§ Communicated by the Physical Society: read December 13, 1895.

The test to be applied is, whether the line-integral of the magnetic force taken round a re-entrant circuit on the surface of the earth is or is not a vanishing quantity.

The irregular form of the United Kingdom makes the application of this test more difficult than it would otherwise be; but as two detailed Surveys of Great Britain and Ireland have been carried out by Dr. Thorpe and myself for the epochs 1886 and 1891 respectively, the data at our disposal are so numerous that I thought it worth while to undertake the inquiry.

The actual work of calculation has been carried out almost entirely by two of my students, Messrs. Kay and Whalley. My best thanks are due to them for the care and skill they have displayed.

The facts on which the investigation is based are as follows.

The first survey (1886) included 205 stations, at all of which observations were made by Dr. Thorpe or myself.

The *true*, and therefore irregular, isomagnetic curves were drawn for the epoch January 1, 1886, and the *terrestrial curves*, from which the local disturbances were eliminated, were also calculated for the same date (Phil. Trans. vol. clxxxi. A, 1890).

The second survey included observations at 677 stations. These were made by ourselves, or, under our superintendence, by Messrs. Briscoe, Gray, and Watson. The results are about to be published by the Royal Society. The terrestrial isomagnetic curves were drawn for the epoch Jan. 1, 1891. The secular change having been carefully determined by special observations and methods, the values of the elements and the terrestrial curves obtained for the earlier date were reduced to Jan. 1, 1891. Thus the whole of the 882 stations were available for drawing the true isomagnetics for the latter date. The two sets of terrestrial curves obtained from the second survey and from the first survey reduced to the second epoch did not agree exactly, and the lines bisecting the intervals between them were taken as our final result for the terrestrial curves in 1891.

The following sets of curves will be considered in this paper:—

(1) The terrestrial isomagnetics obtained in the first survey for Jan. 1, 1886. These will be referred to as *the* 1886 *curves*.

(2) The same curves reduced by the secular change to Jan. 1, 1891. These will be called *the first survey* 1891 *curves*.

(3) The terrestrial curves for 1891 deduced from the second survey. These will be called *the second survey* 1891 *curves*.

(4) The mean terrestrial curves for Jan. 1, 1891, deduced from (2) and (3). These will be called *the mean* 1891 *curves.*

(5) Lastly, the true isomagnetic curves deduced from the results at all the 882 stations for the epoch Jan. 1, 1891. These will be called *the true* 1891 *curves.*

(1) *The* 1886 *Curves.*

The advantage of using the calculated terrestrial curves is that they can be carried across the sea from England to Ireland, or extended a few miles from the coast by extrapolation. The area included can therefore be made as large as possible. On the other hand, the method of obtaining these curves is such that the errors in their positions will probably be greatest near the boundaries of the land area over which the survey was carried. In order therefore that such errors might affect different calculations as differently as possible, it was determined to take two circuits, which should have their greatest extensions N. and S., and E. and W. respectively. They will be called the α and β circuits.

The α circuit was bounded by long. 2° W., lat. 58° N., long. 7° W. and lat. 52° N.

The β current was bounded by long. 1° W., lat. 55° N., long. 9° W. and lat. 52° N.

In the published account of the 1886 survey (*loc. cit.* p. 322), the values of the declination (δ) and horizontal force (H) are given for all points within the United Kingdom defined by the intersection of whole degrees of latitude and longitude. From these the northerly components of the force ($H \cos \delta$) were calculated for all such points on the lines of latitude, and the westerly components ($H \sin \delta$) for all such points on the lines of longitude which bounded the circuits.

The method of calculating the line-integral of the force may best be shown by an example, for which we may select long. 2° W. between lat. 52° and 58°.

Let C be the number of cm. in a degree of latitude, and N the northerly component of the force. Assume that $N = N_{52} + al' + x$, where a is a constant, $l' = l - 52$, and x is a small variable.

Let W be the work done as the unit pole moves due North from lat. 52° to lat. 58°.

Then
$$W = C\int_{52}^{58} N dl = C\{N_{52} \times 6 + 18a + \int_0^6 x dl'\}.$$

The value of a was found from the values of the northerly

components at the points on latitudes 52° and 58° respectively. The integral was calculated by quadrature, graphic methods being employed.

To give an idea of the relative magnitudes of the terms, I append the following data :—

$$6N_{52}+18a+\int_0^6 xdl'=1{\cdot}00896-0{\cdot}06864-0{\cdot}00152$$
$$=0{\cdot}93880.$$

The constant C was taken = 11119320 cm., so that the work done in this part of the circuit is $1{\cdot}04387\times10^7$ ergs.

Treating the other parts of the circuit in the same way, the four quantities, the algebraical sum of which is the work done in completing the circuit, are :—

$$(1{\cdot}04388+0{\cdot}17082-1{\cdot}00637-0{\cdot}20902)\times10^7=-6{\cdot}9\times10^3 \text{ ergs.}$$

Dividing by 4π, we find that the total current within the circuit is −550 C.G.S. units, and, since the area is $2{\cdot}13\times10^5$ square kilometres, this amounts to −0·026 ampere per square kilometre. The negative sign indicates that the current flows downwards.

A similar calculation carried out with respect to the β circuit, of which the greatest extension is East and West, and the area is $1{\cdot}77\times10^5$ square kilometres, indicates a current of only −0·004 ampere per square kilometre.

(2) *The First Survey* 1891 *Curves.*

When the 1886 curves are reduced to the Epoch Jan. 1, 1891, by methods which are fully described in the account of the later survey, the results obtained from the α and β circuits are −0·045 and −0·030 ampere per square kilometre respectively. It would at first sight appear as though the fact that these values are larger than those calculated for Jan. 1, 1886, might be due to errors introduced by the assumed values of the secular change; but, as will immediately be seen, they are not larger than those obtained by another method, which this cause of error does not affect.

(3) *The Second Survey* 1891 *Curves.*

Treated in exactly the same way as the last, these give values of about the same magnitude but of opposite signs; viz. for the α circuit +0·016, and for the β circuit +0·020 ampere per square kilometre. Thus two different methods of calculating the same quantity lead to very different results, which point to the conclusion that the apparent effects of the

hypothetical currents are due to small errors in the determination of the exact positions of the lines.

In the final calculation of the results of our survey, we have taken the means of the positions of these two sets of lines as the isomagnetic lines for 1891, hence the mean values of the currents deduced from them by the

(4) *Mean* 1891 *Curves*

are $+0{\cdot}001$ and $-0{\cdot}005$ ampere per square kilometre for the α and β circuits respectively.

(5) *The True* 1891 *Curves.*

We have further checked the above results by means of the true curves, taking two circuits—one (γ) in England and Scotland, and the other (δ) in Ireland.

The first of these was as follows :—

Long. 1° E. from lat. 51° to lat 53°.
Lat. 53° from long. 1° E. to 1° W.
Long. 1° W. from lat. 53° to 55°.
Lat. 55° from long. 1° W. to 3° W.
Long. 3° W. from lat. 55° to 53°.
Lat. 53° from long. 3° W. to 4° W.
Long. 4° W. from lat. 53° to 51°.
Lat. 51° from long. 4° W. to 1° E.

The area is $1{\cdot}054 \times 10^5$ square kilometres. The values of the horizontal force and declination for every 10′ of latitude or longitude were read off from the maps on which the values at the different stations were entered, and the true isomagnetics drawn. This operation was performed by Messrs. Kay and Whalley and checked by myself. The northerly or westerly component of the force was then calculated for each of these points, and the average value for each short section was assumed to be equal to the mean of the values at its initial and final points.

No difficulty arose except at a point in Wales, where the curves are closed, and where it was therefore necessary to assume an average value for a section of the line on which a maximum occurred.

The result of the calculation gave a current of $-0{\cdot}008$ ampere per square kilometre.

The second circuit was taken in Ireland. It traversed the district of Antrim, in which there are violent local disturbances, and is interesting chiefly as showing to what extent the result may be affected by such causes.

The circuit was as follows :—

Long. 6° 30′ W. from lat. 52° to lat. 55°.
Lat. 55° from long. 6° 30′ to long. 8°.
Long. 8° from lat. 55° to 54°.
Lat. 54° from long. 8° to 9°.
Long. 9° from lat. 54° to lat. 52°.
Lat. 52° from long 9° to long. 6° 30′ W.

The area is 48×10^3 square kilometres.

The current is $-0{\cdot}046$ ampere per square kilometre.

The fact that these different circuits, including areas of very different magnitudes and situated in different parts of the United Kingdom, all give very small values for the hypothetical currents, is strong evidence that the smallness of the calculated numbers is not due to the fact that large positive and negative values mutually cancel each other. It is, for instance, conceivable that the directions of the current-flow might be opposed on what may be called the oceanic and continental sides of the kingdom.

If this were so, it is probable that circuits γ and δ would have given results of opposite signs. By way of testing the matter further, the current was calculated both from the 1886 and the 1891 lines for the relatively small area in the West of Ireland bounded by latitudes 52° and 53°, and longitudes 9° and 10°.

The results were :—

for 1886, $-0{\cdot}04$ ampere per square kilometre,
„ 1891, $+0{\cdot}11$ „ „ „

Hence the difference of direction which characterized the currents deduced from the two surveys when applied to large areas, also distinguishes, and in an exaggerated degree, the results obtained from a small border district. It is therefore evident that either the distribution of the vertical currents has entirely altered in five years, or they are too small to be detected by the method employed.

The former of these hypotheses is negatived by the fact that different calculations, based on the first and second survey 1891 curves respectively, lead to discordant results for the same date, and we are therefore compelled to fall back upon the second alternative.

Effect of the Ellipticity of the Earth.

The question may fairly be raised whether, in dealing with such minute quantities, the ellipticity of the earth ought not to be taken into account.

In answer to this, it may be stated that the work done

when the unit pole traverses the α circuit was also calculated, using the data as to the form of the earth given by Captain Clarke and quoted by Prof. Everett ('C.G.S. System of Units,' ed. 1891, p. 71). The numerical values thus obtained differed from those given above, but the differences between them were of the same order. Both methods of calculation lead to opposite conclusions as to the directions of the hypothetical currents according as the 1886 or 1891 curves are used; thus proving that the small outstanding uncertainties as to the magnetic state of the United Kingdom are the cause of the discrepancies, which are not reduced by using a closer approximation to the form of the earth.

Conclusion.

I can only conclude from these various figures that the local magnetic surveys furnish no evidence of vertical electrical currents in the United Kingdom. The largest number obtained from the larger circuits is less than half that which Dr. Schmidt assigns as the mean value for the whole earth. Different calculations lead to results of different signs for the same quantity; and the data which would *a priori* be accepted as the best give the smallest values.

As far as the terrestrial curves are concerned, the final results for the two surveys are embodied in the 1886 and the mean 1891 curves respectively. The local irregularities in the North of Ireland are so great that the calculations based on the true isomagnetics in that country may be neglected as compared with the probably much better results obtained in Great Britain.

Selecting, then, only the most trustworthy values, the results may be summed up as follows in terms of amperes per square kilometre.

	Circuit.		
	α.	β.	γ.
1886.	−0·026	−0·004	
1891.	+0·001	−0·005	
1891.	.....	.....	−0·008

From these figures we may conclude that there is not in the United Kingdom a vertical current amounting on the average to 0·1 ampere per square kilometre. They are not

inconsistent with the existence of a current of about a tenth or a twentieth of that amount; but on account both of the smallness of the results and of the discrepancy between the values obtained for 1891 by two methods, we cannot assert that such a current actually exists. The calculations taken by themselves do not disprove the hypothesis that electric currents traverse the earth's surface, as we cannot argue from the condition of a small portion of the globe to that of the whole. The most that can be said is that no evidence in favour of the existence of vertical currents can be drawn from one district, which has been very minutely surveyed.

P.S.—No reference to Dr. Schmidt's original paper was given in the short notice published in the Report of the British Association. I had therefore supposed that the latter was a preliminary note. Professor Schuster has, however, recently shown to me Dr. Schmidt's complete investigation, and he has kindly calculated the current-density at latitudes 50° and 55° on the prime meridian from formulæ given by Dr. Schmidt. The result is *upward* currents and 0·20 to 0·15 ampere per square kilometre at lat. 50° and 55° respectively. The mean of these two numbers, viz. 0·175, is nearly equal to Schmidt's mean for the whole earth (0·17). It is opposite in direction to and very much greater in magnitude than any vertical current the existence of which is compatible with the results of the Magnetic Survey.

XIII. *A Continuous and Alternating Current Magnetic Curve Tracer.* *By* John Ely Moore, *M.E., E.E.**

§ 1. Of the various methods that have been proposed and used for measuring the magnetic quality of iron, or the losses of energy in iron due to magnetic reversals, none have been successful as accurate means of measurement that have not given simultaneous and independent readings (capable of interpretation in absolute units) of the two quantities, magnetic induction or intensity of magnetization, and the magnetizing force producing that induction.

In a magnetic-curve tracer intended for accurate work, the first condition to be fulfilled is that its indicating system shall be capable of receiving and recording two simultaneous and independent displacements, always at right angles to each other, and always strictly proportional, respectively, to the intensity of magnetization of the sample under test, and the magnetizing force producing that magnetization.

* Communicated by the Author.

This condition is not fulfilled by any magnetic-curve tracer, so far as I know, that has ever been described. The failure being due to mutual interference (mechanically or magnetically) of the separate displacements of the indicating system, to lack of proportionality of these two displacements to the intensity of magnetization and magnetizing force respectively, or to the inability to interpret these displacements in terms of Absolute Units.

In the following description of a continuous and alternating current magnetic-curve tracer, the order observed will be the actual order of development from the observation of the underlying principle to the embodiment of this principle in the present form of instrument.

§ 2. Several years ago (in the early part of 1892), while making an investigation of the losses in iron magnetized by alternating currents, the writer used, he believes for the first time, an instrument which may be conveniently called an alternating-current magnetometer. A small helix of fine insulated wire was suspended bifilarly, so that the axis of the helix was in the earth's magnetic meridian. The lower ends of the bifilar wires were put in connexion with the free ends of the helix, and the upper ends of the bifilar wires were arranged so that the suspended helix might be made to form part of a circuit, composed of a few cells of secondary battery, a regulating resistance, and a revolving contact-maker carried on the shaft of an alternating-current generator. It is clear that when the armature of the alternating-current generator was set in rotation at a constant angular velocity, an intermittent current of constant mean intensity and of definite period and phase relation was passed through the suspended helix. It was observed, on bringing a long rod magnetized by an alternating current from the generator whose armature shaft carried the contact-maker, that the suspended helix was deflected from its position of rest through a definite angle for any definite position of the rod; and that for any fixed position of the rod, by varying the point in the revolution of the armature-shaft at which the intermittent current was made through the suspended helix, a series of deflexions of the suspended helix to the right and left of the zero position was obtained. Thus, by means of the suspended helix carrying an intermittent current of proper periodic time and of constant mean intensity we are enabled to measure directly the magnetic state of the iron at any definite point of the alternating-current waves. The alternating-current magnetometer method, employed to measure the losses in iron magnetized by alternating currents, has been described in the

London 'Electrician' of Feb. 3, 1893, and also in the New York 'Electrical World' of Feb. 4, 1893, and need not be entered into here.

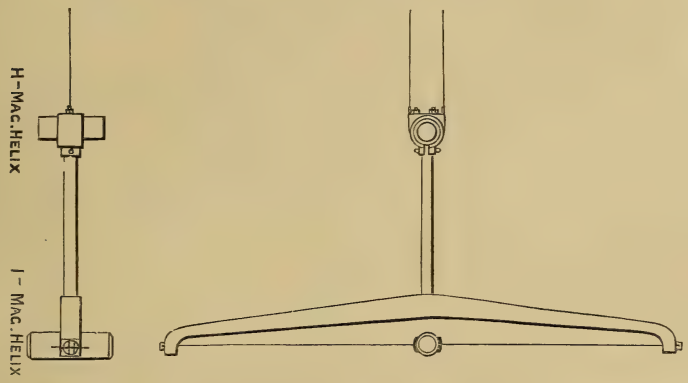
Although the alternating-current magnetometer can plainly be used equally well for continuous and alternating currents of any frequency, yet as a method of measuring losses in iron under widely varying conditions of magnetization and periodicity it necessitates the separate observation of a comparatively large number of quantities, the subsequent calibration of these quantities, and finally the graphical representation of them, before the desired estimation of losses can be made.

§ 3. It seemed desirable, therefore, in order to facilitate the investigation of the nature and amount of the losses in iron, carried through magnetic cycles varying widely in both amplitude and in time, to devise an instrument that would, first, be independent in its action of the time required to complete the magnetic cycles; second, be accurate and reliable, and give indications capable of interpretation in Absolute Units; third, that would reduce the number of observations required to obtain the losses to an absolute minimum; fourth, that would read directly the quantities to be measured; and fifth, that would allow measurements to be made in the least possible time.

It is attempted to fulfil these conditions in the present instrument in the following way:—A system of two small helices is so suspended and mechanically connected that any angular displacement about a vertical axis, given the one, is imparted to the other also, while one of the two helices may receive independent angular displacements about a horizontal axis besides. The arrangement and connexions of the magnetometer system will be immediately seen from the accompanying diagram (fig. 1). If the losses in iron magnetized by an alternating current are to be measured, an intermittent current of the proper periodic time (secured, as previously indicated, by a revolving contact-maker on the shaft of the alternating-current generator) is passed through the helices of the suspended magnetometer system in series, making of the small helices during the very short time the current is flowing in them virtual magnets. If now the sample to be tested, in the form of a long rod supplied with the proper magnetizing coil, be adjusted so that its axis is vertical and lies in the vertical axis of suspension of the magnetometer system, and it be magnetized by an alternating current from the generator whose armature-shaft carries the revolving contact-maker, only that one of the magnetometer helices which is capable of displacement about a horizontal

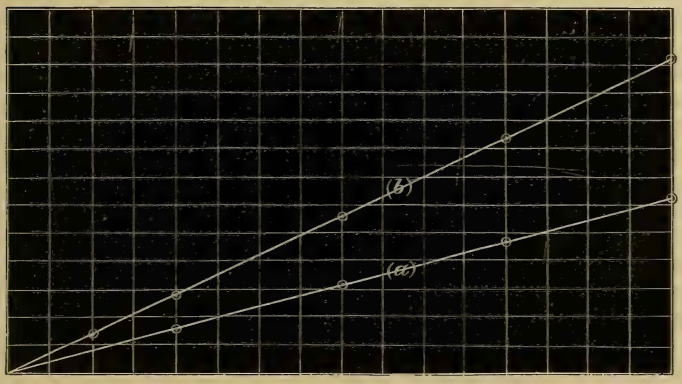
axis (called hereafter the I-magnetometer helix) will be deflected. The horizontal suspension being a silver unifilar

Fig. 1.—Magnetometer System.



torsion-wire, the angular displacements of the helix may be taken proportional in magnitude to the product of the magnetic moments of the long rod and the helix, at the point in the alternating current waves at which the helix becomes a magnet. As will be seen on referring to curve (*a*), fig. 2,

Fig. 2.—Calibration of Magnetic Curve Tracer.



(*a*) Ordinates = deflexion of I-magnetometer helix.
Abscissæ = magnetic force at centre of I-magnetometer helix.
(*b*) Ordinates = deflexion of H-magnetometer helix.
Abscissæ = current in H-deflecting coils.

by keeping the mean strength of the intermittent current constant throughout an experiment, the vertical displacements of the I-magnetometer helix may be taken proportional to the

magnetic moments of the sample, at the points in the alternating-current waves at which the instantaneous direct current passes in the magnetometer system. Since the centre of the I-magnetometer helix lies in the prolongation of the axis of the long rod, and the magnetic field due to the long rod is symmetrical about its axis, the deflecting force exerted on the I-magnetometer helix by the long rod is independent of the angular position of the I-magnetometer helix about the vertical axis of suspension of the magnetometer system. If, then, we adjust a coil of wire so that its centre is in a horizontal line (normal to its plane) passing through the centre of, and perpendicular to that magnetometer helix capable of displacement about a vertical axis only (called hereafter the H-magnetometer helix) ,and pass through this coil the same alternating current that magnetizes the sample, the H-coil will receive a horizontal angular displacement, and through the mechanical connexions of the parts of the moveable system it will carry with it through the same horizontal angle the I-magnetometer helix. This horizontal displacement being made against the restoring couple of the fine silver wire bifilar suspension of the system, is equilibrated partly by the restoring couple of the bifilar suspension proper (which varies as the sine of the angle of displacement), and partly by the torsion of the bifilar wires (which varies directly as the angle of displacement). Hence for small horizontal angular deflexions the angular displacement may be taken proportional to the product of the magnetic moments of the coil carrying the magnetizing current and the H-magnetometer helix, at the point in the alternating current waves at which the helix becomes a magnet. But the magnetic moment of the H-magnetometer helix may be kept constant throughout an experiment; hence, as will be seen from curve (*b*), fig. 2, the horizontal angular displacements may be taken proportional to the magnetic action, or the magnetizing force of the current, at the points in the alternating-current waves at which the instantaneous direct current passes through the magnetometer helices.

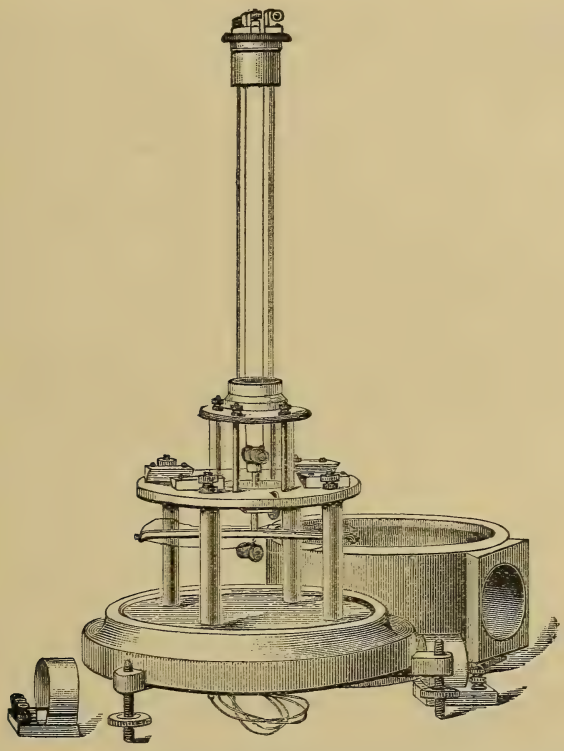
If it is desired to measure the losses of energy in iron magnetized by continuous currents, or the "static" losses of iron, plainly the only change of arrangement in the curve-tracer is to substitute for the intermittent current through the magnetometer system a direct current of the same mean intensity, to replace the source of alternating current by a continuous-current source, and provide a regulating resistance for varying the continuous magnetizing current from zero to the maximum value required.

In either case, by means of one of the mirrors attached to the I-magnetometer helix, we are enabled to record a displacement, the rectangular components to which are independent of each other, and accurately proportional to the intensity of magnetization of the sample and the magnetizing force producing that magnetization, at any point in the magnetic cycle.

Thus we have, in this magnetometer system of two simple helices connected as described, the fulfilment of all the essential conditions for an instrument measuring the dissipation of energy in either "static" magnetic cycles, or in "periodic" magnetic cycles of any frequency.

§ 4. Fig. 3 shows the form of the magnetic-curve tracer

Fig. 3.



now used in the Magnetic Observatory of Princeton University. It consists, first, of a solid wooden base of circular shape, provided with levelling-screws. From this base arise four vulcanite pillars, which support a vulcanite disk, having

a circular opening at its centre. The cover of the instrument rests in a groove in the wooden base, fitting closely about the circular disk at its top, and is provided with a plane glass window for the admission of a beam of light. Placed diametrically opposite, on the upper surface of the vulcanite disk, are two coils of a few turns of stout wire, hereafter to be called the H-deflecting coils. From the centre of this same disk rises a glass cylinder, covered at its upper end by a second vulcanite disk, to which is attached the suspension-tube. The suspension-tube carries at its upper end a torsion head, provided with binding-posts and adjusting devices for regulating the bifilar wires. From this torsion-head is suspended, by means of the silver bifilar wires, the moveable magnetometer system. The upper or H-magnetometer helix is held by a hard rubber clamp, serving at the same time as a terminal block for the lower ends of the bifilar wires. To the lower part of this clamp is fastened a hard rubber shaft, supporting centrally, at its lower end, a horizontal hard rubber bar, to the extremities of which are fastened (by means of suitable adjusting devices) the horizontal unifilar torsion-wires carrying the I-magnetometer helix.

Each of the magnetometer helices having the same number of turns of wire will, when so connected as to develop opposite polarities in the ends lying in the same direction, form an astatic combination for horizontal deflexions. If it should be necessary to make the vertical deflexions also independent of the earth's magnetic field, one can easily arrange a flat coil of wire with its centre in the vertical axis of suspension of the moveable system, and its plane horizontal, and pass through the coil such an electric current as will just equal and neutralize, in its magnetic effect, the vertical component of the earth's magnetic field at the centre of the I-magnetometer helix.

§ 5. In using the instrument, it should be set up on some firm support (generally, though not necessarily, so that the planes of the H-deflecting coils are parallel to the earth's magnetic meridian) and properly levelled until the suspended system is perfectly free, and hangs centrally in the instrument. The length of the bifilar wires should then be adjusted until the centre of the H-magnetometer helix lies in a horizontal line passing through the centre of the H-deflecting coils on either side of the instrument. The stress on the bifilar wires should then be equalized, by means of the adjusting devices at either end of the horizontal bar carrying the I-magnetometer helix, the horizontal torsion-wires should be regulated in length, so that the centre of gravity of the

I-magnetometer helix lies in the vertical axis of the suspended system, and the torsion adjusted in the wires until the axis of the helix is horizontal. The torsion-head of the instrument is then turned until the axes of the magnetometer helices are parallel to the planes of the H-deflecting coils.

Upon one of the mirrors, carried by the I-magnetometer helix, is projected a beam of light from some fixed source, which after reflection is received upon a screen placed normally to the helices in their undisturbed position. By means of cross-hairs placed in the path of the incident beam of light, one is enabled to mark on the screen the initial or zero position of the magnetometer system, and to measure any subsequent angular displacement of either or both of the magnetometer helices.

The sample to be tested (in the form of a long rod supplied with the proper magnetizing coil) should be placed with its axis in the vertical line of suspension of the magnetometer system. This can be accomplished most conveniently by clamping the sample and its magnetizing coil below the support of the instrument. The sample and its magnetizing coil should then be adjusted vertically, until the magnetic action of the sample, for maximum intensity of magnetization, produces the desired maximum vertical deflexion of the I-magnetometer helix. A compensating coil of a few turns of wire should then be adjusted so as to neutralize, at the I-magnetometer helix, the magnetic action of the magnetizing coil about the sample.

The accuracy of the adjustments for horizontal displacements can be tested by making the magnetometer helices active, by means of a constant current of one or two tenths of an ampere, and reversing an independent direct current through the H-deflecting coils. This should give the magnetometer system equal horizontal displacements on either side of the zero position, and, for various current-strengths, the magnitude of these deflexions should be proportional to the respective currents.

The accuracy of the adjustments for vertical displacements depends upon the setting of the sample to be tested (with its magnetizing coil) and the balancing coil. This can be tested before beginning an experiment by placing the magnetizing coil (with the sample removed) in the position it will occupy during an experiment, and reversing continuous currents of various strengths through it. There should be no horizontal displacements, and the vertical displacements, on either side of the zero position, should be proportional to the respective magnetic forces due to the magnetizing coil, at the I-mag-

netometer helix. The compensating coil should be so connected in series with the magnetizing coil as to give an opposing magnetic effect at the I-magnetometer helix, and by keeping its plane horizontal, and its centre in the vertical line of suspension of the magnetometer system, be adjusted in position until the joint magnetic effect of the two coils at the I-magnetometer helix is zero, for all strengths of current through them.

§ 6. The instrument, having thus been set up and adjusted and the adjustments tested, is ready to be used for the measurement of energy-losses in iron magnetized by either direct or alternating currents. If the losses in iron magnetized by continuous currents are to be measured, a current of two-tenths of an ampere from some independent battery (a single cell of storage-battery answers very well) is passed through the helices of the magnetometer system, and kept constant throughout an experiment. The intersection of the cross-hairs in the beam of light reflected upon the recording screen by one of the mirrors carried by the I-magnetometer helix, is taken as the centre of a system of vertical-horizontal rectangular coordinates. The sample is then placed in the magnetizing coil, which, after being properly connected in series with the compensating coil, the H-deflecting coils of the instrument, a regulating resistance, and a source of continuous current, has the magnetizing current through it increased from zero to the maximum value required to produce the desired maximum degree of magnetization, reduced to zero, reversed, the same operation performed in the opposite sense, and the cycle completed by returning finally to the first maximum. The point of intersection of the cross-hairs on the screen will be displaced for every different value of the magnetizing current, a distance from each of the coordinate axes previously drawn, proportional respectively to the magnetizing force acting on the sample and the intensity of magnetization of the sample. Hence by marking down on the screen the point of intersection of the cross-hairs in the reflected beam of light, for any desired number of values of the magnetizing current throughout the cycle of magnetization, we have an accurate outline of the magnetization curve for the sample, or, as it is frequently called, the "hysteresis" curve. By properly varying the magnetizing current, loops can be traced to any part of the hysteresis curve, or a complete set of graded cycles obtained without removing the sample.

In measuring the losses in iron magnetized by an alternating current, an intermittent current of a mean value of

two-tenths of an ampere (obtained by means of a revolving contact-maker on the shaft of the alternating-current generator supplying the alternating magnetizing current) is passed through the magnetometer helices, and kept accurately constant throughout an entire experiment. For the continuous magnetizing current is substituted an alternating current, adjusted to such a mean value as will give the desired maximum intensity of magnetization. As has been shown, by varying the point in the revolution of the armature-shaft at which contact is made, or current passes in the magnetometer helices, different deflexions will be obtained, the rectangular components to which are strictly proportional to the magnetizing force, and the intensity of magnetization respectively, at the points in the alternating-current waves at which the instantaneous direct current passes through the magnetometer helices. Hence by varying the point of contact on the revolving contact-maker, through a circular angle corresponding to 360 degrees of phase in the alternating-current waves, we have, as in the case of "static" cycles, the successive positions on the screen of the intersection of the cross-hairs in the reflected beam of light, marking points in the outline of the energy-loss curve.

§ 7. In calibrating the instrument for either "static" or "periodic" cycles, the current through the magnetometer helices is kept the same in kind and magnitude as during the experiment. The sample is removed from the magnetizing coil, and an accurately measured current sent through either the magnetizing coil or the compensating coil, alone. The deflexion is marked down on the vertical axis of coordinates. Then an accurately measured current is sent through the H-deflecting coils alone, and the resulting deflexion marked down on the horizontal axis of coordinates.

By knowing the distance of the resultant magnetic distribution, or magnetic poles, from the I-magnetometer helix, we can calculate, by well-known magnetometer laws, the value of the magnetic force at the centre of the I-magnetometer helix, due to the long rod, in terms of the distance of the poles from the centre of the helix, the cross section of the sample, and the intensity of magnetization of the sample. It will be plain, by considering the relative position of the long rod and I-magnetometer coil, that this force,

$$F_{lr} = SI\left(\frac{1}{R^2} - \frac{1}{R'^2}\right); \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where S is the cross section of the sample, I the intensity of magnetization of the sample, R the distance from the

centre of the I-magnetometer helix to the nearer pole, and R′ the distance from the centre of the I-magnetometer helix to the farther pole.

By knowing the distance of the compensating coil during calibration from the I-magnetometer helix, its number of turns, mean radius, and the current through it, we are enabled to calculate its magnetic force at the I-magnetometer helix. If the calibrating coil is a flat coil of only a few turns of wire (as is always most convenient), then, as is well known, its magnetic force at the I-magnetometer helix is expressed by :

$$F_{(cc)} = \frac{2\pi n C r^2}{10(d^2+r^2)^{\frac{3}{2}}}; \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where n is the number of turns in the calibrating coil, C the value in absolute units of the current in it, r its mean radius, and d the distance of the calibrating coil from the I-magnetometer helix. Calling D_e any particular vertical deflexion in the course of an experiment, and D_c the vertical deflexion during calibration, we have, since the current in the magnetometer system has been kept constant through the whole experiment, the following relation :—

$$\frac{D_e}{D_c} = \frac{F_{lr}}{F_{cc}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Substituting in equation (3) from equations (1) and (2) we get

$$\frac{D_e}{D_c} = \frac{SI\left(\frac{1}{R^2} - \frac{1}{R'^2}\right)}{\frac{2\pi n C r^2}{10(d^2+r^2)^{\frac{3}{2}}}}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Solving equation (4) for I, we get

$$I = \frac{2\pi n C r^2}{10(d^2+r^2)^{\frac{3}{2}}} \frac{R^2 R'^2}{SR'^2 - R^2} \frac{D_e}{D_c}. \quad . \quad . \quad . \quad . \quad (5)$$

We are thus enabled to find the value of the intensity of magnetization of the sample in Absolute Units for any point on the entire curve of magnetization.

Knowing the number of turns, the length, and the current flowing in the magnetizing coil surrounding the sample, we can at once calculate in Absolute Units the magnetizing force of the coil on the sample. In the calibration of the horizontal deflexions, we know the current flowing in the H-deflecting coils and the displacement it gives along the horizontal axis. We calculate the magnetizing force of the

magnetizing coil about the sample for that current, and by direct proportion between horizontal displacement in calibration, and the horizontal displacement of any point on the curve, we obtain in Absolute Units the magnetic force corresponding to any point on the curve. Having thus calibrated the curves of either continuous or alternating current magnetic cycles in Absolute Units, the energy losses, $\int I dH$, may be obtained by taking the area of the curves in the usual way.

Although this description has only been concerned with the curve-tracer as a means of measuring the magnetic quality of, and the energy losses in iron and other metals, when carried through magnetic cycles, it plainly lends itself to such operations as direct tracing of alternating-current and electromotive-force curves, the investigation of the nature and amount of iron losses in alternating-current transformers, the measurement of the power in any electrical circuit, &c.

XIV. *On the Dissociation Degree of some Electrolytes at* 0°. *By* R. W. Wood*.

THE values obtained by the lowering of the freezing-point for the dissociation-degree of dissolved electrolytes are always a little smaller than those calculated from the electrical conductivity.

Meyer Wildermann † has recently lessened the difference by the use of a more accurate method for the freezing-point determinations, and has expressed the opinion that the cause of the discrepancies lay in the fact that the electrical conductivities have been determined at a higher temperature (18°–25°).

At Prof. Jahn's suggestion, I have determined the conductivity of certain Salts and Acids in solutions of varying concentration at 0°, for the purpose of reckoning the Dissociation-degree at this temperature, and the results indicate that, in dilute solutions, the dissociation-degree is practically independent of the temperature.

The determinations were made according to the Kohlrausch method, and the conductivity of the distilled water used in the experiments was determined at different temperatures and taken into account.

The value expressing the conductivity of infinitely dilute

* Translated from the *Zeitschrift für phys. Chemie*, xviii. p. 3 (1895). Communicated by the Author.

† Phil. Mag. July 1895.

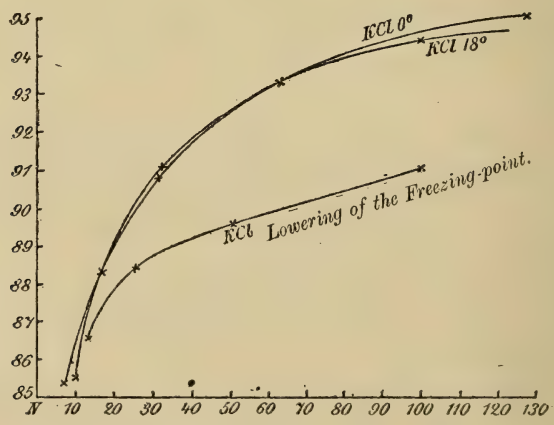
solutions was calculated by means of the temperature-coefficient from the values found by Kohlrausch and Ostwald for 18° and 25°.

In the case of the acids, it was determined by the aid of the Kohlrausch law.

The results of the determinations made with potassium and sodium chloride are as follows :—

Potassium Chloride 0°.			Sodium Chloride 0°.		
V.	λ.	$\frac{\lambda_v}{\lambda_\infty}$.	V.	λ.	$\frac{\lambda_v}{\lambda_\infty}$.
1	61·10	78·53	1	44·59	68·60
2	62·49	80·30	2	48·00	73·85
4	64·46	82·78	4	50·76	78·09
8	66·47	85·44	8	53·31	82·02
16	68·70	88·30	16	55·36	85·17
32	70 66	90·82	32	57·20	88·00
64	72·60	93·32	64	59·13	90·77
128	73·90	94·99	128	60·10	92·46
256	75·30	96·79	256	61·13	94·06
512	75·90	97·56	512	62·43	96·05
1024	76·35	98·14	1024	63·70	98·00
∞	77·80	100·00	∞	65·00	100·00

V signifies the volume containing a gram-molecular weight of the salt, λ the conductivity, and $\frac{\lambda_v}{\lambda_\infty}$ the dissociation-degree.



If we take as ordinates the values obtained by Kohlrausch for the dissociation-degree at 18°, and those which I have

found for 0°, and as abscissæ the values of V, we find that the curves are almost coincident, while the one representing the values obtained by Wildermann from the lowering of the freezing-point runs considerably lower, as the diagram indicates.

The values of λ_∞ for 0° in the following table is calculated by means of the temperature-coefficient of the most dilute of the investigated solutions from the values of λ_∞ given by Ostwald for 25°.

For di- and trichloracetic acids the following values were found:—

V.	λ for 0°.	$\frac{\lambda_v}{\lambda_\infty}$ for 0°.	λ for 25°. Wood.	λ for 25°. Ostwald.	$\frac{\lambda_v}{\lambda_\infty}$ for 25°.	Dissociation-degree by lowering of freezing-point.
Dichloracetic Acid, $\frac{1}{\lambda}\frac{d\lambda}{dT}=0{\cdot}0148$.						
32	174	76·6	252·3	253·1	70·2	
64	194·2	85·5	—	290·7	80·5	74·0
128	207·9	91·5	317·4	317·5	88·0	84·0
256	216·1	95·2	—	337·0	93·4	
512	221·9	97·7	351·8	352·2	97·6	
∞	227·0	100·0	—	361·0	100·0	
Trichloracetic Acid, $\frac{1}{\lambda}\frac{d\lambda}{dT}=0{\cdot}0149$.						
32	206·0	91·7	324·0	323·0	90·1	88·1
64	211·9	94·3	333·5	332·8	93·0	94·0
128	216·6	96·4	341·5	341·0	95·3	
256	219·7	97·7	348·1	348·0	97·0	
512	221·9	98·7	353·0	353·7	98·8	
∞	224·7	100·0	—	358·0	100·0	

For control, the conductivity of the two acids at infinite dilution at 0° was calculated from the values obtained for $\frac{1}{512}$ normal solutions of their potassium salts, potassium chloride, and hydrochloric acid, according to the Kohlrausch law. The following are the values found, which agree very well with those in the table:—

Dichloracetic acid . $\lambda_\infty = 228$.
Trichloracetic acid . $\lambda_\infty = 225{\cdot}2$

The dissociation-degree of the two acids is apparently

independent of the temperature, and Wildermann's supposition has not been confirmed. The deviations of the values for the dissociation-degree calculated by the two methods still remain.

In all probability the cause lies in the inexact determination of the freezing-point of dilute solutions. We know from the investigations of Nernst and Abegg* that its exact determination is attended with great difficulty.

XV. *The Duration of the Flash of Exploding Oxyhydrogen.* *By* R. W. Wood†.

SOME time ago, in endeavouring to photograph the explosion-wave, if it may be so termed, of electrolytic gas by means of the electric spark, I was struck with the fact that the duration of the flash was exceedingly small. In many cases I found that the glass bulbs, which were thinner than paper, were photographed by the light of the incandescent gas within before the walls had given way. The striations on the glass were sharp and the outlines of the bulb were perfectly distinct, the interior being quite filled with a bright glow.

I have recently repeated and enlarged somewhat on these experiments with a view to determine the duration of the flash.

The bulbs were blown very thin, with a quill-like neck at each end. They were filled with a mixture of oxygen and hydrogen by means of an electrolytic apparatus, and two copper wires were then sealed in with wax, with a gap of perhaps half a millimetre between them.

A pendulum, made by hanging a heavy lead ball on a fine wire, was hung in front of the bulb, the copper wires of which were connected with the terminals of a small inductorium.

From the bottom of the ball projected a pin which came in contact with a slip of platinum foil mounted edgewise before the bulb, and the pendulum and foil were put in circuit with the primary coil of the inductorium. This arrangement insured the explosion of the bulb at the moment when the pendulum passed its point of equilibrium, and by calculating the velocity of the pin's head and measuring the amount of blur of the photographic image the duration of the flash could be approximately determined. The camera, provided with a Zeiss lens of large aperture, was placed near enough

* *Zeitschr. für phys. Chemie*, xv. p. 681 (1894).

† Communicated by the Author.

to give an image of natural size in order to simplify the measurements.

With a velocity of 3 metres a second the image of the round pinhead was as sharp as if stationary; but on raising the pendulum to a height sufficient to give it a velocity of 6 metres a second, a blur amounting to a trifle less than ·5 millim. was observed. The pendulum-point moving 6000 millim. in a second would traverse a distance of ·5 millim. in the $\frac{1}{12000}$ of a second, consequently the duration of the flash is somewhat less than $\frac{1}{12000}$ sec., probably about $\frac{1}{13000}$ sec.

Of course the total duration of the flash depends on the speed of propagation of the explosion in the mixture, as well as on the actual duration of the flash of each exploding element; Le Chatelier found that this velocity was about 1000 metres a second.

The bulbs being about 8 centim. in diameter, and the point of ignition in the centre, the explosion would reach the walls in $\frac{1}{25000}$ of a second. This being about one half of the total duration, we may suppose it to be about the duration of the flash of any small element.

In some cases the luminous gas in the photograph was confined within the walls of the bulb, while in other cases a cloud or outburst was noticed at some particular spot outside the wall. The image of the pendulum wire in one case came in front of this cloud and was found to be much more blurred than the pin, indicating that the light of one of these outbursts lasts longer than that of the mixture within. This may be due to an admixture of outside air, which increases the time of combustion.

I find that the light of these exploding bulbs is quite suitable for photographing small objects in rapid motion such as water-jets; and in some cases may be preferable to the spark, the electrostatic strain just before the flash tending to deform the jet. In one of the photographs, where an outburst has taken place directly against the jet, there are curved traces around the drops, apparently where the burning gases are rushing around them.

For the purpose of photographing jets, I think that a better device than the thin bulb would be a shallow, rectangular tin tray with a highly polished bottom, such as a box cover. This could be covered with a thin film of mica fastened on with wax, and the vessel filled through two small tubes. The reflecting back would increase the amount of light without increasing the depth of the exploding layer, which of course tends to prolong the flash.

Note.—On receiving the proof of the above paper I was much surprised to find that the photograph, which I had enclosed as a curiosity with the MS., had been reproduced. I had no idea that the picture would be considered of interest enough to warrant reproduction, or I should have taken pains to secure a better one. My best negative came to grief, and the picture reproduced is the first attempt at securing a water-jet by the flash, taken merely to ascertain whether the method was feasible. Much of the detail has been lost in reproduction, but the upper parts of the jet show well. No especial pains were taken to secure uniform jets, though the right-hand one is fairly good. An oblate spheroid, on the end of a liquid thread, then a chain of fine little drops, formed by the breaking up of a thread, then the prolate spheroid.

The outbursts of gas show fairly well in this photograph, and at the bottom of the left-hand jet two blurred drops are dimly seen, showing the longer duration of this part of the flash.—R. W. W.

XVI. *A very Simple and Accurate Cathetometer.* *By* F. L. O. Wadsworth*.

OF the various standard physical instruments which are usually found in a student's laboratory, the cathetometer may justly be considered as one of the most instructive and valuable, both because of the many principles involved in its adjustment, and because of the number of measurements which may be made with it. Unfortunately, good cathetometers (and it is never good policy to use poor instruments for the purposes of instruction) are so expensive, as made at present, that one or at most two are all that one laboratory can afford. For this reason it may perhaps be of interest to briefly describe a form of cathetometer recently designed by the writer which costs less than one-tenth as much as the best German or English instruments, but which has shown itself in use to be quite as accurate as and in some respects even more convenient of manipulation than the latter.

In the new arrangement, the general method of comparison now employed in nearly all of the most accurate linear measurements is followed: *i. e.*, the images of the observed points and of the lines on a standard bar, placed parallel with the length to be measured, are brought in succession into the field of an observing telescope or microscope, and their relative position determined by means of a micrometer or some equivalent arrangement. In previous forms of cathetometer, this has been done by rotating the observing-telescope itself on a long heavy-vertical axis; in the new form, all of these heavy rotating parts are dispensed with, the observing-telescope is fixed in position and the images of the object and scale brought successively into the field by means of a light silvered mirror mounted on a vertical axis just in front of the objective. A sensitive level attached to the upper end of this axis enables the latter to be set accurately vertical, this adjustment being made as in an ordinary cathetometer.

The complete arrangement is shown in elevation in fig. 1 and in plan in fig. 2. A, fig. 1, is the mirror frame and B is the level, mounted at the lower and upper end respectively of the short, conical axis C. The boss D, in which this axis turns, is attached by means of a geometrical clamp at E to a split cap which slips over the end of the observing-telescope T and is clamped thereon by means of the screw F. This much

* From the American Journal of Science, January 1896.

really constitutes all of the essential parts of the new cathetometer, the telescope-mounting itself—consisting of the two L-shaped bars K, G and the adjusting screws H, I—being merely

Fig. 1.

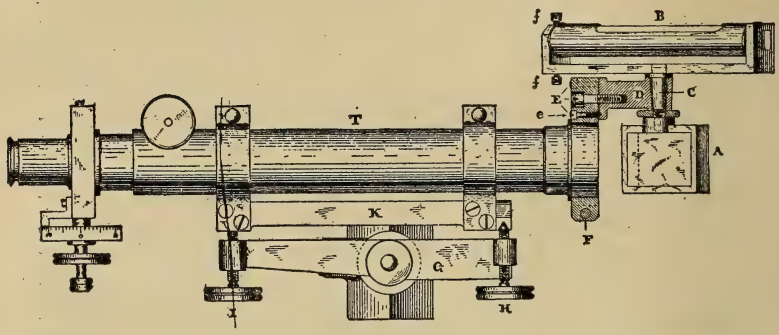
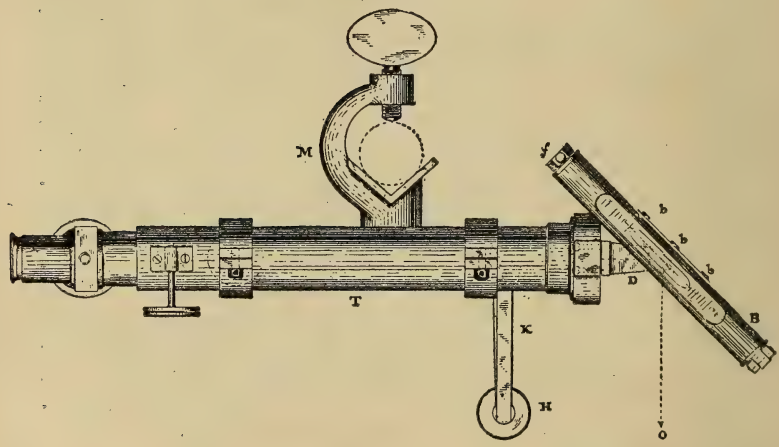


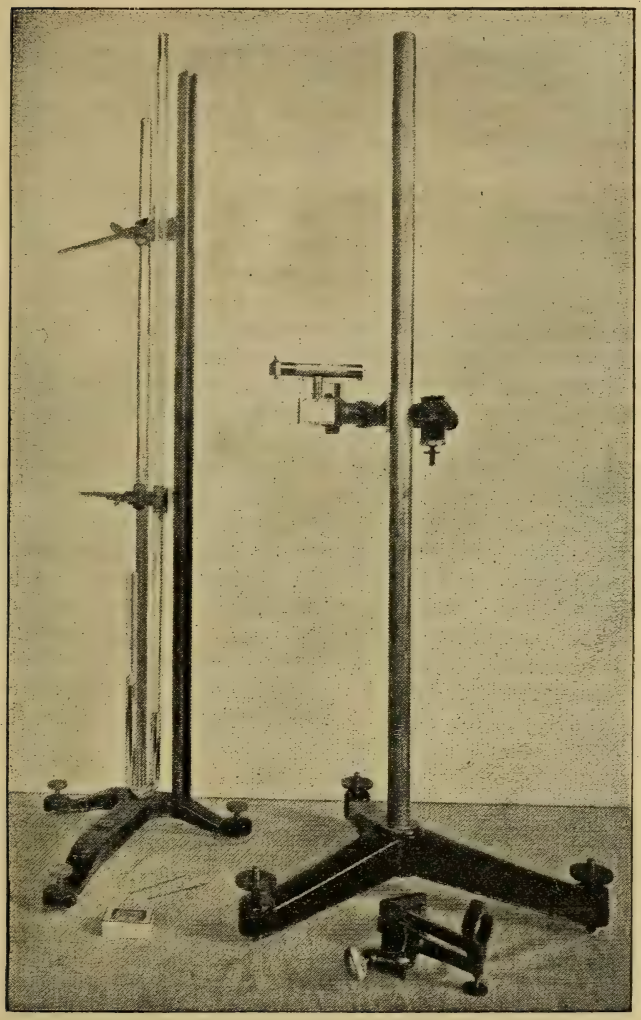
Fig. 2.



accessory to the convenient adjustment of the axis C to verticality. An ordinary open V-clamp M attached to the lower bar G enables the whole arrangement to be clamped to any convenient support; such, for example, as the upright of a heavy retort-stand, or an iron wall-bracket or even a stiff well-secured water- or gas-pipe on the wall of the laboratory. If a retort-stand or other support with adjusting screws in its base is used, the adjustable support G, K, &c., on the telescope

itself may be dispensed with, and the clamp M attached directly to the telescope-tube as in fig. 3, which shows the cathetometer in use.

Fig. 3.



In the use of the instrument, the mirror is first adjusted until it is parallel to the axis of rotation C and perpendicular

to the optical axis of the telescope T. These two adjustments are made simultaneously in the same manner as described in a previous article*; *i. e.* by bringing the reflected image of the cross-wires into coincidence with the wires themselves, revolving the mirror through 180° and correcting one-half of the resulting vertical displacement by means of the adjusting screws *b*, *b*, *b*, against the heads of which the mirror rests, and the other half by means of the screw *e*, which forms part of the geometrical clamp E. The level is then adjusted until it is perpendicular to the axis of rotation by means of the screws *f*, *f*, in the usual manner.

Lastly, the telescope is set at the height of the object to be measured and clamped in position, and the axis C adjusted to verticality by the screws I and H (or the levelling-screws in the base of this support), the level being placed first parallel to the telescope-tube and then at right angles to it†. The first of these adjustments is made once for all; the second is tested at the beginning of each day's work; and the third only is necessary at each setting of the telescope.

It is important to notice that a small error in levelling has the same effect in this new form as in the ordinary form; *i. e.* the error is not doubled by reflexion from the mirror, because the telescope and the latter move together, so far as any movement in a vertical plane is concerned. Let us consider the effect of a small error in levelling, first, in the vertical plane parallel to the axis of the telescope; second, in the vertical plane at right angles to that axis. Let θ be the angle which the line of sight to the object makes with the first plane considered, and α the angle which the axis of rotation *c* makes with the vertical in that plane. Then, if ϵ denote the difference in reading produced by this inclination from the vertical, and *r* the distance of the object from the axis of rotation, we have evidently

$$\epsilon = r \sin \alpha \cos \theta ;$$

for the difference produced by an inclination α in a plane at right angles to this

* "A Simple Method of Determining the Eccentricity of a Graduated Circle with only one vernier," F. L. O. Wadsworth, Amer. Journ. of Sci., May 1894, vol. xlvii. p. 373.

† The mounting shown in figs. 1 and 2 is especially convenient in performing this last operation, as the screws H, I, and the third pivot-point bear respectively in a slot, plane, and conical hole at the three vertices of a right-angled triangle, and the motion of either screw, therefore, affects the position of the axis C only in the vertical plane passing through that screw and the pivot-point.

$$\epsilon' = r \sin \alpha' \sin \theta ;$$

and for the corresponding errors in comparison of object and scale:

$$\Delta = \epsilon - \epsilon_1 = r \sin \alpha (\cos \theta - \cos \theta_1), \quad . \quad . \quad . \quad (1)$$

$$\Delta' = \epsilon' - \epsilon_1' = r \sin \alpha' (\sin \theta - \sin \theta_1). \quad . \quad . \quad . \quad (2)$$

In the new form of instrument the best position for the object and the comparison scale is about 90° from the axis of the telescope, or in the direction *o*, fig. 2. If we suppose the object and scale 15° from one another, and symmetrically placed on the two sides of the 90° position, we have for θ and θ_1 respectively $90° \pm 7\frac{1}{2}°$ or $97\frac{1}{2}°$ and $82\frac{1}{2}°$. Hence

$$\Delta = 0{\cdot}26\, r \sin \alpha \simeq {\cdot}\, \tfrac{1}{4} r \alpha \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$\Delta' = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The general equations (1) and (2) show that care in levelling is only necessary in the vertical plane perpendicular to the line of sight; *i. e.* in the new form the plane parallel to the axis of the telescope; in the usual form *the plane at right angles to that axis.* Hence if the greatest accuracy is to be attained with the ordinary cathetometer, the usual telescope level should be placed at *right angles* to its customary position (or perhaps, better still, a second level added in that position), so as to at once call attention to any error of adjustment in that plane. It is strange that this rather important fact should have been overlooked in previous designs.

The actual magnitude of the error in measurement, due to an error in levelling, is, however, always small, unless either the distance of the object from the telescope is considerable, or the difference between the angles θ and θ_1 is larger than 60°. If $\alpha = 5''$ and $\theta - \theta_1 = 15°$ as in (3), the error, Δ, for objects distant $\frac{1}{2}$M. from the axis of rotation, would be about ·003 millim., or about the limit of accuracy of setting with the best cathetometers under the best conditions.

With a good level sensitive to $5''$ per division (the best cathetometer levels are from two to three times as sensitive as this), there is no difficulty in setting by reversal to within less than $\frac{1}{5}$ div. or $1''$, reducing the error under the above conditions to about $\frac{1}{2000}$ millim.

This shows that we may very considerably increase the angular difference $\theta - \theta_1$, without introducing any appreciable error. For when this difference is 60° the value of Δ is only twice the above values or about ·001 millim. in the last case,

a quantity quite negligible in comparison with the errors of setting. This indicates another method of using this new form of instrument to good advantage; *i. e.*, the method of superposition of object and comparison scale. To accomplish this, the mirror A is half silvered and the scale is viewed directly through the unsilvered half in the direction o' (see fig. 4), the object being at the same time seen by reflexion from the silvered half in the direction o''*. In this case we may make the measurement either by determining the distance between the image of the point and the image of the nearest millim. division on the scale with an ordinary form of micrometer; or better, by bringing these two images into coincidence by means of a Rochon double micrometer, an opthalmometer or a parallel-plate micrometer. The second method of coincidence has the decided advantages both of greater rapidity, only one setting and reading being necessary instead of two, and of greater accuracy for the same reason, since any error, due to a change in position of any part of the apparatus in the interval between two settings, is thus avoided. This last advantage fully balances the disadvantage of the greater effect of a given error of levelling on account of the greater angular distance between scale and object. One additional cause of error is introduced, *i. e.*, that due to a want of parallelism between the mirror A and the axis of rotation C; but since this should not exceed a fraction of a second, if the first adjustment has been properly made, its effect is negligible.

Of the various instrumental means for obtaining coincidence perhaps the simplest and most convenient, as well as one of the most accurate, is the parallel-plate micrometer first invented by Clausen, and quite recently reinvented and much improved by Poynting †, who was the first to adopt it for cathetometric measurements. Figs. 4 and 5 show in plan and elevation a form of this micrometer modified slightly from that described by Poynting, to better adapt it to this particular instrument. It consists simply of a plate of plane-parallel glass, P, rotating on an axis C′ at right angles to the axis C,

* The mirror should be half silvered horizontally, *i. e.* the line of separation of the silvered from the unsilvered portion should be parallel to the axis of rotation, both because the maximum resolution is required in a vertical direction and because, as will be seen later, this management is the better adapted to the use of certain forms of micrometer.

† Phil. Trans. vol. clxxxii. 1891 A, p. 588. See also "On a Parallel Plate Double Image Micrometer," Monthly Not. of the Royal Astron. Soc., vol. liii. No. 8, 1892, p. 556.

and carried in a fork which is a prolongation of the boss D shown in the preceding figures.

The rotation of the plate on its axis shifts the ray from o', which passes through it, and hence also the image of the scale,

Fig. 4.

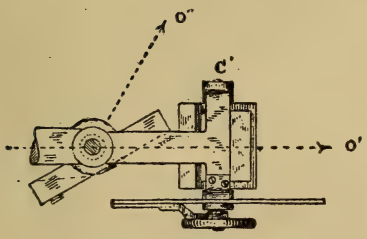
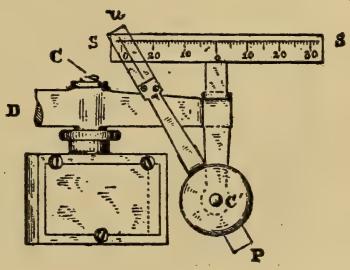


Fig. 5.



by an amount Δ which may easily be shown to be

$$\Delta = t \sin\phi \,.\left[1 - \frac{\cos\phi}{\sqrt{n^2 - \sin^2\phi}}\right].$$

ϕ being the angle of rotation measured from the position in which the plate is normal to the ray. This expression may be written:

$$\Delta = \frac{n-1}{n} t \operatorname{tang}\phi \,.\left\{\left(1 - \frac{\cos\phi}{\sqrt{n^2 - \sin^2\phi}}\right)\frac{n\cos\phi}{n-1}\right\}. \quad (1)$$

$$= \frac{n-1}{n} t \operatorname{tang}\phi \,.f(\phi).$$

The quantity in brackets or $f(\phi)$ may be shown to be very nearly unity for all values of ϕ between 0° and 30°. In order to determine its exact value, we may develop it into a series as Poynting does, but since this series is only rapidly convergent for low values of ϕ, it is on the whole better to compute it directly from (1), which is in a form well adapted to logarithmic computation. I have calculated the values of $f(\phi)$ for values of ϕ from 5° to 30° and for two values of n, viz. $n = 1{\cdot}5$ and $n = 1{\cdot}55$, about the mean indices of the glass

most likely to be used for this purpose. These values are given in the following tables :—

TABLE I.

$$n=1{\cdot}5, \quad \frac{n-1}{n}=\tfrac{1}{3}=0{\cdot}3333.$$

ϕ.	$\frac{n-1}{n}$ tang ϕ.	$f(\phi)=$.	$1+\delta$*.	$rs=$tang α tang ϕ.	$rs-\delta$.
5°	·02916	1·00044=	1+ 00044	+·0011	+·00066
10	·05878	1·00162=	+·00162	+·0021	+·00049
15	·08932	1·00333	+·00333	+·0033	·00000
20	·12132	1·00528	+·00528	+·0045	−·00078
25	·15544	1·00681	+·00681	+·0057	−·00111
30	·19245	1·00708	+·00708	+·00708	+·0000

TABLE II.

$$n=1{\cdot}55, \quad \frac{n-1}{n}=\cdot35484.$$

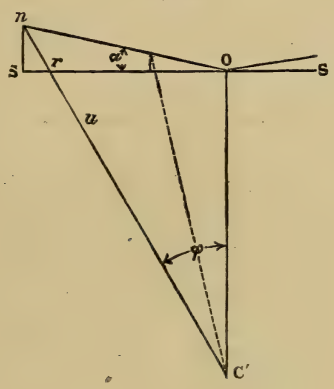
ϕ.	$\frac{n-1}{n}$ tang ϕ.	$f(\phi)$.	$1+\delta$.
5°	·03104	1 00024	1+·00024
10	·06257	1·00085	+·00085
15	·09508	1·00162	+·00162
20	·12915	1·00216	+·00216
25	·16547	1·00188	+·00188
30	·20487	1·00002	+·00002

An inspection of these two tables shows that in the case of

* [In a similar table given by Poynting, the sign of δ is erroneously written negative (probably a typographical error), and there is also a small error in the value of δ for $\phi=10°$, which is, however, unimportant since δ itself is so very small for this angle.]

glass of the higher refractive index ($n=1{\cdot}55$), the maximum value of δ is only 0·2 per cent. of $f(\phi)$, and since the maximum shifting of the image need never exceed 1 millim. (if comparison is made on a millim. scale), the corresponding correction to the tangent value is only $\frac{1}{500}$ millim. and may be disregarded. In this case we may read off the value of Δ directly on a straight scale $s\,s$ by means of a pointer, u, which consists of a thin plate of glass or mica on which is ruled a fine radial line. The distance ro from o, to the point of intersection of this line with the longitudinal line $s\,s$ on the scale, equals $c'o$ tang ϕ, and hence is directly proportional to Δ. If we make $c'o$ equal to $17\frac{3}{4}\,t$ in millim., then each millim. on the scale $s\,s$ corresponds to a shifting of the image through $\frac{1}{50}$ millim. For an angle of 30° $\Delta \simeq \frac{1}{5}\,t$, hence for a shifting of 1 millim. corresponding to this angle, the plate P must be 5 millim. thick, and the distance $c'o$ therefore about 89 millim. as laid off in fig. 6. The scale $s\,s$ of fig. 5 is graduated in

Fig. 6.



2 millim. intervals so that such intervals correspond to $\frac{1}{25}$ millim., but it is easy to set and read the position of the pointer to $\frac{1}{10}$ div. or $\frac{1}{250}$ millim.

In the case of glass of refractive index 1·5 the proportionality between the scale-readings obtained in this manner and the value of Δ is not so exact, the error amounting in case of an angle of 30° to nearly $\frac{3}{4}$ of 1 per cent. or to nearly 0·01 millim. This is a quantity too large to be neglected. Poynting suggests a very ingenious system of link-work, whereby the readings on the scale may be made directly proportional

to Δ for all values of ϕ; but this, as he himself recognizes, is hardly practicable on account of mechanical difficulties. We may, however, obtain the desired result very much more simply. An inspection of the values of δ in Table I. will show that they are roughly proportional to tang ϕ. If we draw a line *no* (fig. 6), inclined at a small angle α, in each direction from *o*, to the longitudinal line on the scale *s s*, and read in each case to the intersection of this line with the radial line *u* on the pointer,—we increase the scale-reading by an amount $sr = os$ tang α tang ϕ, and the new scale-reading is therefore

$$os \simeq c'o \text{ tang } \phi \,.\, [1 + \text{tang } \alpha \text{ tang } \phi].$$

Hence we have only to make tang α tang $\phi = \delta$, in order to make *os* the new scale-reading directly proportional to Δ as before. To find the inclination α of the line *n*, *o*, to the axis of graduation we have only to put

$$\text{tang } \alpha \text{ tang } \phi = \delta$$

for some particular value of ϕ. Suppose we do this for $\phi = 30°$. Then we find

$$\text{tang } \alpha = \frac{\cdot 0071}{\cdot 577} \text{ or } \alpha = 42'.$$

Using this value of α to calculate the values of the scale correction, *rs*, at other points, we find the valnes given in the 5th column of Table I. As will be seen, they differ on the average from the corresponding values of δ by less than $\frac{1}{20}$ per cent.; or only about 0·0008 millim. at the maximum for $\phi = 25°$. By this simple method, therefore, the necessity for making any correction to the scale-reading, even in the most accurate work, is entirely avoided.

The exact constant of the scale-reading for any particular value of the index, differing from those given above, may be either calculated from the above formula or determined experimentally.

In order to always make the value of a 1 millim. scale division correspond to some convenient fractional part of a millim., the support for the scale is made adjustable in height so that the value of $c'o$ may be always made equal to $\frac{n-1}{n} \frac{t}{a}$, where *a* is the fractional value desired. One advantage which

the rotating plate has over the ordinary eye-piece micrometer, is that its constant remains the same for all distances of the scale from the telescope*.

The above form of parallel-plate micrometer may also be advantageously substituted for the eye-piece micrometer in the first instrument described. In this case it should be placed *between* the objective and the reflecting mirror A, or else mounted on the mirror-frame itself so as to rotate with it. The last position enables the same instrument to be used either in the method of comparison or in the method of coincidence, but is objectionable both on account of the increased weight of the moving parts and because of the liability of disturbing the adjustment of the axis or mirror while manipulating the micrometer. It is therefore better to place the rotating plate in the first position indicated and adapt it to either method of use if desired, by making it cover only one-half the field of the telescope, that half of course which is opposite the unsilvered half of the mirror when the coincidence method is used, and opposite the silvered half when the comparison method is employed.

In closing I wish to express my thanks to Messrs. Francis and Kathan, the mechanicians of the laboratory, for the care exercised in the mechanical execution of these designs.

University of Chicago, September, 1895.

XVII. *On the Straining of the Earth resulting from Secular Cooling.* *By* Charles Davison, *M.A., F.G.S., Mathematical Master at King Edward's High School, Birmingham* †.

ESTIMATES of the depth of the surface of no strain have hitherto been founded on the assumptions that the conductivity and the coefficient of dilatation are constant ‡. In the present paper, I propose to calculate the depth on the

* The great practical advantage of this form of micrometer over the ordinary form is its much greater simplicity and cheapness. For these reasons it would have been adopted in all of the above instruments, had it not happened that we already had on hand a number of micrometer eye-pieces which were available for this purpose.

† Communicated by the Author. A paper with the same title was read before the Royal Society on Feb. 15, 1894. The present paper contains the substance of the former, but has been rewritten.

‡ Phil. Trans. 1887 A, pp. 231–249; Phil. Mag. vol. xxv. 1888, pp. 7–20.

supposition that the coefficient of dilatation increases with the temperature, being $\epsilon+\epsilon' v$, where v is the temperature. In assuming this law to hold true up to a temperature as high as 7000° F., it is evident that the numerical results here obtained cannot be regarded as reliable. They are given for their qualitative rather than for their quantitative value.

Let r be the internal radius of a thin spherical shell concentric with the earth*, its thickness δr, density ρ, and coefficient of linear dilatation e. In any given time, let the temperature of this shell be increased by θ; then, if the shell were isolated, r would be increased by $re\theta$. But, in consequence of the expansion of the mass inside it, let the internal radius be further increased by rk, so that r becomes $r(1+\alpha)$, where $\alpha=k+e\theta$. Also, ρ becomes $\rho(1-3e\theta)$ and δr becomes

$$\delta r\left[1+\frac{d}{dr}(r\alpha)\right].$$

Since the mass of the shell remains unchanged, we have therefore

$$2\alpha+\frac{d}{dr}(r\alpha)-3e\theta=0,$$

which may be written

$$\frac{d}{dr}(r^3\alpha)=3e\theta r^2,$$

or

$$\frac{d}{dr}(kr^3)=-r^3\frac{d}{dr}(e\theta).$$

The amount (kr) by which the radius is increased by stretching in the given time is therefore

$$-\frac{1}{r^2}\int_0^r r^3\frac{d}{dr}(e\theta)dr. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let c be the radius of the earth and x the depth of the surface of radius r, v the excess of temperature at this depth above that at the surface of the earth, and θ the increase of temperature in unit time, so that $\theta=dv/dt$; also let ϵ be the

* The method of proof adopted is similar to that given by Prof. G. H. Darwin in his paper in the Phil. Trans. 1887 A, pp. 242–249.

coefficient of dilatation at the surface, and $\epsilon+\epsilon' v$ that at depth x. Then, using Lord Kelvin's well-known solution, expression (1) becomes

$$\frac{1}{(c-x)^2}\int_x^\infty (c-x)^3 \frac{d}{dx}\left[\left\{\epsilon+\epsilon'.\frac{2V}{\sqrt{(\pi)}}\int_0^{x/a} e^{-z^2}\,dz\right\}\frac{V}{2\sqrt{(\pi\kappa)}}\,\frac{x}{t^{\frac{3}{2}}}\,e^{-x^2/a}\right]dx, \quad (2)$$

where $a=2\sqrt{(\kappa t)}$, and infinity is substituted for c in the upper limit, Lord Kelvin's solution being adapted to the case of a sphere of infinite radius.

The shell at depth x will be stretched, unstrained or crushed, according as expression (2) is positive, zero or negative.

Depth of the Surface of No Strain.—Putting $x=ay$ and $\epsilon\sqrt{(\pi)}/2V\epsilon'=\beta$, we obtain, after division by irrelevant factors, the following equation for determining the depth of the surface of no strain,

$$\int_0^\infty (c-ay)^3\left[ye^{-2y^2}+(\beta+\int_0^y e^{-z^2}dz)(1-2y^2)e^{-y^2}\right]dy$$

$$=\int_0^y (c-ay)^3\left[ye^{-2y^2}+(\beta+\int_0^y e^{-z^2}dz)(1-2y^2)e^{-y^2}\right]dy. \quad (3)$$

Since y is a proper fraction, this equation, omitting unimportant terms, becomes

$$c^2a\left(\frac{3\sqrt{2\pi}}{8}+\frac{3\beta}{2}\right)-ca^2\left(\frac{6+3\pi}{8}+\frac{3\beta\sqrt{\pi}}{2}\right)+a^3\left(\frac{15\sqrt{2\pi}}{32}+\frac{3\beta}{2}\right)$$

$$=y\beta c^3+y^2(c^3-\tfrac{3}{2}\beta c^2a)+y^3(-2c^2a+\beta ca^2-\beta c^3)$$

$$+y^4(\tfrac{3}{2}ca^2-\tfrac{4}{3}c^3+\tfrac{9}{4}\beta c^2a)+y^5(\tfrac{16}{5}c^2a+\tfrac{1}{2}\beta c^3)+\ldots\ldots \quad (4)$$

As a first approximation, which is sufficient for most purposes, we have the equation

$$8cy(y+\beta)=3a(\sqrt{2\pi}+4\beta). \quad (5)$$

Putting β equal to infinity in the latter, we get

$$y=3a/2c,$$

or

$$x=6\kappa t/c, \quad (6)$$

which gives the depth of the surface of no strain on the

supposition that the coefficient of dilatation is constant, *i. e.*, an inferior limit to the depth if the coefficient of dilatation increases with the temperature. This agrees of course with the first approximation obtained by Prof. G. H. Darwin and the Rev. O. Fisher*.

Putting β equal to zero, *i. e.*, ϵ'/ϵ to infinity, we get

$$y^2 = \frac{3\sqrt{2\pi}}{8}\frac{a}{c},$$

or

$$x = \left(\frac{18\pi\kappa^3}{c^2}\right)^{\frac{1}{4}} t^{\frac{3}{4}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which gives a superior limit to the depth of the surface of no strain so long as the coefficient of dilatation increases with the temperature.

In Fizeau's table of coefficients of dilatation †, eighteen determinations for non-crystallized bodies are given as especially trustworthy. Making use of these only, and taking V at 7000° F., the average value of β is ·1. With this value, the depth of the surface of no strain after 100 million years is 7·79 miles. At the same time, with a constant coefficient of dilatation, it would be 2·17 miles.

It is evident from equation (4) or (5) that there is no simple relation between the depth of the surface of no strain and the time. If ϵ'/ϵ be small, the depth varies nearly as t: if ϵ'/ϵ be large, the depth varies nearly as $t^{\frac{3}{4}}$.

Volume of Folded Rock above the Surface of No Strain.—Since k and e are both small fractions, the volume of that part of the shell of radius r and thickness δr which is stretched or folded in unit time is

$$4\pi\delta r \,.\, r^2 - 4\pi\delta r(r^2 + 2r \,.\, kr) = 8\pi(c - x)\delta x \,.\, kr,$$

where kr is given by the expression (1) or (2). Denoting this by δU, we have

$$\frac{d\mathrm{U}}{dy} = \frac{16\epsilon\mathrm{V}\sqrt{(\pi\kappa)}}{\beta(c - ay)\sqrt{(t)}} f(y),$$

* Phil. Trans. 1887 A, p. 246; Phil. Mag. vol. xxv. 1888, p. 14.

† Jamin's *Cours de Physique*, vol. ii. 1878, pp. 80–81.

where

$$f(y)=\int_y (c-ay)^3\left[ye^{-2y^2}+(\beta+\int_0^y e^{-z^2}dz)(1-y^2)e^{-y^2}\right]dy,$$

$$\therefore \quad \mathrm{U}=\frac{16\epsilon\mathrm{V}\sqrt{(\pi\kappa)}}{\beta\sqrt{(t)}}\int_0^s \frac{f(y)}{c-ay}\,dy, \quad . \quad . \quad . \quad . \quad (8)$$

where s is the value of y at the depth of the surface of no strain.

Hence, if $\delta\mathrm{W}$ be volume of rock crushed in time δt above the surface of no strain, $\mathrm{U}=d\mathrm{W}/dt$, and therefore

$$\mathrm{W}=\frac{16\epsilon\mathrm{V}\sqrt{(\pi)}}{\beta\sqrt{(\kappa)}}\int_0^a\int_0^s \frac{f(y)}{c-ay}\,dy\,da. \quad . \quad . \quad . \quad (9)$$

This gives the total volume of the rock-folding since consolidation, while equation (8) gives the rate at which rock-folding takes place at time t.

For a first approximation, we may put

$$\frac{1}{c-ay}=\frac{1}{c}$$

and

$$f(y)=\mathrm{A}c^2a-y\beta c^3-y^2c^3,$$

where

$$\mathrm{A}=3(\sqrt{2\pi}+4\beta)/8.$$

We thus get

$$\int_0^s \frac{f(y)}{c-ay}\,dy=\mathrm{A}cas-\tfrac{1}{2}\beta c^2s^2-\tfrac{1}{3}c^2s^3,$$

where

$$s=\tfrac{1}{2}[-\beta+\sqrt{(\beta^2+4\mathrm{A}ac)}],$$

and consequently, after reduction.

$$\mathrm{W}=16\epsilon\mathrm{V}\sqrt{(\pi)}\left[\frac{1}{18}\beta^2c^2a\left(1+\frac{4a\mathrm{A}}{\beta^2c}\right)^{\frac{3}{2}}+\frac{1}{72}\frac{\beta^4c^3}{\mathrm{A}}\left(1+\frac{4a\mathrm{A}}{\beta^2c}\right)^{\frac{3}{2}}\right.$$

$$\left.+\frac{1}{180}\frac{\beta^4c^3}{\mathrm{A}}-\frac{1}{180}\frac{\beta^4c^3}{\mathrm{A}}\left(1+\frac{4a\mathrm{A}}{\beta^2c}\right)^{\frac{5}{2}}-\frac{1}{4}\mathrm{A}ca^2-\frac{1}{12}\beta^2c^2a\right]. \quad (10)$$

If h be the average thickness of this volume of crushed rock spread over the whole globe, then

$$h=\mathrm{W}\ 4\pi c^2.$$

If we make ϵ'/ϵ equal to zero, or β equal to infinity, in these expressions, we get

$$W = 6\sqrt{(\pi)}\epsilon V a^3, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

$$h = 3\epsilon V a^3/2\sqrt{(\pi)}c^2. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This value of h agrees with that obtained by the Rev. O. Fisher on the supposition that the coefficient of dilatation is constant*.

The mean value of the coefficient of dilatation deduced from Adie's experiments † on six rocks at temperatures varying from about 50° to 208° F., is ·0000057. Taking this as the surface-value ϵ, and as before V equal to 7000° F., and β to ·1, we find that after a lapse of 100 million years the total volume of solid rock is about 6,145,000 cubic miles, and the mean thickness of the layer formed by spreading this over the whole earth is ·03120 mile or 164·7 feet. If the coefficient of dilatation were constant and equal to the value above-mentioned, the corresponding figures would be 184,500 cubic miles and 4·95 feet.

Alleged Insufficiency of the Contraction Theory.—If the coefficient of dilatation increases with the temperature, instead of being constant, it thus appears that the result is a considerable increase in the depth of the surface of no strain, and in the total amount of rock-folding due to the cooling of the earth. If the conductivity increases with the temperature, there will be a further increase of both quantities. It is possible, moreover, that besides their variation with the temperature, both the conductivity and the coefficient of dilatation may be greater in the material which composes the earth's interior than they are in the surface rocks. Mr. Rudski has also pointed out ‡ that the depth of the surface of no strain will be much greater if initially the temperature, instead of being uniform, increased with the depth, and there is some reason for supposing that this may have been the case. If, then, we regard the contraction theory as a theory of the formation of mountain-ranges only, and not necessarily of the continental masses, we may, I think, conclude that calculations as to its alleged insufficiency are at present inadmissible.

* Phil. Mag. vol. xxv. 1888, p. 17.
† Edinb. Roy. Soc. Trans. vol. xiii. 1836, pp. 354–372.
‡ Phil. Mag. vol. xxxiv. 1892, pp. 299–301.

XVIII. *Notices respecting New Books.*

Théorie de l'Électricité. By A. Vaschy. Paris: Baudry et Cie., 1896.

IN this treatise M. Vaschy presents a purely mechanical theory of electricity founded on the fact, experimentally demonstrated, that an electric field with a measurable intensity exists at every point of space. The connexion between electric intensity and the medium in which it is manifested is not considered; on this point the author remarks:—"The idea of intensity of the electric field is not more intimately connected with the three fundamental ideas of mechanics than these latter are connected among themselves. We shall therefore consider it as fundamental; but all others with which we meet in electricity will be derived from it, and from length, time and mass."

It is shown that the condition of stability of the electric field is the existence of a potential function; this leads to the discussion of lines of force, equipotential surfaces and conductors. M. Vaschy distinguishes two classes of action at a distance, of which the action between two electric charges and that of an element of a linear current on a magnet pole are the types; in the former case the force is in the line joining the particles, whereas in the latter case one of the attracting "masses" is a vector quantity, and its direction determines that of the force. The two classes may be called Newtonian and Laplacian respectively; the author shows that any field of force whatever can be completely represented by a suitable distribution of Newtonian and Laplacian masses, with an inverse square law of attraction: a stable field is due only to Newtonian masses, while a solenoidal distribution is produced by Laplacian masses alone. The energy which really exists in every unit of volume of the field is referred in action-at-a-distance theories to these fictitious masses, thus giving rise to the ideas of electric charges and currents.

M. Vaschy's explanation of a current in a conductor is analogous to that of Poynting; in every element of volume of the conductor, the energy of the electric field is constantly being transformed into heat, and new energy must be supplied along definite paths from the battery or other source of electric energy. Assuming that the rate of supply of energy to the element of volume depends only on the instantaneous state of the field at its surface, and not on the mode of transformation of the energy within it, the field of a current can be divided into an electrostatic and a magnetic portion. The discussion of variable currents follows, and the detailed treatment of electric waves is a prominent feature in the book.

The aim of the author has been to produce a work bearing the same relation to electrical theory as a treatise on thermodynamics bears to the theory of heat. Temperature in the one case, electric force in the other, are regarded as fundamental quantities; the rest consists entirely of experiment and mathematical deduction. A closer study of the behaviour of heated gases gives us a clue to the physical meaning of temperature; may we not hope to learn from the electrical properties of matter something more definite concerning electric force? JAMES L. HOWARD.

Molecules and the Molecular Theory of Matter. By A. D. RISTEEN, *S.B.* Boston, U.S.A.: Ginn & Co., 1895.

THIS volume is the outcome of a lecture on the same subject, which the author has greatly amplified, retaining, however, the lecture style, which involves the frequent use of the first person singular. The introduction of this personal element enables Mr. Risteen occasionally to put forward his own views without leading students to suppose them orthodox—if such a term is permissible in connexion with the theory of the constitution of matter. He thus imparts to his subject a greater interest and a more vigorous treatment than it has previously received, except in Maxwell's discourse on 'Molecules,' and Lord Kelvin's lecture on the 'Size of Atoms.'

The first part of the work is devoted to the kinetic theory of a perfect gas; the gaseous laws are explained, and Maxwell's law of distribution of molecular velocities is enunciated and discussed. This is followed by the statement of Boltzmann's theorem and the deduction of the relation between the number of degrees of freedom of the molecules of a gas and the ratio of its specific heats, after which diffusion and viscosity are treated according to the principles of the kinetic theory, and molecular free path is illustrated by the radiometer and Crookes's tubes. The second section of the volume deals with the molecular theory of liquids and solids, and is necessarily more superficial and sketchy than the previous one; it concludes with a good account of the various methods for the determination of molecular magnitudes. The rest of the volume contains short descriptions of the various theories concerning the nature of atoms and molecules.

The author has endeavoured to give an outline of the whole subject rather than a detailed mathematical treatment of a portion of it, so that formulæ are occasionally quoted without proof. In these cases the student might with advantage be referred to a treatise or memoir where the proof is to be found.

JAMES L. HOWARD.

A Laboratory Course in Experimental Physics. By W. J. Loudon, *B.A., and* J. C. McLennan, *B.A.* New York: Macmillan & Co., 1895.

Although text-books of practical physics are now fairly numerous, the demonstrators in physics of the University of Toronto appear to have found it impossible to adapt any of the existing ones to the requirements of their students. The same difficulty has been experienced in most English laboratories, partly on account of the diversity which exists between the apparatus of the various institutions, and partly because of the different aims of students working in them. Consequently nearly every laboratory possesses instruction-sheets, either in print or manuscript, which have in some instances been edited and issued in book form; the present volume is a case in point.

The experiments described in the book are arranged in two courses, elementary and advanced, in the former of which weighing and measuring, light and heat, are included, while the advanced course consists of sound, advanced heat, electricity, and magnetism. Some easy experiments in electricity and magnetism might have been included in the elementary course, otherwise the selection is a fairly good and useful one. The section on sound contains figures and descriptions of Helmholtz's and König's apparatus for various experiments, but many students' laboratories in this country are, unfortunately, not equipped with large electrically-driven tuning-forks and chronographs. Experimental details are sometimes omitted; for example, in describing the use of the spectrometer as a goniometer, the student is not told how to focus the telescope and adjust the slit so as to obtain a parallel beam of light from the collimator. In the same experiment it is erroneously stated that the edge of the crystal must coincide with the axis of the instrument, whereas the two need only be parallel to each other if the collimator is properly adjusted. Again, in calorimetric measurements, errors arising from radiation between the calorimeter and its surroundings are not very adequately discussed; the calorimeter is supposed in all experiments to alter its temperature at a uniform rate, and matters are so arranged that the mean between its initial and final temperatures is that of the atmosphere. In specific heat determinations a preliminary experiment is necessary in order to satisfy the latter condition, and the assumption of uniform rise of temperature is only approximately true, even when the body under experiment is a good conductor of heat.

The book is printed in a bold, clear type, and its illustrations are good; it will be found a useful addition to the laboratory bookshelf.

James L. Howard.

XIX. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 78.]

December 4th, 1895.—Dr. Henry Woodward, F.R.S., President, in the Chair.

THE following communications were read:—

1. 'On the Alteration of certain Basic Eruptive Rocks from Brent Tor, Devon.' By Frank Rutley, Esq., F.G.S.

Two microscopic sections of rock occurring on the north side of Brent Tor were examined, and a cursory glance suggested at once the idea that they might originally have consisted to a greater or less extent of extremely vesicular basalt-glass. No unaltered vitreous matter, except perhaps mere traces, can now be detected in these specimens, the interest of which lies in the assemblage of alteration-products which they contain. A third section cut from a small chip collected at the southern side of the base of the Tor consists of a highly vesicular lava of a hyalopilitic character, which may be regarded as an amygdaloidal glassy basalt.

The author gives a detailed account of the microscopic characters of the three sections, and discusses the history of the rocks, comparing them with Tertiary basic glass, and with the Devonian rocks of Cant Hill, which he described previously. He brings forward evidence in favour of the view that the original alteration of both the Brent Tor and Cant Hill rocks was palagonitic, and that while in the Brent Tor rocks the subsequent alteration of the palagonite into felsitic matter, magnetite, secondary felspar, epidote, and probably kaolin, and some serpentine and chlorite was complete, it was only partial in the case of the Cant Hill rocks. We may therefore assume that palagonite is not the ultimate phase of alteration in basic igneous rocks.

2. 'The Mollusca of the Chalk Rock.—Part I.' By Henry Woods, Esq., M.A., F.G.S.

December 18th.—Dr. Henry Woodward, F.R.S., President, in the Chair.

The following communications were read:—

1. 'The Tertiary Basalt-plateaux of North-western Europe.' By Sir Archibald Geikie, D.Sc., LL.D., F.R.S.

The author in this paper gives the results obtained by him in the continued study of Tertiary Volcanic Geology during the seven years which have elapsed since the publication of his memoir on 'The History of Volcanic Action during the Tertiary Period in the

British Isles.' His researches have embraced the Western Islands of Scotland, St. Kilda, and the Faröe Islands.

1. In an account of the rocks of the basalt-plateaux attention s particularly directed in this paper to a type of banded basic lavas which play an important part in the structure of the volcanic districts both of the Inner Hebrides and of the Faröes. The banding sometimes consists in layers of more highly vesicular structure, sometimes in alternations of distinctly different lithological character, such as close-grained basalt and more coarsely-crystalline dolerite. These banded rocks are more particularly developed in the lower portions of the volcanic series. At a distance they might be mistaken for tuffs or other stratified deposits. They occupy a conspicuous place in the great precipices of the west and north of the Faröe Islands.

Numerous examples are cited of the ending-off of basalt-sheets in different directions, indicative of many local vents from which the lavas issued. An account is also given of tuffs and other stratified intercalations which occupy a subordinate place in the structure of the plateaux.

2. A number of examples are adduced of the volcanic vents which form a characteristic feature of the basalt-plateaux. A remarkable row of five such vents was met with by the author at the base of the great cliffs on the west side of Stromö, in the Faröe Islands. They are occupied with agglomerate, and their saucer-shaped craters have been filled in by successive streams of lava from neighbouring vents, the whole being buried under the great pile of basalt-sheets forming the island of Stromö.

An instance of similar structure is described from Portree Bay, the agglomerate in this case being connected with a thick and widespread sheet of tuff intercalated among the basalts. Another example is cited from the eastern end of the island of Canna, where the ejected volcanic blocks are associated with records of contemporaneous river-action.

3. The paper describes in some detail the evidence for the flow of a large river across the lava-fields during the time when volcanic activity was still vigorous. Thick sheets of well-rolled gravel are intercalated among the basalts of the islands of Canna and Sanday. These masses of detritus consist mainly of volcanic material, but they include also abundant pieces of Torridon Sandstone and rocks from the Western Highlands. The current of water which transported them certainly came from the east. That it flowed while the volcanic vents continued in eruption is shown by the bands of tuff and the large blocks of slag contained in the conglomerates, as well as by sheets of vesicular basalt interstratified in the same deposits. From the terrestrial vegetation whereof the macerated remains are enclosed in the tuffs and shales, and from the entire absence of marine organisms, it may be confidently inferred that the water was that of a river. The large size of many of the rounded blocks that

were swept along proves that this river must have been of considerable volume and rapidity. The stream probably drained some part of the Inverness-shire Highlands, and wandered over the volcanic plain, following the inequalities of the lava-fields, sweeping away the loose detritus of volcanic cones, and continually liable to have its course altered by fresh volcanic eruptions. An interesting section is cited from the island of Sanday, where what appears to be a portion of the ravine of one of the tributaries of this river. trenched in the basalts, filled with coarse shingle and buried under later basalts, remains in a picturesque sea-stack.

An account is given of the little islet of Hysgeir, about 18 miles to the west of the island of Eigg, which has been identified by Dr. Heddle with the rock of the Scuir of Eigg, and which the author has visited in two successive years. The 'pitchstone' is precisely like that of the Scuir down to the most minute structure, as is shown by an examination of the rock under the microscope by Mr. A. Harker. There can be little doubt that this rock was a superficial lava like that of Eigg, and there seems every probability that it is really a westward continuation of the Scuir. The Hysgeir pitchstone everywhere dips under the sea, so that its bottom cannot be seen.

The author considers that the Canna river may have been the same as that which at a later part of the volcanic period had its channel filled up by the pitchstone of the Scuir of Eigg.

4. Many additional details are given to illustrate the structure and behaviour of the basic sills which are so abundantly developed, especially at the base of the plateaux. Some of these sheets are of colossal proportions, as in the Shiant Isles, where a single columnar bed is 500 feet thick. Others descend to extremely minute proportions, such as the slender layers and threads which have been injected into the coal and shale between the lower basalts in the Portree district. A remarkable instance of a sill traversing the centre of another is cited from the south-east of Skye, the younger sheet having a strongly developed skin of black glass on its upper and under surfaces. One of the most striking instances of a sill rising obliquely across a thick mass of the plateau-basalts is described from Stromö in the Faröe Islands.

5. The author adds some additional particulars, more especially from Skye and St. Kilda, to his published account of the dykes which have taken so important a place in the origin and structure of the plateaux.

6. Further observations are narrated regarding the great bosses of gabbro in the Inner Hebrides. In particular, the peculiar banded structure, already described from a part of the Cuillin Hills, is shown to have a wide distribution in Skye, and to occur also in Rum. The remarkable alteration of the plateau-basalts as they approach the gabbros of Loch Scavaig is referred to, and the microscopic structure of these metamorphosed rocks is described in notes supplied to the author by Mr. Harker. An account is also given of the

gabbros of St. Kilda, which display a considerable variety of texture and composition and include basalts and dolerites.

7. The author, having been able to visit St. Kilda, describes the junction of the granophyre of that remote island with the basalts and gabbros. He brought away a series of specimens and photographs which demonstrate that the acid rock has been injected into the basic masses, traversing them in veins and enclosing angular pieces of them. The granophyre is precisely like that of Skye and Mull, and is traversed by veins of finer material, as in these islands. Where it has penetrated the basic rocks it sometimes forms a kind of breccia-matrix in which the pieces of dark material are enclosed. It has taken up a certain quantity of the basalt or gabbro, as is shown by the abundant brown mica and hornblende which have been developed in the acid rock, especially round the enclosed basalt-fragments. The results of a microscopic examination of thin slices of the rock at the junction are furnished by Mr. Harker.

From Skye examples are given of triple dykes and sills, wherein a central band consists of granophyre or spherulitic felsite, while the two marginal bands are of basalt, diabase, or other basic material. There does not appear to be any ascertainable connexion between the acid and basic parts of such compound intrusions. In some cases the basic, in others the acid portion is the older.

8. By way of illustrating the probable history of the basaltic plateaux of North-western Europe, the author gives a short summary of the results of recent investigations of the modern volcanic eruptions of Iceland, especially by Th. Thoroddsen and A. Helland. He shows in how many ways the phenomena of that island explain the facts which are met with in the study of our Tertiary plateaux, and how, in some respects, the enormous denudation of these plateaux throws light on parts of the mechanism of the Icelandic volcanoes which are still buried under the erupted material.

9. Reference is made to the evidence of considerable terrestrial movement since the Tertiary volcanic period, as shown by the tilting of large sections of the plateaux in different directions, and also by the existence of actual faults. Besides the normal faults, which are not infrequent among the Western Isles, there occur among the Faröe Islands instances of reversed faults, which probably indicate disturbance of a more serious character.

10. The concluding section of the paper deals with the effects of denudation on the plateaux. With possibly some minor intervals of partial depression, the present Tertiary volcanic tracts of the British and Faröe Isles have remained as land ever since the volcanic period. Their valleys were probably begun before the close of the eruptions, and these hollows have been continuously widened and deepened ever since. The result is a stupendous memorial of the potency of the agents of geological waste. While the Inner Hebrides abound in most impressive illustrations of this denudation,

they are inferior in that respect to the Faröes. The long level lines of basalt-sheets furnish, as it were, datum-lines from which the extent of erosion can be estimated and even measured. There is certainly no other area in Europe where the study of the combined influence of atmospheric and marine denudation can be so admirably prosecuted, and where the imagination, kindled to enthusiasm by the contemplation of such scenery, can be so constantly and imperiously controlled by the accurate observation of ascertainable fact.

2. 'The British Silurian Species of *Acidaspis*.' By Philip Lake, Esq., M.A., F.G.S.

January 8th, 1896.—Dr. Henry Woodward, F.R.S., President, in the Chair.

The following communications were read:—

1. 'A Delimitation of the Cenomanian, being a Comparison of the Corresponding Beds in Southern England and Western France.' By A. J. Jukes-Browne, Esq., B.A., F.G.S., and William Hill, Esq., F.G.S.

The object of the authors is to compare the beds which form the lower part of the Upper Cretaceous Series in those parts of Southern England and Western France which are nearest to one another. They briefly trace the history of English and French geological research, and remark that even at the present time French geologists are not agreed as to the beds to be included in their 'étage Cénomanien.'

The authors feel justified in taking the English succession as a standard, and endeavour to show that the French succession is in accord with it, believing that the confusion of French geologists has arisen from their having taken a set of arenaceous shallow-water beds as the standard of their Cenomanian stage, in a district where these form the local base of the Cretaceous System, and where the typical Albien fauna does not exist.

Commencing with the English sections, they describe such as serve to establish the succession in the Isle of Wight, Dorset, and Devon, pointing out that the Gault and Upper Greensand are everywhere so inseparably united that it is difficult even to assign limits to the component zones; further, that the Lower Chalk is clearly marked off from this group, and that no classification can be accepted in England which does not recognize the clear and natural line of division at the base of the Chalk.

In Devonshire the representative of the Lower Chalk is found in a set of arenaceous deposits which contain a remarkable fauna, some of the fossils being such as occur in the Upper Greensand, some in the Chalk Marl, while many have not been found elsewhere in England, but occur in the Cenomanian of France and in the Tourtia of Tournay. This Devonshire 'Cenomanian' includes

the beds numbered 10, 11, 12, and 13 by Mr. Meÿer in his Beer Head section, Quart. Journ. Geol. Soc. vol. xxx. (1874) p. 369.

Passing over to France, the fine section in the cliffs between Cap la Hève and St. Jouin is described in detail, and the bed which is regarded as the base of the Cenomanian by M. Lennier and Prof. A. de Lapparent is shown to be the representative of the Chloritic Marl of the Isle of Wight; the Greensand and the Gault below forming, as in England, a separate and independent group of beds.

An account is then given of a traverse made through the departments of Calvados and Orne as far as Mortagne; succeeded by a brief account of the lateral changes which take place as the Cenomanien is traced through the Sarthe, this being derived from the publications of MM. Guillier and Bizet.

A critical study of the fossils found in Devonshire and Normandy follows, with tabulated lists comparing the Devonshire fauna with that of the French Cenomanian, and the fossils of the Norman Cenomanian with those of the Warminster Greensand and of our Lower Chalk. In this part of the work the authors have received much assistance from Mr. C. J. A. Meÿer and Dr. G. J. Hinde.

Finally, they claim to have defined the limits of the Cenomanian stage in Western France, and to have shown that this group of beds is simply a southern extension of our Lower Chalk, formed in a shallower part of the Cretaceous Sea and nearer to a coast-line.

2. 'The Llandovery and Associated Rocks of Conway.' By G. L. Elles and E. M. R. Wood, Newnham College.

The discovery of beds with *Phacops appendiculatus*, Salt., near Deganwy, and of shales with a fauna of Upper Birkhill age close to the town of Conway, indicates that the break between Ordovician and Silurian is smaller in this area than has hitherto been supposed.

In the paper a full description of the representatives of the Birkhill, Gala (Tarannon), and Wenlock beds is given, and the distribution of the fossils (chiefly graptolites) in the various subdivisions is recorded. Many of the graptolites are forms which have been described from Swedish deposits, but have hitherto been unrecorded in this country.

3. 'The Gypsum Deposits of Nottinghamshire and Derbyshire.' By A. T. Metcalfe, Esq., F.G.S.

The gypsum deposits of these counties occur in the Upper Marls of the Keuper division of the Triassic system. The author describes their occurrence in thick nodular irregular beds, large spheroidal masses, and lenticular intercalations, and their association with satin-spar, alabaster, selenite, and anhydrite.

XX. *Intelligence and Miscellaneous Articles.*

CONTRIBUTIONS TO THE KNOWLEDGE OF TROPICAL RAIN. BY PROF. T. WIESENER.

AN incentive to this series of researches carried out in Buitenzorg, Java, in the winter 1893-94, was the question as to the direct mechanical effect of tropical rain on plants, on which subject entirely erroneous views are current.

The author first determined the rainfall per second, and found the highest value to be 0·04 millim. If a rain of such intensity had continued, within a day it would almost have equalled a year's fall at Buitenzorg.

The masses of rain in tropics even in the heaviest falls are very small in comparison with those from the rose of an ordinary watering-pot. The former are to the latter as 1 : 25 up to 100.

From the heaviest rainfalls and the smallest number of drops falling on a surface of 100 sq. cm. in a second, the weight of the greatest possible raindrop was calculated at 0·4 gramme. This number is far too great, for the largest drops which can be made (of 0·25-0·26 gramme) break up at a height of fall of over 5 metres into a larger one weighing 0·2 gramme and into several smaller drops. The weight of the largest raindrop in Buitenzorg, measured by the method of absorption, is still smaller, amounting namely to only 0·16 gr.

Experiments by the author on free fall have shown that drops of water of 0·01–0·26 gramme, at heights of more than 5–10 metres fall with approximately uniform velocity of about 7 metres in a second. The acceleration is therefore very soon after the commencement of the fall almost entirely compensated by the resistance of the air.

The *vis viva* of the heaviest drops of rain calculated by the formula

$$\frac{pv^2}{2g}$$

amounts to 0·004 kgmetre. In heavy falls no doubt several drops fall in rapid succession on a leaf—2–6 large drops per square centimetre per second,—but the impact of each falling drop is diminished by the elastic attachment of the leaf to the stem.

It follows from the experiments, that the force which tropical rain falling in a still atmosphere can exert is far too small to cause injury to vegetation. The mechanical action of the strongest tropical rain on plants is seen in a violent agitation of the leaves and twigs. Injuries only occur in isolated cases in the more tender parts of the plants which cannot give way to the impact, for instance on the tender shoots of tobacco when they lie on a coarse soil consisting of angular particles of earth and sand.

The statements that leaves are torn and separated from the stem by the mere impact of the rain, in a still atmosphere, that upright leafy plants are shattered, and the like, depend on errors.—*Wiener Berichte,* November 21, 1895.

ON THREE DIFFEREN TSPECTRA OF ARGON. BY DR. J. M. EDER AND E. VALENTA, OF VIENNA.

In this paper it is demonstrated that besides the "red" and "blue" argon spectrum discovered by Crookes, there is a third separate spectrum of argon which is characterized by different lines as well as by partial displacement of certain groups of lines toward the red. The authors describe also the spectrum of the glow-light at the + and — poles of the tubes filled with argon.—*Wiener Berichte,* December 19, 1895.

ON THE RED SPECTRUM OF ARGON. BY DR. J. M. EDER AND E. VALENTA.

Through the kindness of Lord Rayleigh we became possessed of some argon gas which was very carefully filled into vacuum-tubes by Hr. Goetze in Leipzig. The pressure in those tubes which we used for our investigations amounted to 1–3 millim. For the purpose of analysis by the spectrum, we used a very powerful concave grating of $\frac{3}{4}$ metre curvature, and the photographic method. We measured the spectrum of the second order and referred the wave-length to the lines on Rowland's standards. We examined the red-and-blue argon spectrum which was obtained in accordance with the statements of Mr. Crookes, with feeble sparks without leyden-jars or with sparks from the jars themselves.

For the red argon spectrum we obtained the following numbers. The lines marked with a star are also present in the blue argon spectrum; the other lines are characteristic of the red spectrum itself. As specially characteristic lines of the red spectrum may be mentioned the principal lines $\lambda=4628{\cdot}56$, $4596{\cdot}22$, $4522{\cdot}49$, $4510{\cdot}85$, $4300{\cdot}18$, $4272{\cdot}27$, $4259{\cdot}42$, $4251{\cdot}25$, especially the groups $4200{\cdot}76$, $4198{\cdot}42$, $4182{\cdot}07$, $4164{\cdot}36$, $4158{\cdot}63$, and, further, $4044{\cdot}56$, $3949{\cdot}13$, $3834{\cdot}83$. This, of course, only applies to the region examined by us, and we shall shortly publish further measurements in the Memoirs of the Vienna Academy. It may be remarked that the red argon spectrum is resolved if the double line $\lambda \begin{cases} 4191{\cdot}15 \\ 4190{\cdot}75 \end{cases}$ appears well separated. If the blue and the red argon spectrum belong to two elements, which is by no means improbable, the above lines would be the characteristic principal ones.

We give the following preliminary list for the region $\lambda=5060$ to $\lambda=3319$, in which i is the intensity of the lines (the feeblest $=1$, the strongest $=10$).

Wave-lengths of the Lines in the Red Spectrum of Argon.

λ.	*i.*	λ.	*i.*	λ.	*i.*
5060·27	1	4321·77	1	*3892·10	1
5054·07	1	4312·27	2	3875·27	1
4888·27	1	4300 18	10	*3868·68	3
4876·52	1	4288·06	1	3866·44	1
*4847·95	3	4284·24	1	*3850·70	5
*4806·10	5	4278·21	1	3834·83	8
4768·80	2	4272·27	10	*3809·58	2
*4764·99	4	*4266·41	10	3781·46	3
4753·02	2	4265·40	2	*3781·07	2
4746·82	1	4259·42	10	3775·62	2
*4736·03	6	4251·25	6	*3770·81	4
*4726·96	5	4247·68	1	*3765·43	2
4702·38	3	*4228·30	4	3760·43	1
*4658·01	4	4212·37	1	3743·89	1
4647·75	1	4210·14	1	*3738·03	1
4628·66	8	4200·76	10	*3729·44	3
*4609·69	4	4198·42	10	*3718·39	1
4602·63	1	4191·15	6	3696·66	2
4596·25	10	4190·76	6	3691·07	4
4590·03	3	4182·07	9	3675·38	2
4589·40	5	4164·36	9	3670·81	3
*4579·49	3	4158·63	10	3659·70	3
*4545·28	4	4150·18	1	3649·95	4
4523·54	1	4147·30	2	3643·27	3
4522·45	8	4141·65	1	3635·60	4
4510·83	10	4134·48	1	3632·83	4
4501·66	1	*4131·95	2	3606·77	1
*4498·62	1	*4104·10	3	*3588·58	2
*4482·03	3	*4082·59	1	*3582·51	2
*4475·15	1	*4079·83	1	*3581·83	1
4460·90	1	*4077·47	1	*3576·79	3
4434·22	1	*4072·15	1	*3571·89	3
*4431·13	2	4055·91	1	*3567·84	4
*4430·35	4	4054·68	4	3564·48	2
*4426·15	6	4046·01	4	*3563·46	2
4424·09	3	4044·56	8	*3561·13	1
*4421·06	1	*4043·02	2	*3559·66	3
*4401·17	5	4033·11	3	3556·16	2
*4400·20	3	*4013·97	4	3554·47	4
*4379·79	4	3979·81	2	*3546·07	2
*4376·15	2	*3968·54	1	*3545·87	2
*4371·46	3	3960·24	1	*3514·67	2
*4370·89	2	3949·13	10	3506·59	2
4363·93	4	3947·70	5	*3191·71	3
*4348·11	8	*3932·71	1	*3476·94	2
*4345·27	10	*3928·82	4	3461·21	2
*4335·42	8	*3925 98	1	3393·90	2
*4333·64	10	*3914·93	1	3392·99	1
*4332·15	2	3900·04	8	3373·64	1
*4331·31	1	3894·76	4	3319·35	1

—*Wiener Berichte*, October 24, 1895.

INTERFERENCE EXPERIMENT WITH ELECTRICAL WAVES. BY PROF. VON LANG.

The experiment described corresponds to the well-known acoustical one of Quincke. The electrical wave from a Righi's exciter is divided into two parts with unequally long paths ; after joining, the partial waves interfere, which is demonstrated by a Branly's Coherer. If the one wave is successively lengthened, both waves will be alternately strengthened and weakened, and as many as four maxima and minima can be easily shown. The experiments were made with tubes of nearly 60 millim. diameter. Tubes with half that diameter did not show the phenomenon. If in one of the two tubes a paraffin cylinder is introduced which entirely fills it, the position of the maxima and minima is displaced, from which the refractive index of paraffin can be calculated. The author found for it 1·65–1·70. Similar experiments with sulphur gave for its refractive index 2·33–2·37. These numbers are considerably higher than the values found by Righi, which were, for paraffin 1·43 ; for sulphur 1·87. The length of the electrical waves in these experiments was 80 millim., the diameter of the knobs of the exciter 106 millim.—*Wiener Berichte*, Oct. 24, 1895.

ON ELECTRIFIED ATOMS.

To the Editors of the Philosophical Magazine.

GENTLEMEN,

I find that the view expressed by me in my paper "On the Relation between the Atom and its Charge" (Phil. Mag. Dec. 1895, p. 541), that a negatively electrified atom moved more rapidly than a positively electrified one in the same electric field, was arrived at from different reasons by Professor Schuster, and given by him more than five years ago in the Bakerian Lecture for 1890. I am glad to find that this view is held by so eminent an authority, and regret that I had previously overlooked such strong evidence in its favour. There is another result given in the same Bakerian Lecture which I ought to have quoted. I allude to Mr. Stanton's interesting experiments on the escape of electricity from a hot copper rod surrounded in one case by oxygen and in another by hydrogen.

I am, Gentlemen,

Yours very sincerely,

J. J. THOMSON.

Cavendish Laboratory, Cambridge,
January 20, 1896.

ON THE DOUBLE REFRACTION OF ELECTRICAL WAVES.

BY PROF. AUGUSTE RIGHI.

In No. 2, 1895, of Wiedemann's *Annalen* is a memoir by Prof. Mach (Phil. Mag. July 1895), in which are described experiments showing that a plate of pine-wood with faces parallel to the fibres behaves towards electrical radiation just as a double refracting plate towards luminous radiations. This memoir is dated Nov. 1894.

Now on May 27, 1894, I communicated to the R. Accademia di Bologna a long paper in which are described experiments equivalent to those of Prof. Mach, and others of analogous import. In my experiments I not only confirmed the double refraction produced by wood, and the varying absorption of electrical rays according as they are parallel or perpendicular to the fibres, but I also obtained elliptical and circular polarization, using for this latter a wooden plate of such a thickness as to represent a quarter of a wave-length. Thus, like Prof. Mach, I compared the behaviour of wood to that of tourmaline rather than to that of calc-spar or of quartz. One experiment of Prof. Mach's I did not make, namely, that in which it is proved that there is no double refraction if the wooden plate is cut perpendicularly to the fibres. I did not think it necessary to make this experiment, the idea of which did indeed occur to me, because from reasons of symmetry the result seemed *a priori* evident.

I am satisfied that Prof. Mach did not know of my memoir, or at any rate not when I made my communication to the *Annalen*, and I am glad that his experiments so completely confirm those I had already published. I may add that, like him, I do not believe in the possibility of producing double refraction of electrical radiations by means of a plate of Iceland spar, for the reasons I have explained in my memoir. In any case I intend soon to resume my researches in these directions, which I have been prevented from doing by other work.—*Nuovo Cimento* [4] vol. i., April 1895. *Communicated by the Author.*

NOTE ON ELEMENTARY TEACHING CONCERNING FOCAL LENGTHS.

To the Editors of the Philosophical Magazine.

GENTLEMEN,

In Dr. Barton's communication, page 59, I see he follows the usual practice with regard to signs. My experience is that students find it simpler and more natural to take the positive sign for real images always, and the negative sign for virtual ones. Similarly, converging instruments are to have their focal length or their "potency" considered positive, and diverging instruments negative, whether they be mirrors or lenses. Does Dr. Barton see objection to this course?

Yours faithfully,

January 3, 1896. OLIVER J. LODGE.

THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[FIFTH SERIES.]

MARCH 1896.

XXI. *On Magnetic Tractive Force* *. *By* E. TAYLOR JONES, *D.Sc., Assistant Lecturer in Physics in the University College of North Wales, formerly Science Scholar of Royal Commission for Exhibition of* 1851 †.

IN a former paper ‡ an account was given of experiments in which the value of the magnetic tractive force of a divided ellipsoid was found to agree within 1 per cent. with Maxwell's expression $B^2/8\pi$ for the electromagnetic stress in the narrow air-space between the two halves, for inductions between 6000 and 20,000 C.G.S.

In the present paper further experiments will be described, the chief object of which was to test whether the same agreement existed at much higher inductions. Since the magnetizing force H in the iron was very small in comparison with the induction B in the former experiments, these cannot be considered as deciding between Maxwell's expression $B^2/8\pi$ and, for instance, the expression $2\pi I^2$, which has also been given as the value of the tractive force §.

In order to complete the test it was necessary to continue the experiments to such high inductions that the magnetic force H was a considerable part, numerically, of the induction $H+4\pi I$.

* Partly communicated to the *Physikalische Gesellschaft* of Berlin on June 28, 1895.

† Communicated by the Author, in continuation of the paper on "Electromagnetic Stress," Phil. Mag. March 1895.

‡ E. T. Jones, Phil. Mag., March 1895; Wied. *Ann.* liv. p. 641 (1895).

§ Stefan, *Wien. Ber.* lxxxi. 2 Abth. p. 89 (1880).

Apparatus.

These inductions were obtained by means of the Ring Electromagnet designed by Dr. H. du Bois *.

The chief parts of the apparatus are shown in fig. 1, by a vertical axial section; the left half is not all shown, but is symmetrical with the right. The pole-pieces were conical, and had a vertical angle of 78° 28′, so that the field near the vertex was as nearly uniform as possible†.

Both pole-pieces were axially perforated, the width of the openings being about 5 millim.

Into one of the pole-pieces fitted a short cylindrical iron bar, C_1, provided with a flange, so that when pressed home it just reached to the vertex of the cone. Into the opening in the other pole-piece fitted a longer bar, C_2, of the same iron as C_1 and of equal diameter; this bar could be moved to and fro in its socket with very little friction. The two bars thus met in the equatorial plane EE′ and formed an "isthmus" between the pole-pieces. Similar bars were also made of steel.

This part of the apparatus was carefully centred, and the inner ends of the bars were turned plane and polished; in this process the pole-pieces themselves served as "guard-rings" before they were completely turned down to their conical form. The surfaces were optically tested as in the former experiments (*l. c.* p. 262).

The bar C_2 was connected, by a long brass rod M and a cord which passed over a pulley, with a scale-pan, so that the tractive force could be directly measured. For measuring the induction and magnetic force in the isthmus the method of Ewing and Low was employed. A small brass bobbin, on which two coils, each of five turns of thin copper wire, were wound, could be placed on the isthmus near the plane of section, so that when the bar C_2 was pulled out the bobbin automatically fell out to a distance and was turned so that its plane remained parallel to the field. The diameter of one of the coils was slightly greater than that of the isthmus, that of the other about 2 millim. greater, so that an annular space about 1 millim. wide existed between the coils. Each coil could be connected through a resistance-box with an Ayrton-Mather galvanometer. The movable coil of this instrument was specially weighted to make it suitable for ballistic measurements.

* Du Bois, Wied. *Ann.* li. p. 537 (1894); see fig. 11 there.

† Ewing, 'Magnetic Induction in Iron and other Metals,' § 97 (1893).

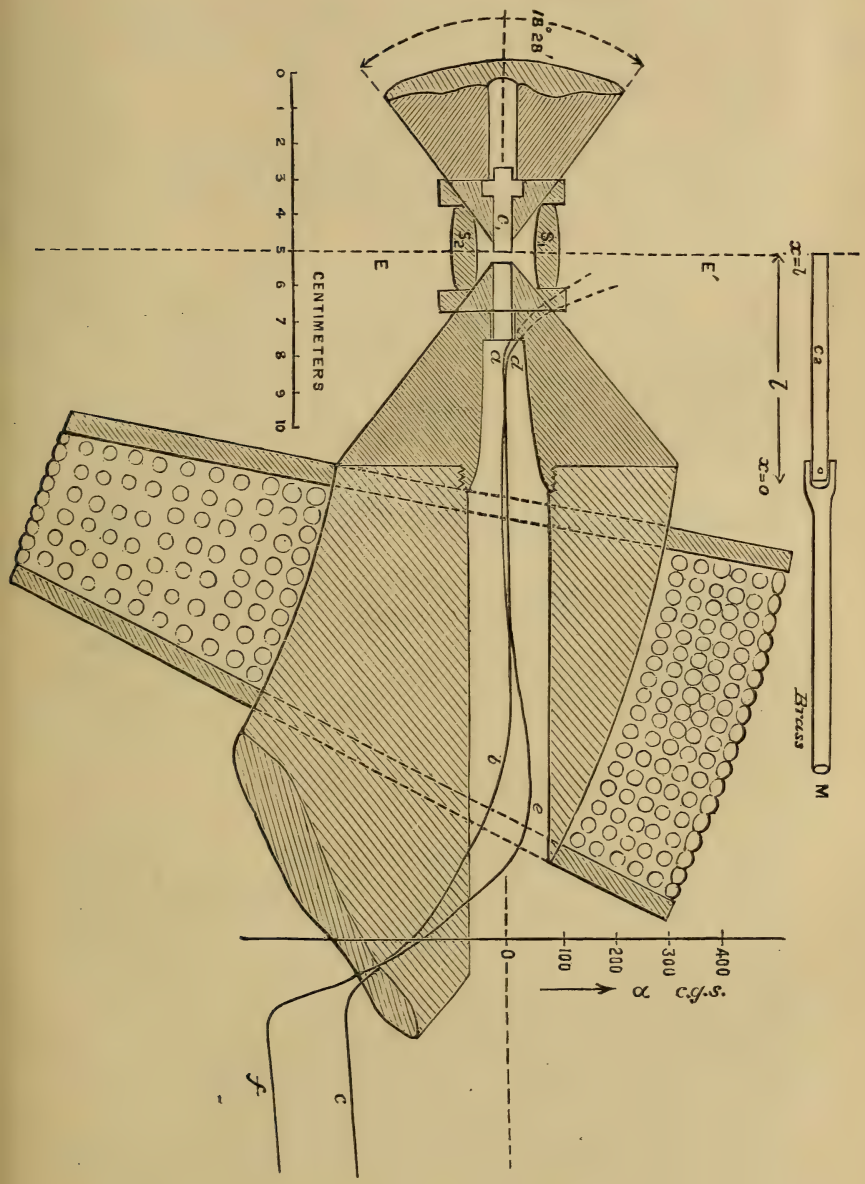

18°28′
CENTIMETERS
0 1 2 3 4 5 6 7 8 9 10
E
E′
S_2
S_1
c_1
c_2
$x=l$
l
$x=0$
Brass
M
a
d
b
e
f
c
0
100
200
300
400
α c.g.s.

Fig. 1.

From the induction-throws obtained with the two coils, the mean induction in the isthmus and the field-intensity just outside the isthmus (which by the principle of continuity of tangential magnetic force is equal to the magnetic force just inside the isthmus) could be calculated.

It was assumed that the field in and near the isthmus was so nearly uniform that the induction and magnetic force were constant over the section of the isthmus, and the latter equal to the field in the annular space between the two coils.

Theory.

The results obtained with this arrangement can be interpreted in the following way according to Maxwell's theory.

The mechanical force X acting in the direction ox on a body magnetized with an intensity whose components are A, B, C, and placed in a field of magnetic force of components α, β, γ, is *

$$X = X_1 + X_2 + X_3 = \iiint \left(A\frac{d\alpha}{dx} + B\frac{d\beta}{dx} + C\frac{d\gamma}{dx} \right) dx\, dy\, dz \quad . \quad (1)$$

Applying this to the case of the bar C_2, taking the origin at the outer end (*cf.* fig. 1) and ox along the axis of the bar—also considering for the present only the first term X_1 of the integral, and assuming that A and α are independent of y and z in each section of the bar—we have after integrating by parts,

$$\frac{X_1}{S} = \Big[A\alpha \Big]_0^l - \int_0^l \alpha \frac{dA}{dx} dx,$$

where l = length, and S = section of the bar.

At the end ($x = l$) between the pole-pieces we have $A = I$, while $\alpha = H + 2\pi I$, corresponding to the action of the pole-pieces and of the free end of the bar C_1.

The distribution of α along the axis was found by throwing out a small exploring coil from different points on the axis. The results showed that a certain region existed in which α was always very small and nearly constant. The ordinates of the curves $\overline{abc}$ and $\overline{def}$, fig. 1, represent the values of α at points on the axis, with magnetizing currents of 7 and 25 amp. respectively.

The length of the bar C_2 was so chosen that it reached to

* Maxwell, 'Electricity and Magnetism,' vol. ii. § 639.

this region where the field was practically zero. We have, therefore, at $x=0$, also $\alpha=0$, and the equation becomes

$$\frac{X_1}{S}=I(H+2\pi I)-\int_0^I H dI,$$

or, remembering the relation $B=H+4\pi I$,

$$\frac{X_1}{S}=\frac{B^2-H^2}{8\pi}-\int_0^I H dI. \quad . \quad . \quad . \quad . \quad (2)$$

This part of the integral in (1) depends only on the longitudinal components of the field and magnetization.

It may be shown that the part of X depending on the transverse components of field and magnetization vanishes if the value of α is constant at the outer end of the bar, *i. e.* if $\frac{d\alpha}{dx}=0$ at $x=0$* ; since this was very nearly the case in the experiments we may neglect these terms.

Equation (2) gives, therefore, the tractive force per unit area; calling this P grammes weight per sq. centim., we have

$$P=\frac{B^2-H^2}{8\pi g}-\frac{1}{g}\int_0^I H dI\,; \quad . \quad . \quad . \quad . \quad (3)$$

or †

$$\frac{B^2}{8\pi g}=P+\frac{H^2}{8\pi g}+\frac{1}{g}\int_0^I H dI\,;$$

so that the expression

$$\sqrt{P+H^2/8\pi g+\frac{1}{g}\int_0^I H dI}, \quad . \quad . \quad . \quad . \quad (4)$$

* An approximate calculation based on Laplace's equation in cylindrical coordinates gives

$$X_2+X_3=\iiint\left(B\frac{d\beta}{dx}+C\frac{d\gamma}{dx}\right)dx\,dy\,dz$$
$$=-\frac{1}{16}\pi k r^4\left(\frac{d\alpha}{dx}\right)^2_{x=0},$$

where $r=$ radius of section of bar, and $k=$ ratio of mean radial magnetization to mean radial "external" magnetizing force.

† In the immediately following section of his treatise (*l. c.* § 641), Maxwell explains these forces by his well-known theory of stresses in a medium, and develops the corresponding stress equations. From these the equation (3) might be more shortly deduced. In somewhat different ways similar expressions have been obtained by du Bois, Wied. *Ann.* xxxv. p. 146 (1888), and by Adler, *Wien. Ber.* c. 2 Abth. p. 897 (1891).

in which the first term is generally the greatest, is in the present arrangement, according to Maxwell's theory, proportional to the induction B. In the former experiments B was proportional, neglecting small corrections, simply to $\sqrt{P}$.

The integral $\int_0^I H dI$ is clearly equal to the area enclosed by the magnetization curve, the axis of I, and the line parallel to the H-axis at distance I from it*. This area is very different in the cases of hard steel and soft iron. For this reason, and in order to make the test as general as possible, two bars of glass-hard steel were also experimented on.

Preliminary Experiments.

The ballistic galvanometer was standardized in the usual way by means of a long solenoid and a secondary coil.

The core of the solenoid consisted of a glass tube covered with ebonite†; this was turned accurately cylindrical, and its diameter measured with a Zeiss "thickness measurer," both before and after it was wound with one layer of silk-covered copper wire. From these measurements the field inside the solenoid and the area of the solenoid were calculated in the usual way, The current in the solenoid was measured by a Kelvin "platform"-galvanometer, which was moreover standardized by a copper voltameter‡.

From the throw of the galvanometer-needle on reversing the primary current, the ballistic sensitiveness of the galvanometer was calculated. This was not constant but varied, firstly somewhat with the time and temperature, secondly with the resistance of the galvanometer circuit. The first changes were observed and corrected for by withdrawing a small coil from a very constant permanent magnet before and after each set of observations.

This was then repeated with various resistances in the galvanometer circuit. At the same time the damping was also observed, *i. e.* the ratio m of two successive swings measured from the zero. The results of these observations are contained in Table I., in which R = total resistance of galvanometer circuit, σ = product of R into the throw (σ being thus proportional to the ballistic sensitiveness of the galvanometer), and

* Du Bois, Wied. *Ann.* xxxv. p. 146 (1888); Max Weber, Wied. *Ann.* liv. p. 35 (1895).

† Ebeling, *Reichsanst.-Bericht, Zschr. Instr.-Kunde*, xv. p. 331 (1895).

‡ The instructions given in A. Gray's 'Absolute Measurements in Electricity and Magnetism,' vol. ii. part ii. p. 421, were observed.

$$\sigma' = \sigma\{1 + 0{\cdot}5(m-1) - 0{\cdot}277(m-1)^2 + 0{\cdot}13(m-1)^3\}^*.$$

σ' is proportional to the ballistic sensitiveness of the galvanometer corrected for the damping of the oscillations.

TABLE I.—Temp. 21°·3 C.

R.	σ.	m.	σ'.
7·82	2112	1·231	2333
10·82	2142	1·187	2325
12·82	2149	1·184	2327
14·82	2164	1·180	2340
17·82	2165	1·161	2326
22·82	2191	1·143	2335

The value of σ' is therefore within $\frac{1}{2}$ per cent. constant; in other words, the influence of the resistance of the secondary circuit on the ballistic sensitiveness of the galvanometer is completely, though indirectly, explained by its well-known influence on the damping of the oscillations, and can be eliminated, was as done in all the galvanometer observations. Moreover, the damping was too small to have any effect on the proportionality of throw to quantity of electricity flowing through the galvanometer, as I convinced myself by special observations.

The areas of the two isthmus coils were compared electromagnetically in the uniform field between flat pole-pieces with a standard coil made of a single strip of phosphor bronze wound on a circular glass plate which had been carefully turned, ground, and measured. The connecting strips of the standard coil were exactly laid one above the other so as to enclose no area parallel to the plane of the coil.

In using this coil the influence of its resistance on the sensitiveness of the galvanometer was of course, as explained above, taken into account.

The total areas of the two isthmus coils were thus found to be 1·235 and 2·553 sq. centim. respectively, that of the standard coil being 1·530 sq. centim. These areas were also calculated from the previously measured diameters of the brass bobbin and of the wire, and the calculated values

* Ayrton, Mather, and Sumpner, Phil. Mag. [5] xxx. p. 69 (1890).

agreed within $\frac{1}{2}$ and $\frac{1}{3}$ per cent., respectively, with the values obtained electromagnetically.

A series of observations was then made of the tractive force and at the same time of the induction-throws caused by the coils falling from the isthmus, the electromagnet being excited by various currents ranging from 0 to 25 amp.

One great difficulty in measuring the tractive force was due to the fact that the iron bar, and, more especially, the steel bar (which in the process of hardening might easily be slightly bent), often got jammed in its socket. If the bar was pulled exactly in the direction of its axis and that of its socket the jamming was avoided, and the friction was very slight. This was determined by removing the other pole-piece and demagnetizing the electromagnet as far as possible in order to remove all magnetic action. By properly guiding the brass rod M, the position was then found in which the weight necessary to pull the bar out was a minimum, and this was found to be less than 5 grammes weight, which is negligible in comparison with the tractive forces obtained later.

But if the bar was pulled only slightly to one side, it became jammed in its socket, and about 700 grms. weight more were necessary to pull it out. In all subsequent measurements therefore the position was found, by properly guiding the brass rod, in which the tractive force was a minimum, and this was taken as the magnetic tractive force. The question as to whether the magnetization had any appreciable effect on the friction was tested by pushing the longer bar C_2 through both pole-pieces (C_1 being removed), so that it lay in a symmetrical position of minimum potential energy. No effect on the friction could be detected when the exciting current was made or broken*.

For the graphical evaluation of the integral $\int_0^I H dI$ it was necessary to have the magnetization-curves for the iron and steel.

The higher values of H and I were already known from the isthmus experiments. For the determination of the earlier parts of the curves, two bars were used which were cut from the same pieces as the isthmus bars. The curves were then

* This arrangement, in which no gap existed in the isthmus, afforded at the same time a means of testing how far the gap in the usual arrangement caused disturbance by spreading the lines of induction. It was found that in the strong fields here used this was a considerably smaller source of error than might be supposed after certain experiments at less intense fields.

determined for these by means of a du Bois Magnetic Balance*, which had been tested at the Reichsanstalt. These experiments, as well as all the tractive-force measurements, were made with increasing magnetization †. This integral formed only a comparatively small correction, so that it was not necessary to know it very accurately.

Final Experiments.

From the galvanometer-throws the values of B and H were calculated, and their difference gave, on being divided by 4π, the magnetization I. The greatest values of I obtained for the soft iron and the hard steel were 1818 C.G.S. and 1556 C.G.S. respectively, and these occurred at a field of about 10,600 C.G.S. At stronger fields I appears to have rather smaller values ‡; a similar apparent diminution of I was also observed by Ewing §.

It must, however, be remarked that the isthmus method is never very suitable for accurate measurements, and that the arrangement described above is particularly unfavourable to accurate measurements of I from the difference of B and H; firstly, because of the disturbing influence of the gap in the isthmus; and, secondly, because the bars are not continuous with the pole-pieces, as in the original arrangement of Ewing and Low.

No valid conclusions can therefore be drawn from these experiments as to the course of the magnetization-curve; the induction in the isthmus, however, which is the quantity of chief importance in these experiments, was, in consequence of its direct determination, known with sufficient accuracy.

With each value of the current, the measurements were repeated several times. For given values of the galvanometer-throws, the minimum tractive force was very constant, so that even at the highest values it could be determined to within about 30 grammes weight.

The greatest magnetic tractive force measured was 9430 grms. weight, the cross section of the iron bar being 0·1896 sq. centim. This gives a pull of 49·73 kilogs. weight per sq. centim. (707·4 lb. wt. per sq. inch).

* See du Bois, *Zeitschr. für Instr.-Kunde*, xii. p. 404 (1892).

† The curves obtained with the balance will be published in another connexion.

‡ In later experiments made with a continuous isthmus (see p. 160, footnote) this diminution of I was not observed.

§ Ewing, *l. c.* § 93; this was attributed by him to non-uniformity of the field near the isthmus.

The greatest value of the integral $\frac{1}{g}\int_0^I HdI$ was in the iron 550, in the steel 840 grms. weight per sq. centim. This only gives, therefore, a correction of a few per cent. In my former experiments the corresponding correction (there due to the slight non-uniformity of the field inside the coils) was only about ½ pro mil. (Wied. *Ann.* l. c. p. 649, footnote).

TABLE II.—Former results (Phil. Mag. *l. c.* p. 265).

I. C.G.S.	B. C.G.S.	$\frac{B}{\sqrt{8\pi g}}$. Calculated.	G. Observed.	Difference.
493	6198	39·2	39·2	0·0
551	6929	43·9	44·4	+0·5
646	8122	51·6	51·6	0·0
853	10730	68·1	68·4	+0·3
996	12520	79·4	79·0	−0·4
1163	14630	92·9	92·0	−0·9
1291	16260	103·3	102·9	−0·4
1346	16970	107·7	107·3	−0·4
1400	17690	112·3	111·9	−0·4
1463	18540	117·7	117·1	−0·6
1550	19730	125·3	123·5	−1·8
1585	20230	128·4	126·7	−1·7

TABLE III.—Annealed Soft Styrian Iron.

Current, Amperes.	H. C.G.S.	B. C.G.S.	I. C.G.S.	$\frac{B}{\sqrt{8\pi g}}$. Calc.	G. Obs.	Difference.
0	260*	17740	1392	113·0	109·5	−3·5
0·13	613	22050	1707	140·5	138·0	−2·5
0·37	1440	23370	1745	148·9	147·0	−1·9
0·93	3470	26140	1805	166·5	162·0	−4·5
1·33	5070	27420	1779	174·7	172·0	−2·7
2·6	8560	31220	1803	198·9	195·0	−3·9
3·8	10520	33370	1818	212·5	207·5	−5·0
5·6	12720	34660	1746	220·8	219·5	−1·3
7·9	14230	35790	1716	227·9	228·3	+0·4
9·8	14850	36450	1719	232·1	232·8	+0·7
14·9	16030	38270	1770	243·8	239·8	−4·0
25·0	17690	39260	1796	250·1	250·9	+0·8

* Due to residual magnetism in the pole-pieces.

TABLE IV.—Glass-hard Wolfram Steel.

Current, Amperes.	H. C.G.S.	B. C.G.S.	I. C.G.S.	$\frac{B}{\sqrt{8\pi g}}$. Calc.	G. Obs.	Difference.
0	181*	11390	892	72·5	71·8	−0·7
0·13	558	17550	1353	111·8	107·3	−4·5
0·37	1615	19750	1444	125·8	121·3	−4·5
0·91	3470	22510	1516	143·4	139·0	−4·4
1·31	5050	24000	1508	152·9	148·7	−4·2
2·6	8670	27950	1535	178·0	174·6	−3·4
3·8	10750	30290	1556	193·0	187·6	−5·4
5·7	12830	31950	1522	203·5	201·1	−2·4
8·0	14220	32860	1484	209·3	207·3	−2·0
9·9	14860	33470	1481	213·1	211·3	−1·8
14·9	15970	35380	1545	225·4	221·9	−3·5
25·0	17830	36110	1456	230·0	230·2	+0·2

* Due to residual magnetism in the pole-pieces.

Tables III. and IV. contain the values of the exciting current, H, B, I, $B/\sqrt{8\pi g}$, of the quantity

$$G = \sqrt{P + \frac{H^2}{8\pi g} + \frac{1}{g}\int_0^I H dI},$$

and, lastly, of the difference

$$G - \frac{B}{\sqrt{8\pi g}},$$

in iron and steel.

Table II. contains the results of my former experiments (*l. c.* p. 265), which are here reproduced in order to show the complete range experimented over.

The results are also exhibited graphically in fig. 2, in which the points +, ⊙, show the values of G as a function of B in iron and steel respectively, while the straight line represents the function $B/\sqrt{8\pi g}$.

The points ⁎ are taken from the former experiments, and represent the square root of the tractive force per sq. centim., which in that case was theoretically equal to $\frac{B}{\sqrt{8\pi g}}$.

All the observed points lie near the straight line; the greatest difference is about 3 per cent. Most of the points lie under the line; this is probably due to the spreading action of the gap in the isthmus, which diminishes the tractive

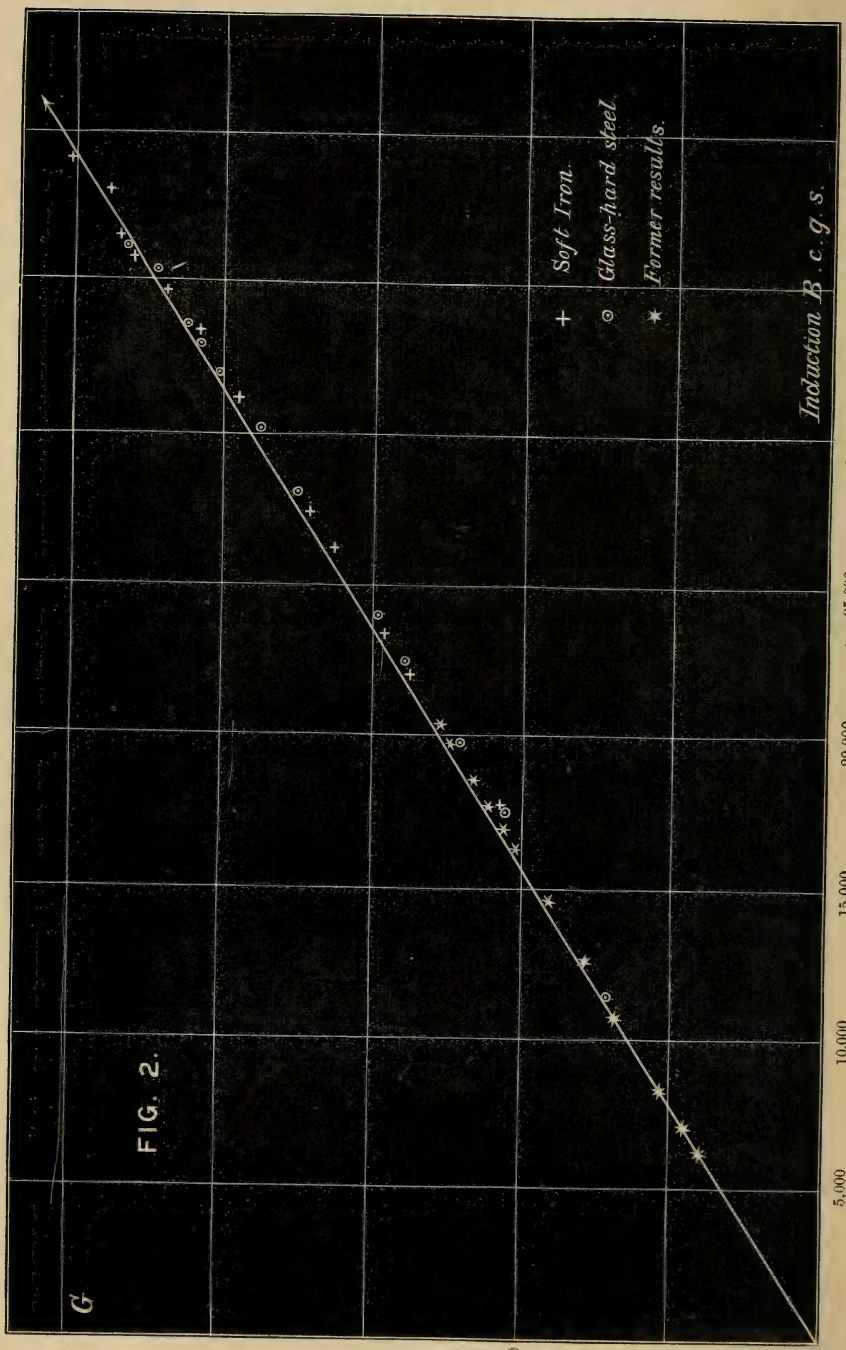

G
Soft Iron.
Glass-hard steel.
Former results.
Induction B. c. g. s.
0
50
100
150
200
250
5,000
10,000
15,000
20,000
25,000
30,000
35,000
40,000

FIG. 2.

force. The agreement is, however, on the whole such that the equation (3)

$$P = \frac{B^2 - H^2}{8\pi g} - \frac{1}{g}\int_0^I H dI$$

can be regarded as verified.

According to Maxwell's theory (*l. c.* § 643) the first term $\frac{B^2}{8\pi g}$ is the total electromagnetic tension existing in the narrow air-gap between the ends of the two bars; similarly, $H^2/8\pi g$ is the tension in the gap, depending only on the "external" field H. The correctness of this interpretation of the term $H^2/8\pi g$ follows from the well-known experiments of Quincke and others.

This latter part of the stress is not included in the tractive force in the present arrangement: it merely serves to compress the brass pillars S_1, S_2, S_3.

If the bar C_2 were firmly fixed in its pole-piece (instead of sliding in it), and if the two halves of the electromagnet were pulled from each other, the pull in grammes weight in this case, corrected for the tension due to all those tubes of induction which do not pass along the isthmus, would be equal to

$$\frac{B^2}{8\pi g} \times \text{area of contact.}$$

This case corresponds theoretically to the arrangement of my former experiments, but cannot of course be practically carried out.

Measurement of High Inductions.

After Maxwell's law of tractive force, or electromagnetic stress, had been verified as explained above for inductions up to 40,000 C.G.S., its general truth could be assumed. And just as various physicists have conversely used the tractive force for the measurement of lower inductions, it now seemed feasible to apply a similar process to the measurement of the highest attainable inductions, all other known methods failing in this case.

I therefore arranged a few more experiments with this object. Two pole-pieces of 120° vertical angle were made of the very best annealed Swedish iron. The point of one was slightly flattened, ground plane, and polished, so that it presented a small circular pole-face whose diameter was only a fraction of a millimetre. Through the opposite pole-piece a slightly conical hole was bored, through which a long wire of

annealed soft iron (of diameter 0·586 millim.) could be drawn, which fitted accurately in front and more loosely behind. The friction was thus only a fraction of 1 gramme weight.

The end of the wire was also carefully polished and, as well as the opposite small pole-face, examined and measured with a Zeiss microscope.

With a magnetizing current of 25 amperes, the weight supported in this case was 249 grms. weight; this corresponds to a pull of 92·39 kilogs. weight per sq. centim. (1314 lb. wt. per sq. inch).

The maximum value of the magnetization was, moreover, determined by means of an isthmus consisting of a bundle of 50 pieces of the same wire pushed through the former pole-pieces of 78° 28′ vertical angle. This value was found to be about 1800 C.G.S.

Taking also 500 grms. weight as the approximate value of the integral $\frac{1}{g}\int_0^I H dI$, we have from these data the values of B^2-H^2, and of $B-H=4\pi I$; from these were deduced the values

$$B=61{,}900 \text{ C.G.S.}$$

$$H=39{,}300 \quad ,,$$

These experiments were extended still further by drawing out the wire to a thickness of 0·2412 millim., and narrowing down the hole in the pole-piece correspondingly.

After the wire had been once more annealed in a spirit-flame and its end ground plane, the weight supported with a current of 40 amperes was 52·5 grms. weight, corresponding to a pull of 114·9 kilogs. weight per sq. centim. (1634 lb. wt. per sq. inch).

Hence follow, as above, the values

$$\mathbf{B = 74{,}200 \text{ C.G.S.}}$$

$$\mathbf{H = 51{,}600} \quad ,,$$

and the permeability is only

$$\mu = B/H = 1{\cdot}44.$$

The calculated value of H from the Stefan-Ewing logarithmic formula is 50,450 C.G.S., assuming that the conical pole-pieces are uniformly magnetized with a maximum intensity of 1800 C.G.S. To this must be added about 750 C.G.S. due to the direct action of the coils of the electromagnet with

a current of 40 amperes. The total calculated field is thus 51,200 C.G.S., which agrees well with the value found above.

In this experiment the area of the isthmus was about $\frac{1}{200,000}$ of the area of the base of the conical pole-piece.

This induction ($B=74,200$) exists also by continuity in the narrow air-gap at the contact, and here it is to be regarded as "external" field. This will not be much altered at the moment of separation as long as the gap is narrow in comparison with its transverse dimension. It was then observed with a magnifying glass that, unless the apparatus was very carefully cleaned, small microscopical iron filings flew into the gap when separation took place and formed new "isthmuses" across it, whose thickness was very small in comparison with that of the $\frac{1}{4}$ millim. iron wire, so that their self-demagnetizing effect could be neglected. The induction B' in these is thus

$$B'=74200+4\pi\,.\,1800=96,800 \text{ C.G.S.}$$

We may therefore conclude that inductions of nearly 100,000 C.G.S. can exist in iron without any special occurrence.

If the wire is pulled back so that its end is in the plane of the edge of the opening, the field between the pole-faces is then about 60,000 C.G.S.; it only extends, however, over a few tenths of a millimetre*.

At the instant of separation a tension of $(74200)^2/8\pi g=224$ kilogs. weight per square centim. (3185 lb. wt. per sq. inch) is transmitted across the air-gap; this is about the limit of elasticity of lead. In this connexion it is interesting to remember that the tension between the plates of an air-condenser at atmospheric pressure cannot far exceed 2 grms. weight per square centim. without a spark appearing. An absolute vacuum, however, or a body of high specific inductive capacity can transmit much greater tensions.

My best thanks are due to Dr. H. du Bois, in whose laboratory the above experiments were carried out, for his help and advice.

Berlin, Christmas, 1895.

* If the field extends over several millimetres an intensity of over 40,000 C.G.S. can scarcely be reached. See du Bois, Wied. *Ann.* 1. p. 547 (1894); J. B. Henderson, Phil. Mag. Nov. 1894.

XXII. *Researches in Acoustics.*—No. X. *By* Alfred M. Mayer *.

[Contents.—The Variation of the Modulus of Elasticity with Change of Temperature, as determined by the Transverse Vibration of Bars at Various Temperatures. The Acoustical Properties of Aluminium.]

Summary of the Research.

POISSON, in his *Traité de Mécanique* (Paris, 1833, t. ii. pp. 368–392) † discusses the laws of the transverse vibrations of a bar free at its ends and supported under its two nodes. He shows that the frequency of the vibrations of the bar is given by an equation, which, reduced to its simplest expression, is $N = V \times 1{\cdot}0279 \frac{t}{l^2}$; in which N is the number of vibrations per second of the bar, t its thickness, l its length, and V the velocity of sound in the direction of the length of the bar.

To ascertain how nearly the frequency of the transverse vibrations of a bar, computed by Poisson's formula, agrees with the result obtained by experiment, the following method of experimenting was used.

Rods of steel, aluminium, brass, glass, and of American white-pine (*Pinus Strobus*)—substances differing greatly in their moduli of elasticity, densities, and physical structures—were carefully wrought so as to have the length of $1{\cdot}5 \pm$ metre, the thickness of 0·5 cm., the width of 2 cms., and a uniform section throughout their lengths. The velocity of sound in these rods was determined by vibrating them longitudinally at a temperature of 20°, while held between the thumb and forefinger, and their frequencies of vibration ascertained by the standard forks of Dr. R. Kœnig's tonometer.

Out of each of these long rods were cut three bars of the length of 20 cms., and these bars, also at 20°, were supported on threads at their nodes, vibrated transversely by striking them at their centre with a rubber hammer, and their frequencies of vibration determined by the forks of the tonometer.

The mean departure of the observed from the computed numbers of transverse vibrations (see Table I.) is $\frac{1}{328}$; the computed frequency being always in excess of the observed,

* From an advance proof from the American Journal of Science for February 1896 communicated by the Author, having been read before the British Association at Oxford, August 1894.

† See also 'The Theory of Sound,' by Lord Rayleigh, 1894, vol. i. chap. 8.

except in the case of glass, where the computed is $\frac{1}{275}$ below the observed frequency.

In Table I. l = length and t = thickness of bar in centimetres at 20°; V = velocity of sound in centimetres in bar at 20°; N = number of vibrations per second at 20°.

The close agreement of the computed and observed values shows that, by vibrating a bar at various temperatures, the variation of its modulus of elasticity with change in its temperature can be obtained. We observe N at various temperatures of the bar; then $V = \frac{N}{1{\cdot}0279\frac{t}{l^2}}$ is computed, and the modulus $M = \frac{V^2 d}{g}$. As t, l, and d (the density of the bar) vary with the temperature, the coefficient of expansion of each bar and its density at 4° were determined, so that the dimensions and density of the bar could be computed for each of the temperatures at which it was vibrated.

Experiments were made on five bars of different steels, on two of aluminium, on one of St. Gobain glass, one of brass, one of bell-metal, one of zinc, and one of silver. The results of these experiments may be summed up as follows:—

The modulus of elasticity of	St. Gobain glass	is	1·16	per cent.	less at	100°	than at 0°.
,,	,,	the five steels	,, 2·24–3·09	,,	,,	,,	,,
,,	,,	brass	,, 3·73	,,	,,	,,	,,
,,	,,	bell-metal	,, 4·3	,,	,,	,,	,,
,,	,,	aluminium	,, 5·5	,,	,,	,,	,,
,,	,,	silver	,, 2·47	,,	,,	60	,,
,,	,,	zinc	,, 6·04	,,	,,	62	,,

The decrease of the modulus of elasticity of glass, aluminium, and brass is proportional to the increase of temperature; straight lines referred to coordinates giving the results of experiments on these substances. The five steels, silver, and zinc give curves, convex upwards, showing that the modulus decreases more rapidly than the increment of temperature; while bell-metal alone gives a curve which is concave upwards, its modulus decreasing less than the increment of temperature. (See Curves, fig. 5, p. 185.)

The more carbon a steel contains, the less is the fall of its modulus of elasticity on elevating the temperature of the steel. Thus, the modulus of the steel with 1·286 per cent. of carbon is 2·24 per cent. less at 100° than at 0°, while the steel containing 0·15 per cent. of carbon has a modulus at 100° which is 3·09 per cent. lower than its modulus at 0°.

So far as experiments on a single steel containing nickel

permit of any general deductions, it appears that the presence of nickel in a low carbon steel lowers its modulus of elasticity. Thus, steels No. 3 and 4, having respectively ·47 and ·51 per cent. of carbon, have a modulus of 2130×10^6; while steel No. 5, containing ·27 per cent. of carbon and 3 per cent. of nickel, has a modulus of 2080×10^6, which is 2·35 per cent. lower than that of steels Nos. 3 and 4.

The presence of nickel in a steel may, in a diminished degree, have the effect of carbon in lessening the lowering of the modulus when the temperature of the steel is increased. Thus the percentage of the lowering of the modulus, by heating from 0° to 100°, of steel No. 5 containing 0·27 of carbon and 3 per cent. of nickel, is the same as that of steel No. 3 with 0·47 per cent. of carbon.

If a bar of any one of the substances experimented on is struck with the same energy of blow, by letting fall on the centre of the bar a rather hard rubber-ball from a fixed height, the sound emitted by the bar diminishes in intensity and in duration as the temperature of the bar is raised. Thus:

Brass	at 0°	vibrates	during 75	secs.;	at 100°	it vibrates	during 45	secs.
Bell-metal	,,	,,	55	,,	,,	,,	15	,,
Aluminium	,,	,,	40	,,	,,	,,	12	,,
J. & C. Cast Steel	,,	,,	80	,,	,,	,,	5	,,
Bessemer Steel	,,	,,	45	,,	,,	,,	1·5	,,
St. Gobain Glass	,,	,,	6	,,	,,	,,	3·5	,,

Zinc at 0° vibrated during 5 secs.; at 20° only during 1·5 sec. At 62° it vibrated for so short a time that it only gave three beats with forks of 1090 and 1082 v. s. At 80° it was not possible to determine the pitch of the bar, and at 100° the bar when struck gave the sound of a thud. The bar of silver acted in a similar manner to the bar of zinc—it was even less sonorous than zinc,—thus flatly denying the "silvery tones" attributed to it.

These phenomena do not depend on the fall of modulus, but on changes in the structure of the metal on heating it, which cause the blow to heat the bar and not to vibrate it.

Bell-metal was found to be an alloy peculiarly well suited for bells, as the intensity and duration of its vibrations were the same at 50° as at 0°, all other substances showing a marked diminution of intensity and duration of sound at 50°.

A bar of unannealed drawn brass, after it has been heated to 100°, has its modulus at 20° increased $\frac{36}{100}$ per cent. (See Table III. and fig. 11, p. 188.)

In this research I had the good fortune to have the assistance of Dr. Rudolph Kœnig, of Paris. He not only placed at my service the resources of his laboratory and

workshop, but generously gave me constant assistance during the experiments—making the determinations of the numbers of vibration of the rods and bars with the standard forks of his tonometer. Without his aid this work could not have been done. For instance, in the cases of the bars of silver and zinc the beats they give with a fork are so few that they cannot be compared with a chronometer; but Dr. Kœnig, from his long experience in the estimation of beats, was enabled to form an accurate judgment of their number per second from the *rhythm* of the beats. The determination of pitches extending through such a range of vibrations as occurs in this research can only be made with Dr. Kœnig's "grand tonomètre"—a unique apparatus of precision, giving the frequency of vibrations from 32 to 43690 v. s., and really indispensable to the physicist who would engage in precise quantitative work in Acoustics.

We now proceed to give accounts of the several operations performed in the progress of this research.

Determination of the Velocity of Sound in Rods.

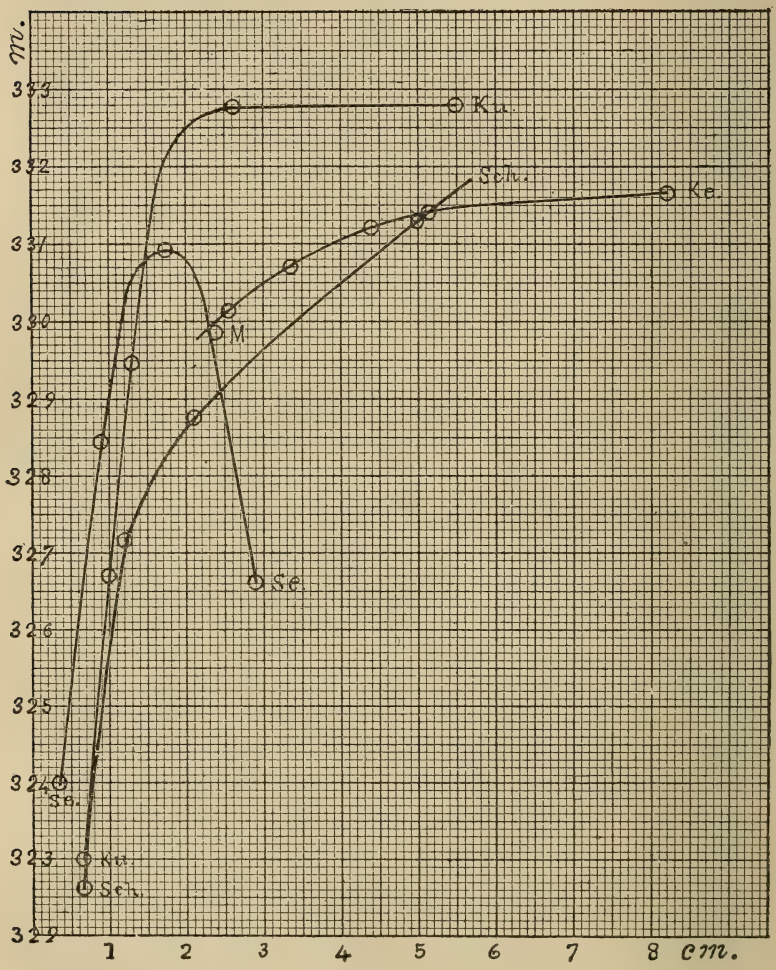
In the determinations of the velocity of sound in the rods of 1·5 m. in length, I used the method of Chladni*. Kundt's method of obtaining nodal lines of fine powders in a tube, by vibrating a rod whose end carries a cork which fits loosely the end of the tube, is not accurate. The weight and friction of the cork, the necessity of firmly clamping the rod at a node, and, above all, the want of knowledge of the velocity of sound in the air of glass tubes of different diameters, renders this method, so beautiful and ingenious, worthless for accurate measures of the velocity of sound in solids.

The curves in fig. 1 show the very diverse determinations of the velocity of sound in the air in tubes of different diameters by the physicists Kundt†, Schneebeli‡, Seebeck§, and Kayser‖. The velocity of sound in metres is given on the axis of Y; the diameter of the tube in centimetres on the axis of X. Ku stands for Kundt, Sch for Schneebeli, Se for Seebeck, and Ke for Kayser. The most precise measures of velocities are those of Kayser, who closed the end of the tube with a cork attached to the end of a steel bar, while the other end of the bar was securely clamped. The frequency of the transverse vibrations of the bar was registered by a style

* *Traité d'Acoustique*, Paris, 1809, p. 318 *et seq.*
† *Bericht. der Akad. der Wiss. zu Berlin*, 1867.
‡ Pogg. *Ann.* 1869, t. 136. § Pogg. *Ann.* 1870, t. 139.
‖ Pogg. *Ann.* 1877, t. 2. p. 218.

describing the sinusoids of the vibrating bar. Thus the weight and friction of the cork introduced no error. In a similar manner I obtained the velocity, marked M in fig. 1, by vibrating a rod of aluminium. The frequency of the

Fig. 1.



longitudinal vibrations of the rod was measured while the cork at the end of the rod was vibrating in the mouth of the tube. The result agrees closely with Kayser's. It is needless to discuss the curves of fig. 1.

The method of Chladni, used exactly as that eminent man used it, remains the best we have. It is important, however, to note that the rod must be held between the thumb and forefinger when it is vibrated and not *clamped* when vibrated. When clamped it always gives a higher frequency, as shown by the following experiments :—

Steel rod clamped	3429·2
Steel rod held between fingers . .	3428·4
Aluminium rod clamped	3377·0
Aluminium rod held between fingers.	3376·4

The frequency of the vibrations of the rods of steel, brass, aluminium, glass, and pine wood, when held at the middle of their lengths and vibrated so as to give their fundamental tones, gave exactly the octaves of these fundamental tones when held at one-quarter of their lengths and vibrated.

Determination of the Lengths of the Long Rods and of the Lengths and Thicknesses of the Bars.

The lengths of the rods of 1·5 ± metres were ascertained by comparison with the rod of steel whose length was measured at the Bureau International des Poids et Mesures. The lengths and thicknesses of the bars which were vibrated transversely were measured with micrometer calipers. The readings of these calipers were tested by comparison at 20° with a series of standards of various lengths of inches and fractions of inches, made for me with great care by Mr. George M. Bond, who has charge of the gauge department of the Pratt, Whitney Co. In reducing the comparisons to centimetres, I adopted the value of the inch as equal to 25·4 millimetres. In obtaining the length of a bar, the mean of several measures in the axis of the bar and in directions parallel to the axis and at various distances from it was adopted. The thickness of a bar was taken as the mean of measures taken throughout the length of the bar at points designated by the intersections of lines drawn parallel and at right angles to the axis of the bar.

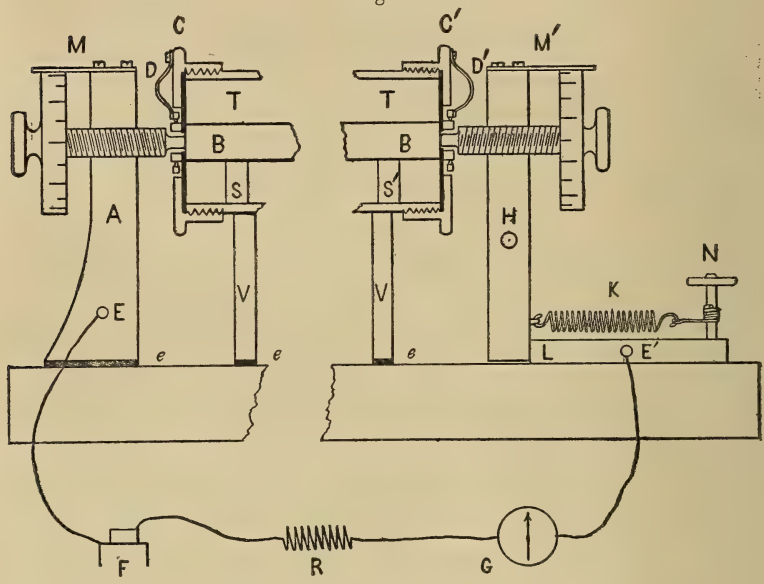
The dimensions of the bars were measured at 20 , except those of steels Nos. 3, 4, 5, which were measured at 18°·25.

Determinations of the Coefficients of Expansion of the Bars.

To determine the coefficients of expansion of the bars, I devised the apparatus shown in fig. 2. In a brass tube, T,

the bar, B, rests in slots in the supports, S, S′. The tube T is slightly shorter than the bar B. Washers of rubber (shown in black in the figure), of the same diameter as the outside diameter of the tube, are placed in the screw-caps, C, C′. These washers are perforated with holes of diameters smaller than the thickness of the bar. When the caps are screwed up, the rubber washers press against the ends of the bar. This

Fig. 2.



pressure is further increased by flat rings which surround the holes in the washers, and are pressed against these washers by means of the springs, D, D′. By this arrangement the surfaces of the ends of the bars are exposed, while the contact of the washers on the bars makes a water- and steam-tight joint. Thus the bar may be surrounded with ice, or with steam, or with a current of water of different temperatures, and be cooled or heated up to its terminal planes, while the holes in the washers allow the micrometer-screws, M, M′, to be brought to contact with the terminal planes of the bar. Two helical springs are attached to the column A. The other ends of these springs are fastened to rods projecting from the tube T. Thus the same pressure of contact is always made between the bar and the end of the micrometer-screw M. The tube T is supported in Vs, V, V′, and the greater part of

the weight of the tube is taken off the Vs by helical springs fastened to a frame above the apparatus. The tension of these springs can be so regulated that the tube rests on the Vs with the same pressure when it has steam passing through it and when it is filled with ice. The column, A, and the Vs, V, V′, are insulated from the base of the apparatus by thin plates of ebonite, *e*. Between the binding-screws, E and E′, and connected by wires, are the voltaic cell F, the galvanometer G, and a box of resistance-coils, R. The micrometer-screw, M′, with which the variations in length of a bar are measured, is mounted as follows :—The screw passes through its nut in a massive brass plate which rotates around nicely fitted centres at H. These centres are supported by two side plates not shown in the figure. A spring, K, is fastened to the lower part of the swinging nut-plate and brings this plate against the plate, L, firmly fastened to the base of the apparatus. When the swinging plate is vertical and the axis of the screw horizontal, the swinging plate fits accurately the surface of the fixed plate, L. By turning the rod, N, the swinging plate and its screw can be rotated away from the bar. This arrangement allows the screw to be swung out of the way while the tube, T, is being placed in the Vs. Also, it prevents any strain between the micrometer-screw, M′, and the column, A ; which would take place if M′ were fixed and it should be brought in contact with a hot bar in the tube, T.

With careful manipulation, successive electric-contacts can be made on a bar in the tube, T, surrounded by ice, so that the variations in a series of measures will not exceed $\frac{1}{2000}$ mm., with a resistance of about 200 ohms placed in the circuit.

It may be reasonably objected to this apparatus that when the micrometer-screw touches the bar at 0° it is cooled and shortened, and that when it touches the bar at 100°, or at temperatures higher than that of the screw, the latter is heated and elongated. This error, however, is quite small, and may be neglected in our work. If we assume that one centimetre of the screw is heated 10°, which is a large estimate, considering the duration of contact of screw and bar during a measure, the shortening or elongation of 1 cm. of the screw by cooling or heating it 10° amounts to only ·0012 mm., or $\frac{1}{166666}$ of the length of the bar. This change in the length of the screw will affect the coefficient of expansion of the bars only ·00000006.

Determination of the Densities of the Bars at 4°.

The bar whose density was to be determined was immersed in water at 4° for a couple of hours. The bar was then suspended by a platinum wire in water at 4° and weighed. The bar was then removed from the wire and a quantity of water equal in volume to the volume of the bar was added to the water in the vessel, and the platinum wire, now immersed exactly as it was when the bar was attached to it, was weighed. This weight, subtracted from the previous weighing, gave the weight of the bar in water. Every precaution was taken to prevent, by means of screens, the action on the balance of the currents of cold air in the balance-case, which are produced by the constant descent of air from the sides of the cool vessel.

The Apparatus in which the Bars were Heated and Cooled. On the precautions used so that one is sure of having the real temperature of the bar when it is vibrated.

The apparatus used to heat and cool the bar is shown in fig. 3. In a brass box, C, is inclosed a box, C′, containing the bar, B, supported on its nodes, N, N, by threads held by upright rods. From this central box two tubes, T, P, pass through the outer box C. The inner box is made water-tight and steam-tight by a rubber washer which is pressed between the top of the box and its cover by means of screws. Through the tube, T, the bar is vibrated by letting fall upon its centre a rubber ball fastened to a light wooden rod. On the blow of the ball it rebounds, and the rod is caught by the fingers in its upward motion. The cork is then at once replaced in the tube, T. The sound from the bar is conveyed to the ear, at E, by means of a tube (fig. 4). One branch of this bifurcated tube leads through a rubber tube to the pipe, P, of the box, fig. 3. The other branch leads to the fork, F, the number of whose beats per second made with the vibrating bar is measured by a chronometer. The pipe, S, allows the steam to issue when water is boiled in the box, C′, by a gas lamp. The flow of gas through this lamp was neatly regulated by a stop-cock turned by a long lever. The box, C, is covered, except at the bottom, with thick felt.

To determine the frequencies of vibration of a bar through a range of temperature from 0° to 100°, the following method was used. The box, C, was filled with ice, surrounding the inner box, C′. It thus remained for an hour so that the boxes were cooled down to 0°, and the moisture in the inner

Fig. 3.

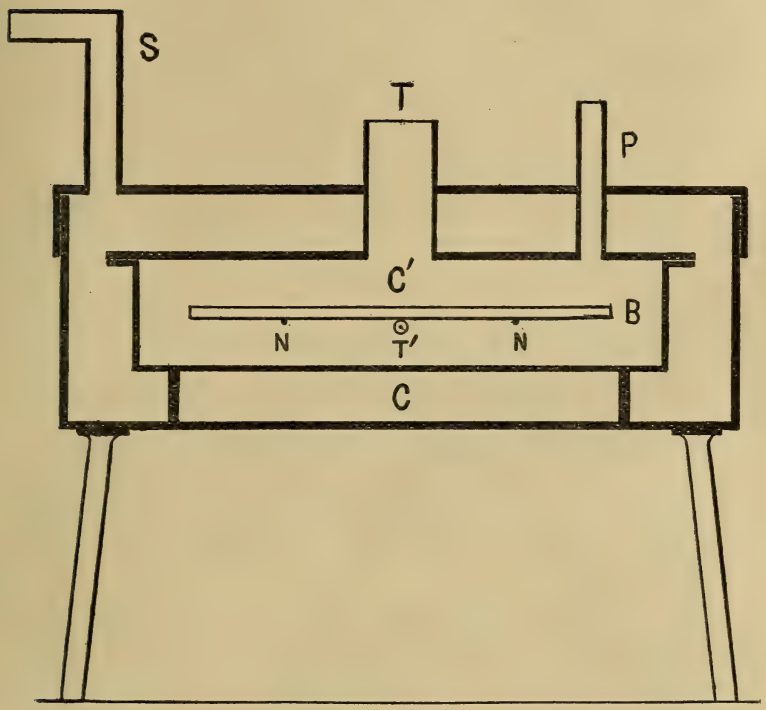
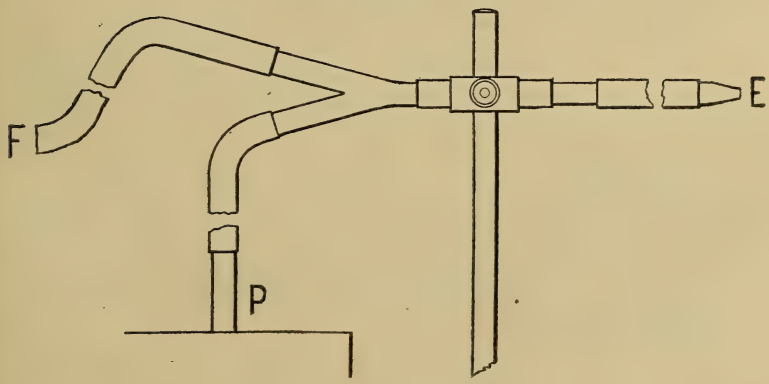


Fig. 4.



box had been condensed so far as it could be at 0°. The bar, which had been in ice for two hours, was wiped dry and quickly introduced into the inner box. A thermometer, T (made by Baudin and corrected), which entered the boxes through stuffing-boxes, and whose bulb touched the under surface of the bar, was read till it became stationary. The bar was now vibrated, and its frequency of vibration determined for the temperature given by the thermometer.

The lamp was now placed under the box, and the water in it boiled till the thermometer reached its maximum reading and the reading remained stationary during a half-hour. The vibration frequency at this temperature was taken. The flame of the lamp was now lowered and the box allowed to cool very slowly, at the rate of 1° fall of temperature in about eight minutes. When the thermometer read 80°, 60°, 40°, the flame of the lamp was carefully adjusted, so that these successive temperatures were maintained during 15 minutes. We then took the frequency of vibration of the bar.

The numbers of vibrations of the forks used in the determinations of the pitches of the bars were corrected for temperature by the coefficient ·0001118, determined by Dr. Kœnig in 1880 (*Quelques Expériences d'Acoustique*, Paris, 1882, p. 172 *et seq.*).

The subsequent tables show the results of the experiments and give the computations of velocities and moduli founded on them. The curves express graphically the effect of change of temperature on the modulus of elasticity of all the bars experimented on. The circles, on or near the curves, give the data as determined by the experiments.

In Table III., T = temperature of bars, l = the length, t = the thickness, and V = the velocity of sound through the bars, in centimetres. M = the modulus in grammes per square centimetre section of the bar. g, at Paris, equals 980·96. D = the density, and N = the number of vibrations of bar per second at temperature, T.

All of the bars were annealed, except those of Jonas and Colver steel, of the French aluminium, and of brass; these were experimented on just as they came from the draw-bench.

For the analyses of the substances of the bars experimented on, I am indebted to my colleagues, Professors Stillman and Leeds.

TABLE I.

Bar.	l.	t.	V at 20° =	N computed by $N = V\ 1{\cdot}0279 \frac{t}{l^2}$.	N observed at 20°.	Diff.
Steel No. 1	20·022	·5025	150·02 × 3427·4 v. s. = 514178 cms.	662·49 v. d.	660·8	$+1{\cdot}69 = \frac{1}{390}$
„ No. 2	20·0246	·5037		663·91 „	661·0	$+2{\cdot}91 = \frac{1}{227}$
„ No. 3	20·0225	·5022		662·07 „	660·3	$+1{\cdot}77 = \frac{1}{379}$
Aluminium.						
No. 1	20·0253	·4993	150·05 × 3377 v. s. = 506719 cms.	648·51 „	646·6	$+1{\cdot}91 = \frac{1}{133}$
No. 2	20·0296	·4991		647·97 „	647·0	$+0{\cdot}97 = \frac{1}{656}$
No. 3	20·0233	·4998		649·80 „	648·0	$+1{\cdot}8 = \frac{1}{360}$
Brass.						
No. 1	20·02	·50116	150·05 × 2386·4 v. s. = 358079 cms.	460·23 „	459·0	$+1{\cdot}23 = \frac{1}{378}$
No. 2	20·02	·50147		460·53 „	458·95	$+1{\cdot}58 = \frac{1}{290}$
No. 3	20·02	·50108		460·16 „	458·35	$+1{\cdot}81 = \frac{1}{253}$
St. Gobain Glass	23·516	·747	152·2 × 3582 v. s. = 538016 cms.	747·03 „	749·75	$-2{\cdot}72 = \frac{1}{275}$
White Pine Density = ·365.	41·15	·803	171·18 × 3072·75 v. s. = 525993 cms.	256·38 „	256·0	$+0{\cdot}38 = \frac{1}{673}$

Mean departure of computed from observed value $= \frac{1}{328}$ of observed value.

Table II.

Tables of Analyses, of Densities at 4°, and of Coefficients of Expansion of Bars.

	Iron.	Carbon.	Silicon.	Phos.	Sulph.	Mang.	Nickel.	Density at 4°.	Coeff't. of Expans.
J. & C. Steel.	98·259	1·286	0·015	0·059	0·031	0·350	.....	7·827	·0000110
No. 3 ,,	98·738	0·47	0·15	0·022	.....	0·62	.....	7·848	·0000118
No. 4 ,,	98·628	0·51	0·158	0·024	.....	0·68	.....	7·845	·0000120
No. 5 ,,	95·719	0·27	0·101	0·031	.....	0·69	3·189	7·851	·0000119
Bess'r. ,,	99·03	0·15	0·02	0·09	0·06	0·65	.....	7·841	·0000122

Brass.

Copper	64·34
Zinc	34·97
Lead	·58
Iron	·11
Density	8·476
Coefft. expan	·0000185

Bell Metal.

Copper	80·08
Tin	18·97
Lead	·12
Zinc	·49
Density	8·347
Coefft. expan	·0000187

Aluminium (Amer.).

Aluminium	98·99
Free Carbon (graphite)	·19
Combined Carbon	·16
Tin	·21
Silicon	·32
Iron	·15
Density	2·702
Coefft. expan	·0000232

Aluminium (French).

Aluminium	97·80
Carbon with Si	·14
,, free	·04
,, with Copper	·09
Copper	1·29
Silicon	·64
Density	2·730
Coefft. expan.	·000022

Silver, Pure.

Density	10·512
Coefft. expan	·0000184

Zinc.

Zinc	99·75
Iron	·10
Lead	·04
Density	6·8107
Coefft. expan	·0000296

St. Gobain Glass.

Silicon	72·3
Alumina	·8
Lime	15·3
Soda	11·8
Density	2·545
Coefft. expan	·00000777 (Fizeau).

TABLE III.

Bar.	T.	l.	t.	$\frac{t}{l^2}$.	D.	N.	$V = \frac{N}{1{\cdot}0279\,\frac{t}{l^2}}$.	$M = \frac{V^2 d}{g}$.	M.
Jonas & Colver Cast Steel.	0°·2	20·0207	·5036	·0012564	7·828	662·0	512602	2097585540	2097×10^6
	20	20·0246	·5037	·0012562	7·823	661·0	511908	2089805486	2090×10^6
	40	20·0286	·5038	·0012561	7·818	659·6	510112	2080863837	2081×10^6
	61	20·0328	·5039	·0012560	7·814	658·6	509389	2072936852	2073×10^6
	80	20·0366	·5040	·0012554	7·809	657·1	508470	2064153151	2064×10^6
	99·8	20·0406	·5041	·0012551	7·804	655·14	507065	2050537870	2050×10^6
Steel No. 3	0	20·3513	·64049	·0015464	7·849	820·38	516124	2131430318	2131×10^6
	18·25	20·3558	·64063	·0015460	7·844	818·91	515329	2123515662	2123×10^6
	34	20·3594	·64074	·0015457	7·8395	817·50	514539	2115797406	2116×10^6
	60	20·3657	·64094	·0015453	7·832	814·70	512907	2100383667	2100×10^6
	80	20·3705	·64109	·0015449	7·8265	812·40	511586	2088031273	2088×10^6
	99·5	20·3752	·64124	·0015446	7·821	810·00	510175	2075047288	2075×10^6
Steel No. 4	0	20·3517	·64295	·0015523	7·846	824·71	516238	2131556968	2131×10^6
	18·25	20·3562	·64312	·0015520	7·841	822·71	515708	2125826869	2126×10^6
	40	20·3614	·64328	·0015517	7·8345	821·25	514893	2117356581	2117×10^6
	51·5	20·3643	·64337	·0015514	7·831	819·67	514002	2109091886	2109×10^6
	81	20·3715	·64360	·0015508	7·823	816·30	512088	2091259452	2091×10^6
	97·6	20·3755	·64373	·0015505	7·819	814·00	510745	2079256951	2079×10^6
Steel No. 5	0	20·3513	·64188	·0015498	7·852	813·29	510527	2086250350	2086×10^6
	18·25	20·3559	·64202	·0015494	7·847	811·84	509757	2078637463	2078×10^6
	40	20·3609	·64218	·0015490	7·841	809·68	508529	2067052961	2067×10^6
	60	20·3657	·64233	·0015486	7·835	807·31	507168	2054430194	2054×10^6
	80	20·3706	·64248	·0015482	7·829	805·02	505856	2042249532	2042×10^6
	99·5	20·3753	·64264	·0315479	7·824	802·71	504506	2030066264	2030×10^6

Table III. (*continued*).

Bar.	T.	l.	t.	$\frac{t}{l^2}$.	D.	N.	$V=\frac{N}{1{\cdot}0279\frac{t}{l^2}}$.	$M=\frac{V^2d}{g}$.	M.
Bessemer Steel	0·4	20·451	·60144	·0014380	7·8421	761·37	515093	2121057163	2121×10^6
	20	20·456	·60160	·0014377	7·8364	759·80	514032	2110783287	2111×10^6
	40	20·461	·60176	·0014374	7·8306	757·70	512960	2100440462	2100×10^6
	60	20·466	·60190	·0014370	7·8248	755·41	511409	2086216940	2086×10^6
	80	20·471	·60205	·0014367	7·8192	752·90	509710	2070893085	2071×10^6
	100	20·476	·60220	·0014363	7·8134	749·90	508026	2055702283	2056×10^6
Brass when bar was cooled from 99°·6 to 1°.	1	22·0128	·5009	·0012506	8·4774	460·64	358337	1109672142	1109×10^6
	21·8	22·0208	·5011	·0012501	8·4677	458·95	357307	1102010322	1102×10^6
	40	22·0274	·50126	·0012497	8·4592	457·40	356103	1093479554	1093×10^6
	60	22·0348	·50144	·0012492	8·4498	455·70	355052	1085864045	1086×10^6
	89	22·0422	·50162	·0012488	8·4404	454·00	353727	1076578495	1076×10^6
	99·6	22·0494	·50180	·0012483	8·4312	452·30	352678	1069044150	1069×10^6
Brass before bar was heated to 99°·6.	0·4	20·0127	·5008	·0012504	8·4778	460·04	357928	1107191685	1107×10^6
	20	20·0208	·5011	·0012501	8·4685	458·35	356698	1098388907	1098×10^6
Bell-Metal	0	22·2402	·82054	·0016589	8·3490	572·05	335300	956864118	$956{\cdot}8\times10^6$
	21	22·2490	·82114	·0016588	8·3390	569·50	333806	947214927	$947{\cdot}2\times10^6$
	40	22·2568	·82168	·0016587	8·3302	567·12	332411	938324839	$938{\cdot}3\times10^6$
	50	22·2610	·82196	·0016587	8·3256	566·02	331766	934172231	$934{\cdot}2\times10^6$
	60	22·2650	·82224	·0016586	8·3208	564·94	331133	930074354	$930{\cdot}1\times10^6$
	74	22·2780	·82266	·0016586	8·3144	563·61	330354	924989117	$925{\cdot}0\times10^6$
	99·15	22·2818	·82338	·0016584	8·3025	561·21	328946	915914824	$915{\cdot}9\times10^6$

Aluminium (American)	0·4	21·612	·55176	·0011813	2·7027	621·00	511423	720621232	$720·6 \times 10^6$
	20	21·622	·55200	·0011807	2·6990	618·10	509032	712929083	$712·9 \times 10^6$
	45	21·6346	·55232	·0011800	2·6943	613·90	505993	703215278	$703·2 \times 10^6$
	60	21·6422	·55250	·0011796	2·6915	611·60	504516	698390199	$698·4 \times 10^6$
	82	21·654	·55278	·0011789	2·6874	607·88	501447	688868686	$688·8 \times 10^6$
	99·5	21·662	·55300	·0011785	2·6840	604·71	499247	681974176	$681·9 \times 10^6$
Aluminium (French) .	0·5	20·0170	·49914	·0012457	2·7306	650·00	507632	717306505	$717·3 \times 10^6$
	20	20·0253	·49930	·0012451	2·7270	646·60	505375	710002959	$710·0 \times 10^6$
	40	20·0340	·49950	·0012445	2·7232	642·78	502388	700672761	$700·6 \times 10^6$
	60·5	20·0428	·49965	·0012438	2·7194	639·19	499977	692975831	$692·9 \times 10^6$
	81	20·0518	·49980	·0012430	2·7156	635·50	497477	685124447	$685·1 \times 10^6$
	100	20·0600	·50000	·0012425	2·7120	632·00	494734	676700317	$676·7 \times 10^6$
Silver	0·3	17·2176	·4614	·0015564	10·5142	437·93	273736	803135399	$803·1 \times 10^6$
	20	17·2250	·46158	·0015557	10·5022	437·35	273489	800762642	$800·7 \times 10^6$
	30	17·2284	·46168	·0015554	10·4962	436·80	273201	798729900	$798·7 \times 10^6$
	40	17·2316	·46176	·0015551	10·4900	435·80	272626	794797589	$794·8 \times 10^6$
	60	17·2380	·46194	·0015545	10·4778	433·00	270979	783307979	$783·3 \times 10^6$
Zinc	0·3	18·2094	·44517	·0013426	6·8130	559·84	405663	1142925404	1143×10^6
	20	18·2200	·44534	·0013415	6·8010	557·84	404501	1134426237	1134×10^6
	40	18·2308	·44552	·0013405	6·7890	553·76	401844	1117556599	1117×10^6
	50·5	18·2364	·44560	·0013399	6·7826	551·22	400294	1107903320	1108×10^6
	62	18·2426	·44570	·0013390	6·7758	543·61	394291	1073843790	1074×10^6
St. Gobain Glass	0·3	23·496	·74898	·0013566	2·5452	750·65	538313	751865836	$751·8 \times 10^6$
	24·5	23·501	·74902	·0013562	2·5436	749·67	537769	749885431	$749·8 \times 10^6$
	40	23·503	·74910	·0013561	2·5424	749·12	537403	748511034	$748·5 \times 10^6$
	60	23·507	·74922	·0013558	2·5411	748·35	536909	746742620	$746·7 \times 10^6$
	80	23·510	·74934	·0013557	2·5397	747·62	536497	745196510	$745·2 \times 10^6$
	99·5	23·514	·74945	·0013554	2·5384	746·70	535955	743311215	$743·3 \times 10^6$

Table IV.

Variation of Modulus of Elasticity with Change of Temperature.

In this Table the modulus of each substance is taken as 100 at 0°. In computing this table the moduli taken were those obtained from the curves passing through the mean positions of the points determined by the experiments. The results contained in this table are expressed graphically in fig. 5.

T.	J. & C. Steel.	Steel No. 3.	Steel No. 4.	Steel No. 5.	Bessemer Steel.	Brass.	Bell Metal.
0°	100·00	100·00	100·00	100·00	100·00	100·00	100·00
20.....	99·61	99·57	99·70	99·58	99·53	99·26	99·03
40.....	99·25	99·10	99·31	99·10	98·99	98·51	98·09
60.....	98·87	98·58	98·8	99 54	98·35	97·76	97·21
80.....	98·42	98·00	98·22	97·98	97·63	97·03	96·38
100.....	97·76	97·34	97·54	97·35	96·91	96·27	95·70

T.	Aluminium (American).	Aluminium (French).	Silver.	Zinc.	St. Gobain Glass.
0°.....	100·00	100·00	100·00	100·00	100·00
20.....	98·92	98·86	99·73	99·26	99·76
40.....	97·83	97·73	98·97	97·78	99·53
60.....	96·75	96·58	97·53	93·96 (62°)	99·30
80.....	95·67	95·42	.....	.....	99·07
100.....	94·59	94·31	.....	.....	98·84

Fig. 5.

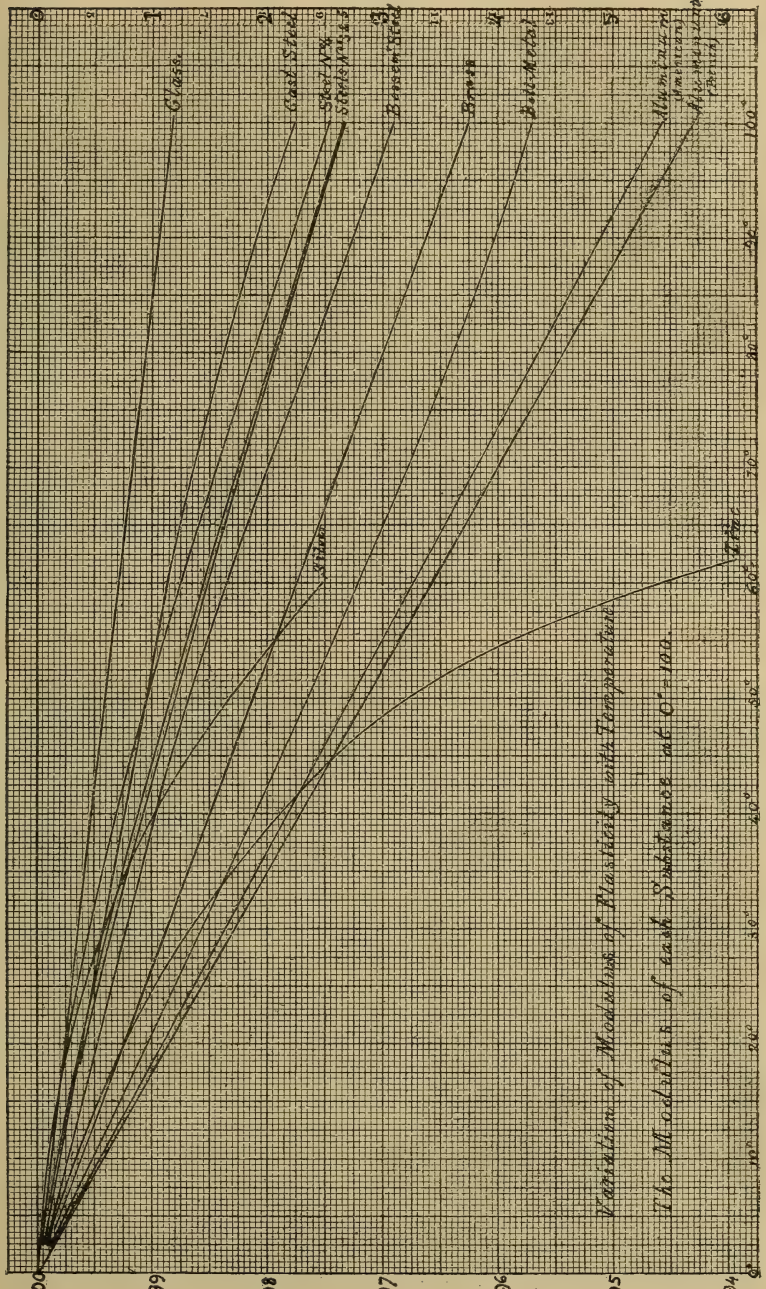

Glass
Cast Steel
Steel No. 1
Steel No. 2 & 3
Brass Steel
Brass
Bell Metal
Aluminum
Aluminum
Brass
Zinc
Variation of Modulus of Elasticity with Temperature
The Modulus of each Substance at 0° = 100.
100
99
98
97
96
95
94
0°
10°
20°
30°
40°
50°
60°
70°
80°
90°
100°

Fig. 6.

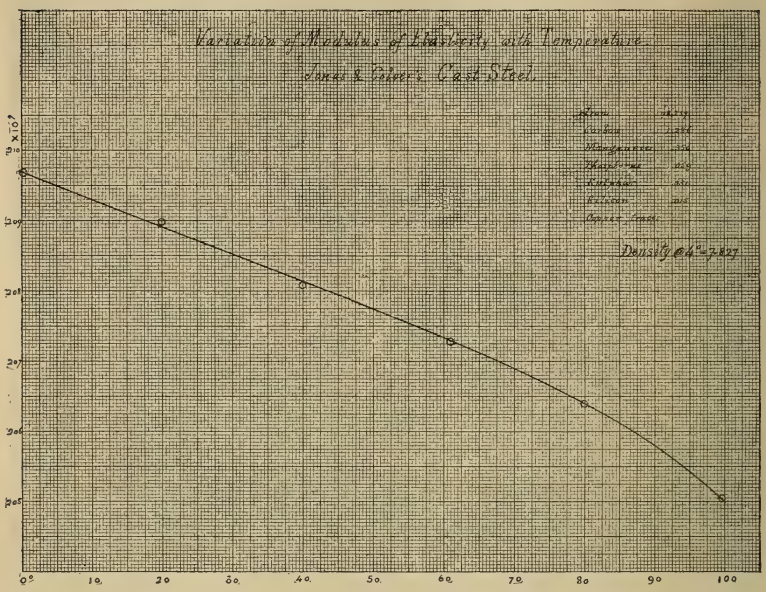

Variation of Modulus of Elasticity with Temperature.
Jones & Taber's Cast Steel.
Density @ 62° = 7.827
0°
10.
20
30.
40.
50.
60.
70.
80
90
100

Fig. 7.

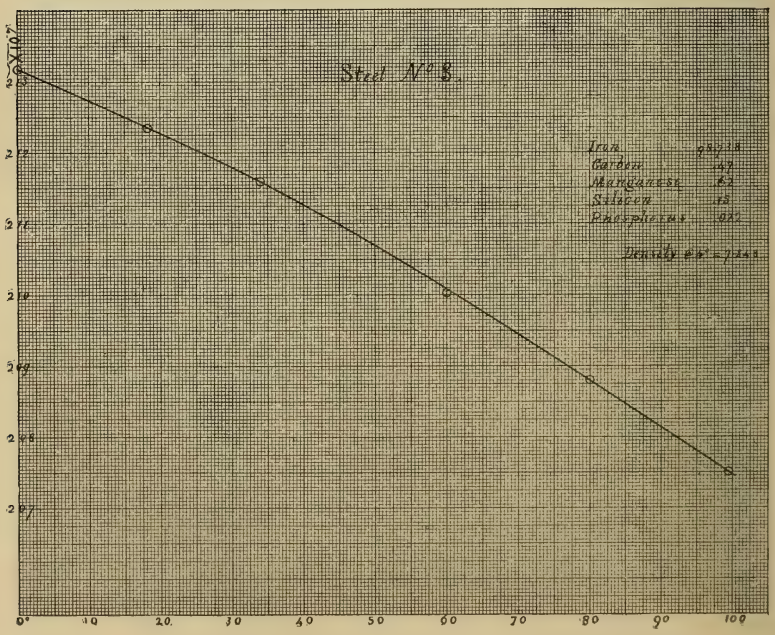

Steel No. 8.
Iron
Carbon
Manganese
Silicon
Phosphorus
Density @ 62° = 7.245
0°
10
20.
30
40
50
60
70
80
90
100

Fig. 8.

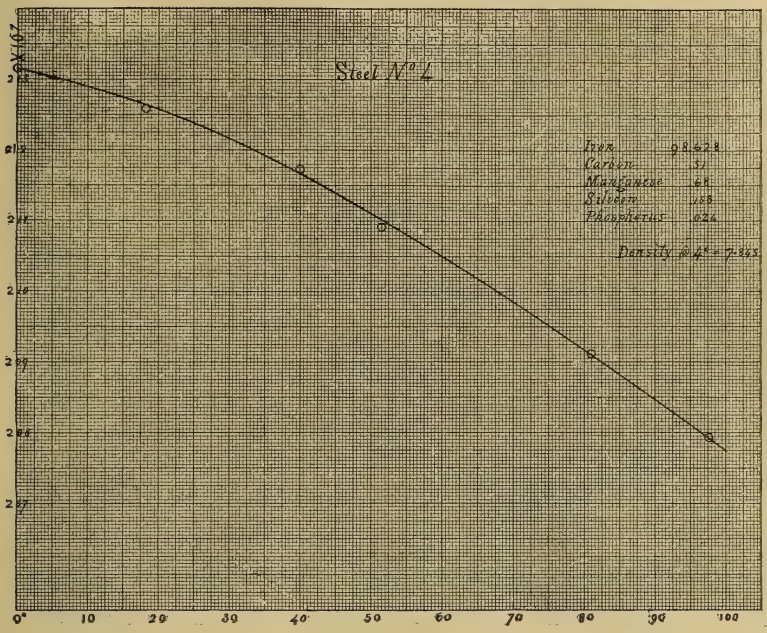

Steel N° 2
Iron 98.628
Carbon .51
Manganese .68
Silicon .188
Phosphorus .024
Density @ 4° = 7.845
0° 10 20 30 40 50 60 70 80 90 100

Fig. 9.

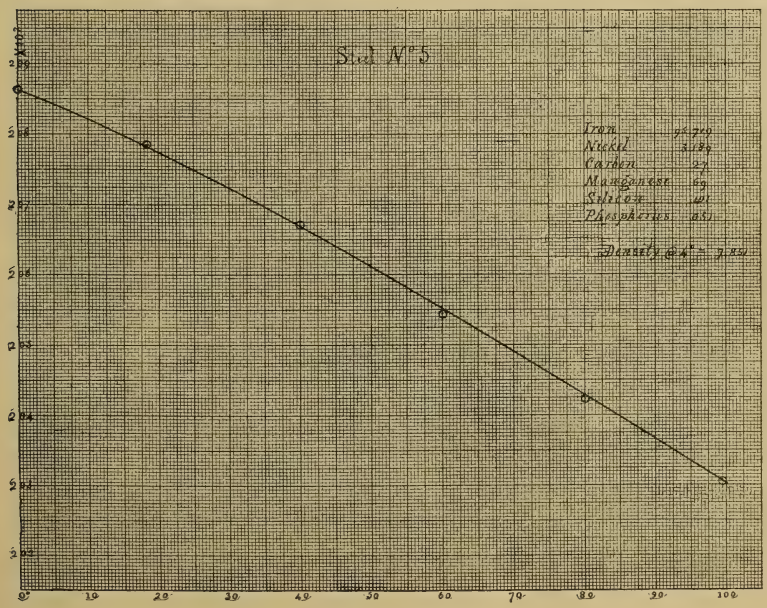

Steel N° 5
Iron 96.700
Nickel 3.184
Carbon .27
Manganese .69
Silicon .181
Phosphorus .051
Density @ 4° = 7.854
0° 10 20 30 40 50 60 70 80 90 100

Fig. 10.

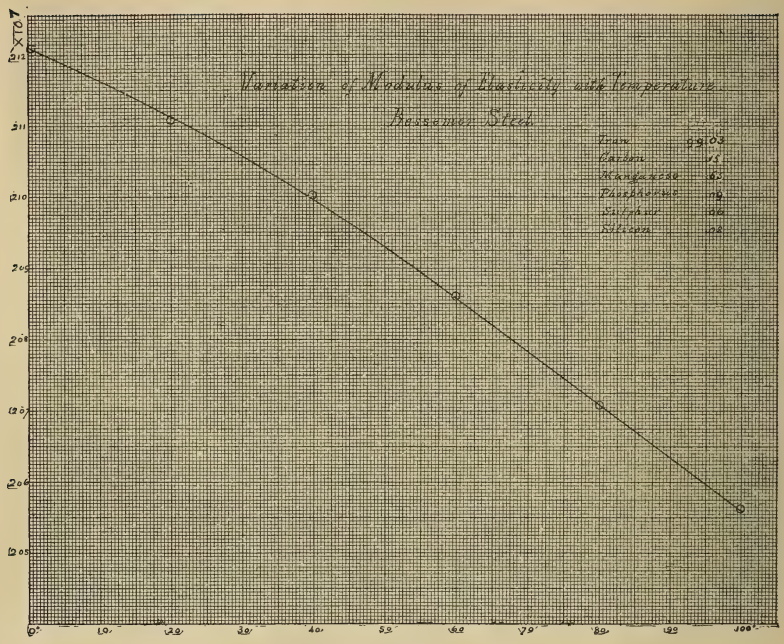

Variation of Modulus of Elasticity with Temperature
Bessemer Steel
Iron
Carbon
Manganese
Phosphorus
Sulphur
Silicon
0°
10
20
30
40
50
60
70
80
90
100

Fig. 11.

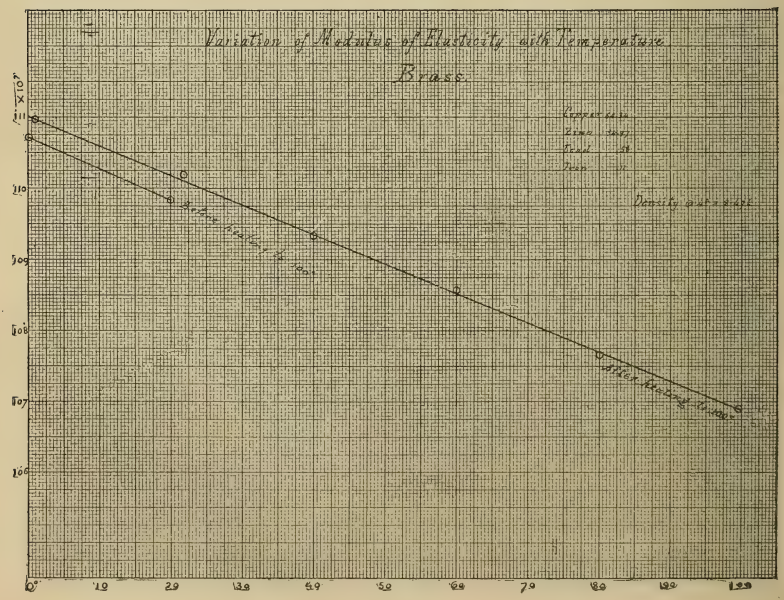

Variation of Modulus of Elasticity with Temperature
Brass
Before heating to 100°
After heating to 100°
0°
10
20
30
40
50
60
70
80
90
100

Fig. 12.

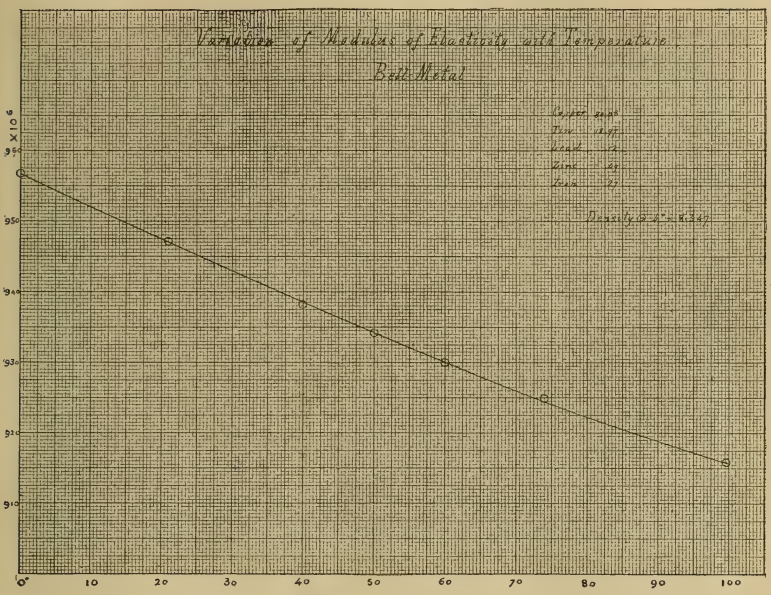

Variation of Modulus of Elasticity with Temperature
Bell Metal
Copper
Tin
Lead
Zinc
Iron
Density @ 4° = 8.347
X 10⁶
960
950
940
930
920
910
0°
10
20
30
40
50
60
70
80
90
100

Fig. 13.

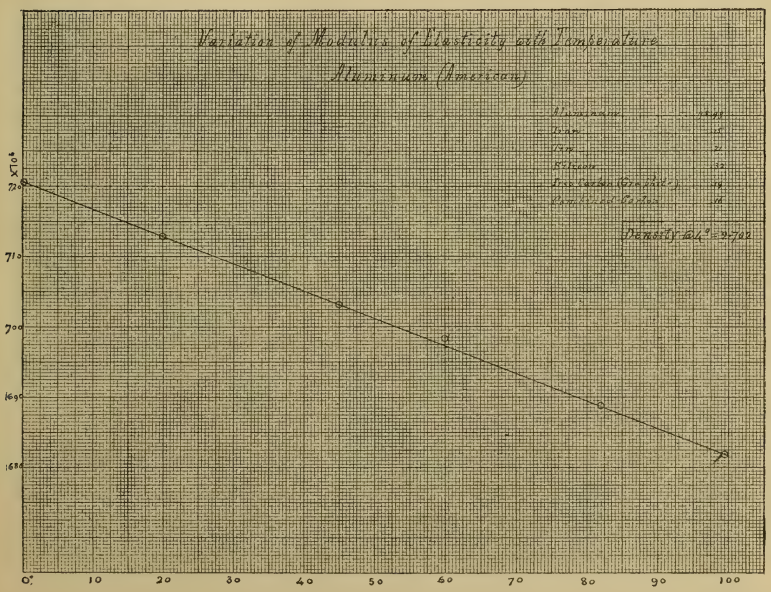

Variation of Modulus of Elasticity with Temperature
Aluminum (American)
Aluminum
Iron
Tin
Silicon
Iron Carbon (Graphite)
Combined Carbon
Density @ 4° = 2.702
X 10⁶
720
710
700
690
680
0°
10
20
30
40
50
60
70
80
90
100

Fig. 14.

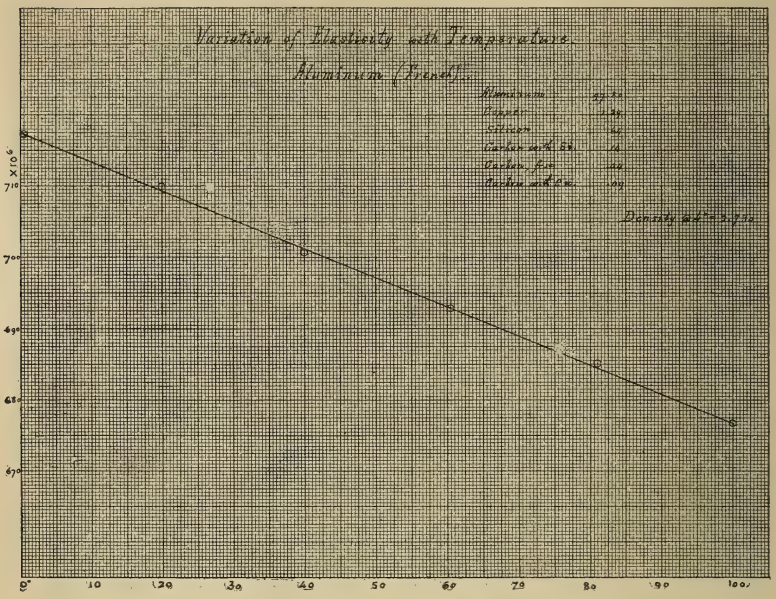

Variation of Elasticity with Temperature
Aluminum (French)
Aluminum
Copper
Silicon
Carbon with Si
Carbon, fine
Carbon with Cu
Density at 4° = 2.7
X10⁶
710
700
690
680
670
0°
10
20
30
40
50
60
70
80
90
100

Fig. 15.

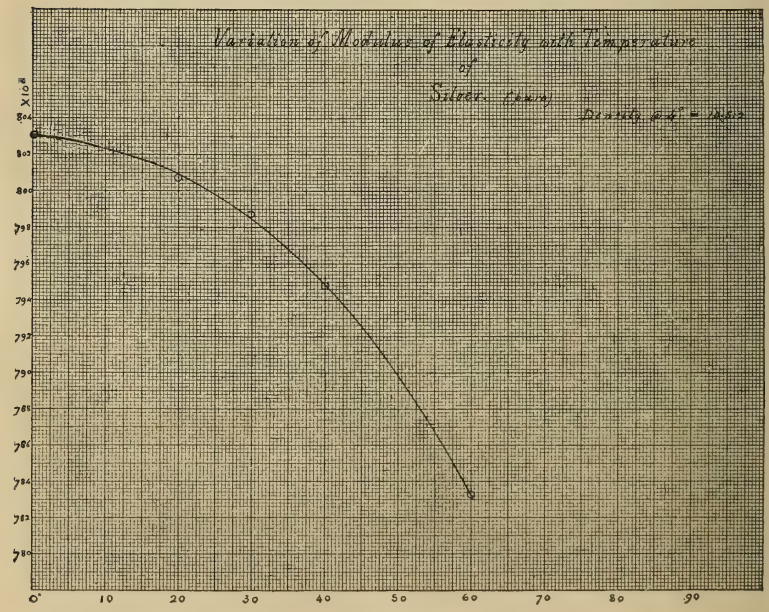

Variation of Modulus of Elasticity with Temperature
of
Silver
Density at 4° = 10.5
X10⁶
804
802
800
798
796
794
792
790
788
786
784
782
780
0°
10
20
30
40
50
60
70
80
90

Fig. 16.

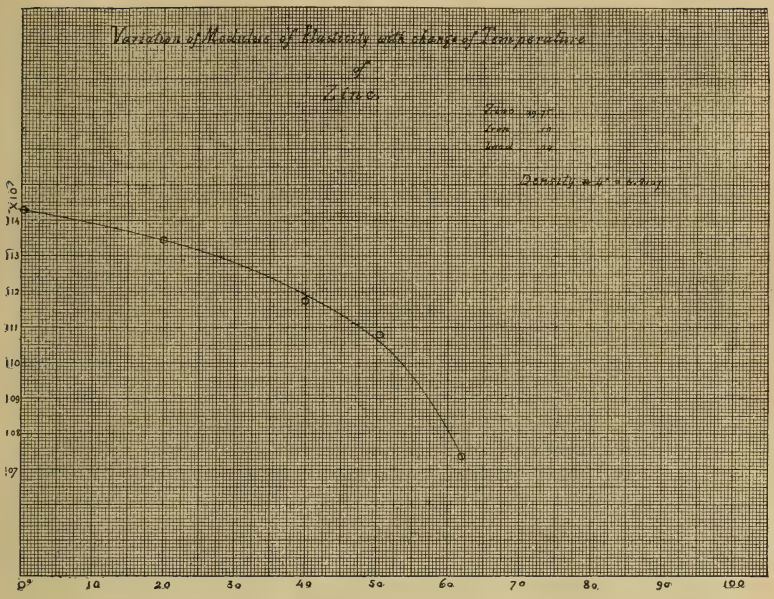
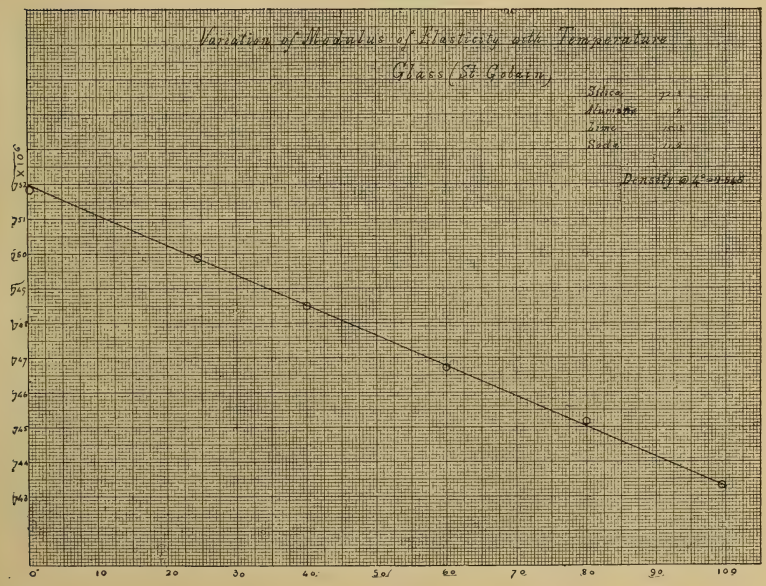

Variation of Modulus of Elasticity with change of Temperature
of
Zinc.
Density @ 4° = 6.9107
0°
10
20
30
40
50
60
70
80
90
100

Fig. 17.


Variation of Modulus of Elasticity with Temperature
Glass (St. Gobain)
Silica
Alumina
Lime
Soda
Density @ 4° = 0.2465
0°
10
20
30
40
50
60
70
80
90
100

Results obtained by other Experimenters on the change of the Modulus of Elasticity with change of Temperature.

I have found five researches on this subject.

Wertheim, 1844. *Ann. de Chim. et de Phys.*

IRON.

Modulus 5·2 per cent. greater at 100° than at 18°.
Modulus 19·1 per cent. less at 200° than at 100°.

IRON WIRE.

Modulus 4·9 per cent. greater at +10° than at −11°·6.
Modulus 7·42 per cent. greater at 100° than at 18°.

WIRE OF ENGLISH CAST-STEEL.

Modulus 23·23 per cent. greater at 100° than at 18°.
Modulus 9·46 per cent. less at 200° than at 100°.
Modulus at 200° is 11·57 per cent. higher than modulus at 18°.

STEEL WIRE TEMPERED TO BLUE.

Modulus 1·97 per cent. higher at +10° than at −10°.
Modulus 5·1 per cent. higher at 100° than at 18°.

CAST-STEEL.

Modulus 2·8 per cent. less at 100° than at 18°.
Modulus 5·73 per cent. less at 200° than at 100°.

SILVER.

Modulus 5 per cent. less at +10° than at −13°·8.
Modulus 1·87 per cent. greater at 100° than at 18°.
Modulus 12·87 per cent. less at 200° than at 100°.

COPPER.

Modulus 6·53 per cent. less at +10° than at −15°.
Modulus 6·58 per cent. less at 100° than at 18°.
Modulus 20 per cent. less at 200° than at 100°.

WIRE OF BERLIN BRASS (Cu=67·55, Zn=32·35).

Modulus 7·95 per cent. less at +11° than at −10°.

Kupffer, 1856. *Mem. de l'Acad. de St. Pétersb.*

Modulus of iron wire 5·5 per cent. less at 100° than at 0°.
Modulus of copper wire 8·2 per cent. less at 100° than at 0°.
Modulus of brass wire 3·9 per cent. less at 100° than at 0°.

Kohlrausch and Loomis, 1870. Pogg. *Ann.*
Modulus of iron wire 5 per cent. less at 100° than at 0°.
Modulus of copper 6 per cent. less at 100° than at 0°.
Brass 6·2 per cent. less at 100° than at 0°.

H. Tomlinson, 1887. Phil. Mag. xxiii.
Says, "my own experiments show that both the torsional and longitudinal elasticities of iron and steel are decreased by about $2\frac{1}{2}$ per cent. when the temperature is raised from 0° to 100°."

M. C. Noyes, 1895. The Physical Review.
Modulus of a piano wire of $\frac{4}{10}$ mm. diam. 5 per cent. less at 100° than at 0°.

The results of Wertheim's experiments giving an increase to the modulus, as the temperature rises, of iron, iron wire, wire of English cast-steel, steel wire drawn to blue, and silver, have not been confirmed in any instance by subsequent experiments; only for cast-steel *rod* and copper did he obtain a diminution of modulus for a rise of temperature from 18° to 100°. Yet he found that a *wire* of English cast-steel had a modulus 23 per cent. higher at 100° than at 18°.

On the Acoustical Properties of Aluminium.

The low density (2·7) of aluminium combined with a modulus of elasticity of only 712×10^6 render this metal easy to set in vibration; a transverse blow given to a bar of this metal causes it to vibrate with an amplitude of vibration greater than that which the same energy of blow gives to a similar bar of steel or of brass. This fact has given rise to the popular opinion that aluminium has sonorous properties greatly exceeding those of any other metal. This opinion is erroneous. If a bar of aluminium and a bar of brass having the same length and breadth and giving the same note, are struck transversely so that the bars have the same amplitude of vibration, the bars give equal initial intensity of sounds; but the bar of aluminium from its low density and because of its internal friction will vibrate less than one-third as long as the bar of brass. Thus, a bar of aluminium and a bar of brass of the same length and width and of such thickness that they gave the same note, SOL_4 of 768 v. d., were vibrated so that the sounds at the moment of the blows were, as near as could be judged, of the same intensity. The duration of the sound of the brass bar was 100 seconds; the sound of the aluminium bar lasted 30 seconds.

The readiness with which a bar of aluminium vibrates when acted on by aërial vibrations of the same frequency as those

given by the bar, gives one the means of making many charming experiments in which "sympathetic vibrations" come into play.

I here describe an experiment which I devised to show the interference of sound in a manner similar to analogous experiments in the case of light. The resonant box on which Kœnig mounts his UT_5 (1024 v. d.) fork is open at both ends and has a length of nearly a half wave of the sound of the fork. If this resonant box is held with its axis vertical, above an aluminium bar in tune with the vibrating fork, the bar does not enter into sympathetic vibration with the fork, because the sonorous pulses, on reaching the aluminium bar from the two openings of the resonant box, differ in phase by one half wave-length. But if the axis of the box is held parallel to the axis of the bar, then the sonorous waves reaching the bar have travelled over equal lengths from the openings at the ends of the box, and these waves conspire in their action and the aluminium bar enters into sympathetic vibration.

As this experiment is an interesting one I here give details as to the manner of making it. The bar of aluminium has a large surface, having a length of 17 cms. and a width of 5 cms. The two nodal lines, which are at a distance from the ends of the bar equal to $\frac{2}{9}$ths of its length, are drawn on the bar. The bar is supported under these nodal lines on threads stretched on a frame. This frame is of such a height that the under surface of the aluminium bar is 8·4 cms., or one quarter wave-length, above the surface of the table, so that the vibrations of the bar and those of the waves reflected from the table will act together. The upper surface of the bar is covered with a piece of thick cardboard, in which is cut a rectangular aperture, having for length the distance between the nodal lines and a width equal to that of the bar. As this piece of cardboard rests on supports which lift it a slight distance above the surface of the bar, the latter, when it vibrates, does not send to the ear the vibrations of the surfaces of the bar included between its nodal lines and its ends, which vibrations are opposed in phase to those given by the central area of the bar. Thus the sound emitted by the bar is much increased and the experiment rendered more delicate and improved in every way. I have found that the experiment succeeds best when the centre of the resonant box is held about 58 cms., or $7\frac{\lambda}{4}$ above the surface of the aluminium bar. This experiment works best in the open air, away from the action of sound-waves reflected from the walls and ceiling of a room.

The fact that aluminium gives, from a comparatively slight blow, a great initial vibration, and that its vibrations last for a short time, render this metal peculiarly well suited for the construction of those musical instruments formed of bars which are sounded by percussion and the duration of whose sounds is not desirable.

I had hopes that aluminium would prove to be a good substance out of which to make plates on which to form the acoustic figures of Chladni. Experiments have shown that aluminium is not suited to this purpose. I had plates of aluminium carefully cast, with 2 cms. of thickness. These plates were turned down on the face-plate of a lathe to thicknesses of 2 mm. and 3·8 mm. Three of these plates were quite homogeneous in elasticity, for the Chladni figures when obtained on them were symmetrical. Yet the Chladni figures were difficult to produce, because it is difficult to obtain a pure tone from an aluminium plate. The sound is generally more or less composite; therefore the plate in its vibration tends to form two or more figures at the same time, and the consequence is that either no figure is formed or one is given that is not sharply defined. One square plate of 30·8 cms. on the side and ·38 cm. thick, gave quite clearly the three following tones:—UT_2 (1), SOL_2 (2), and SOL_4 (3). Corresponding respectively to the Chladni figures of (1) two lines drawn between opposite points of the centre of sides of plate; (2) figure formed of the two diagonals drawn between the corners of plate; (3) figure similar to (1) but with corners of plate cut off by curved lines. Figure 3 corresponded so nearly to the sound of SOL_4 that a vibrating SOL_4 fork when held near the plate set the latter into vigorous vibrat on.

Another difficulty met with in using plates of aluminium for Chladni's figures is that sand, even when entirely free from salt and from the globular grains of wind-blown sand, does not move freely over a vibrating surface of aluminium, whether this surface has been polished or has been slightly tarnished and roughened by the action of alkali.

There is one serious objection to the use of aluminium in the construction of musical and acoustical instruments, and that is the great effect that change of temperature has upon its elasticity. If a bar of aluminium and a bar of cast-steel be tuned at a certain temperature to exact unison, a change from that temperature will affect the frequency of vibration of the aluminium bar $2\frac{1}{2}$ times as much as the same change of temperature will affect the bar of cast-steel.

XXIII. *On the Freezing-points of Dilute Solutions.* *By* W. Nernst *and* R. Abegg*.

THE lowering of the freezing-point of dilute solutions has been recently carefully investigated by two observers, Mr. Jones† and Mr. Loomis‡, but they have found very different values for the lowering in the case of non-electrolytes.

No reader of the two researches could fail to see that Mr. Loomis had worked with great care, and that Mr. Jones, on the other hand, had neglected some very obvious precautions.

Indeed we found that a very dangerous source of error, viz. the influence of the external temperature, which Loomis approximately avoided, made Mr. Jones's results in the case of non-electrolytes entirely worthless. At the same time we showed how the influence of the external temperature is to be computed, since we developed a mathematical-physical theory for its effect upon the freezing-point.

To our surprise Mr. Jones sought to defend his results by attacking our research in a most condemnatory manner. How far he is justified in this can be seen from the following brief observations.

Mr. Jones had nothing to say against the above-mentioned theory, except (page 386 of this Journal, 5th series, vol. xl.) that "the introduction of a correction-term when amounting to more than 20 per cent. would not tend to increase our confidence in the final results." Ought such an argument to be taken seriously when a very elementary acquaintance with physical measurements shows that exact determinations can be made in spite of large correction-terms, provided these are correctly computed. So long as Mr. Jones has not proved any error here he must discard his own results. In reference to the same point Mr. Jones remarks, "Their correction-term . . . . . appears to me to involve the assumption that K for solutions is the same as for water, which assumption is gratuitous and unallowable" [p. 385]. Evidently Mr. Jones has not correctly understood our theory. We have in no way assumed that K, the rate of dissolving (*Lösungsgeschwindigkeit*) solid substances in solutions and in water is the same; but, on the other hand, we have emphasized the fact of the great difference in the values for water and solutions, and we have computed them.

* Communicated by the Authors.

† *Zeitschr. f. physikal. Chemie*, xi. p. 529; and xii. p. 623.

‡ Wiedemann's *Annalen*, li. p. 500.

We had ourselves called attention to the fact that we had at our disposal thermometers reading only to $\frac{1}{100}°$ direct, and to $\frac{1}{1000}°$ by estimation, so that our determinations of the lowering of the freezing-points could for this reason be accurate to within only *one* to *two thousandths* of a degree. Mr. Jones's numbers, which read to $\frac{1}{10,000}°$, are, as we have shown, untrustworthy to within at least hundredths of a degree.

Mr. Jones then remarked that it was not "apparent" to him why our two series for NaCl in very dilute solutions should vary by 5 per cent. He neglected to remark that in reality such differences are limited to the most dilute solutions, and on that account lie within the limits of experimental errors given by us. Moreover in the second series for NaCl the greater values for the lowerings may be accounted for by the presence of larger quantities of the solid substance, since in this method larger quantities of undercooled liquid are introduced. In any case it is self-evident to every one who knows the elements of the computation of errors, that no conclusions can be drawn presupposing a greater degree of accuracy in our results than that given by us. Nevertheless in spite of this Mr. Jones concluded, from our results with ethyl alcohol, that a rise of the molecular lowering took place which was far within the experimental errors.

That we used substances of sufficient purity for the purpose we had in view, that is, substances whose possible impurities were absolutely unessential, we certainly did not especially mention. Every one knows that comparatively pure NaCl, ethyl alcohol, and cane-sugar are easy to obtain.

We refrain from criticizing the few new experiments of Mr. Jones, partly because we have not the least interest in carrying on a further controversy, partly because the criticism given in our earlier research would simply need to be repeated word for word*. Moreover Mr. Jones admits indeed the influence of the external temperature (page 389), and in that the existence of a great source of error which he formerly entirely neglected.

In conclusion a purely personal remark. Mr. Jones spoke (page 385) of "the unusual lack of courtesy." That Mr. Jones

* For instance, Mr. Jones maintained that the influence of his "gentle" stirring was imperceptible, *i. e.* determined from his feeling instead of quantitatively. That the thermometer-reading remained unchanged is no proof for his assertion according to what our theory as well as experiments showed.

introduces no proof for this assertion no one could wonder from the above. But that each one may judge for himself we insert the following passage from a paper in which we refer to the results of other authors, and to which alone Mr. Jones's severe reproach could relate :—

"It is not without interest to test the earlier values for this source of error through which partly they have been rendered so considerably inaccurate: for instance, for the molecular lowering of the freezing-point (computed from Raoult) for dilute (about 1 per cent.) cane-sugar solutions the following values were found by

Arrhenius.	Raoult.	Jones.	Loomis.
2·02	2·07	2·18	1·81

but we find, as was mentioned, 1·86 (uncorrected from 1·6 to 2·1). Arrhenius used the usual Beckmann apparatus with quite an energetic cooling-mixture: this explains why his value is considerably too large. Raoult gives more experimental details, and from these one can conclude that he kept the cooling-bath about 3° below the freezing-point of the solution. This investigator seems to have appreciated the essential importance of the cooling-bath, for he says, 'If the influence of the cooling-bath upon the temperature of the liquid at the moment of freezing is not nought, yet it is indeed the same in the experiments to be compared and vanishes from the differences, so that the lowering of the freezing-point is not influenced by it.' The assumption which Raoult here makes is identical with the supposition that K has the same value for pure water and for solutions, which is certainly not the case with cane-sugar according to our experience. His values must accordingly be considerably too high. Still more erroneous are the values of Jones, who used a cooling-mixture of ice and salt, therefore an exceedingly strong cooler. If Jones had used a single time in the case of cane-sugar another cooling-mixture, or even only a freezing-vessel of other dimensions, he would have observed with the great accuracy with which he read the apparent freezing-points the influence of these factors, and would have refrained from publishing his essentially accidental numbers. [Note:—"Since the corrections for Jones's values amount to hundredths of a degree, if the accuracy is to be increased to within 0·0001° with the same external temperature and rate of stirring only by using a *greater volume of the liquid*, in order to reduce K (*cf.* equations (3) and (7) to its hundredth part, it is necessary to increase the linear dimensions of the

freezing-vessel ($\frac{\text{volume}}{\text{surface}}$, *cf.* page 683) an hundredfold. The accuracy for which Jones strives would have been attained therefore, *cæteris paribus*, by using a vessel not of a 1 litre's capacity but of that of a million litres!"] Loomis undoubtedly worked with precaution . . . . ."

Göttingen, November 1895.

XXIV. *On Resultant Tones.*
By Professor J. D. Everett, *F.R.S.**

1. THE received theory of the generation of resultant tones in the ear may be summed up with rough accuracy as follows †:—

The drumskin being pulled inwards by the end of the handle of the "hammer," which is attached to its centre, offers unsymmetrical resistance to displacement in the inward and outward directions, so that the equation for the movement of its centre in free vibration would be

$$-\ddot{x}=\omega^2x+\alpha x^2,$$

or to a closer approximation

$$-\ddot{x}=\omega^2x+\alpha x^2+\beta x^3,$$

ω, α, β being constants.

The value of ω when the second is the unit of time is less than 60, hence the frequency of free vibration, being $\omega/2\pi$, is less than 10.

When two harmonic forces of frequencies m and n act upon the drumskin, they produce, in addition to their own tones, certain "resultant tones," the one of largest amplitude being the "first difference-tone," of frequency $n-m$. The next largest amplitude is about $\left(\frac{n-m}{n+m}\right)^2$ of this, and belongs to the "first summation-tone," of frequency $n+m$. Neither of these tones will be audible unless the excursion x is so large that αx^2 is sensible in comparison with ω^2x. There will also be difference-tones of frequencies $2m-n$ and $2n-m$, but neither of these will be audible unless βx^3 is sensible compared with ω^2x.

2. This theory does not appear sufficient to account for the loud resultant tones which are sometimes heard. When a Helmholtz siren is driven rapidly, with the rows of holes 9,

* Communicated by the Physical Society: read January 24, 1896.

† See *Tonempfindungen*, Appendix xii.; Rayleigh on Sound, art. 68; Bosanquet, Proc. Phys. Soc. vol. iv. p. 240, arts. 57–69.

12, 15, and 18 open, the resultant tone of frequency 3 on the same scale is the most prominent tone in the whole volume of sound.

3. The view which I desire to put forward is closely connected with the well-known theorem of Fourier, that every periodic variation can be resolved in one definite way into harmonic constituents, whose periods must be included in the list 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c., where 1 denotes the period of the given variation itself. The corresponding frequencies will be as 1, 2, 3, 4, &c.

In the majority of cases, when this analysis is carried out, the fundamental constituent, represented by 1 in the above lists, is the largest or among the largest; but in the case of a variation compounded of two simple tones with frequencies in the ratio of two integers, neither being a multiple of the other, the fundamental will be absent, and the Fourier series will consist of only two terms, which in the language of acoustics are harmonics of the fundamental.

4. Clearness of thought is facilitated in these matters by supposing a curve to be drawn, in which horizontal distance represents time and vertical distance represents the quantity whose variation is in question. Since the variation is periodic, the curve will consist of repetitions of one and the same form, in other words it will consist of a number of equal and similar waves, and the wave-length stands for the complete period of the variation.

The point on which I wish to insist is, that if such a curve representing the superposition of two harmonics of the fundamental is in the first instance very accurately drawn, and is then inaccurately copied in such a way that all successive waves are treated alike, the inaccuracy is morally certain to introduce the fundamental.

Let y denote any ordinate, and θ the time (or abscissa) expressed in such a unit that 2π is the numerical value of the wave-length or period; then the amplitude of the fundamental is the square root of A^2+B^2, where

$$A=\frac{1}{\pi}\int_0^{2\pi} y\cos\theta d\theta, \qquad B=\frac{1}{\pi}\int_0^{2\pi} y\sin\theta d\theta.$$

In the original curve both A and B vanish. Let y' be the altered value of y in the new and inaccurate curve, and let z denote $y'-y$; then we have, between the above limits,

$$\int y'\cos\theta d\theta=\int y\cos\theta d\theta+\int z\cos\theta d\theta=\int z\cos\theta d\theta,$$

since $\int y\cos\theta d\theta$ is zero.

But z may be regarded as a random magnitude, hence it is infinitely improbable that its different values exactly fulfil the condition $\int z \cos \theta d\theta = 0$. Therefore the new A is finite; and similar reasoning shows that the new B is finite.

If all the ordinates were changed in one uniform ratio, A and B would remain zero, and no new constituent would be introduced; but any other change, unless specially planned to avoid introducing A and B, is practically certain to give $A^2 + B^2$ a finite value.

5. I maintain that such a change is effected in the form of sonorous waves during their transmission from the external air to the sensory fibres by which we distinguish pitch. The waves are transmitted first from the air to the drumskin, then through two successive levers, the hammer and anvil, to the head of the stirrup, while the foot of the stirrup sits upon the membrane of the oval window, and passes on the vibrations through the membrane to the liquid on the other side in which the sensory fibres are immersed. The levers turn upon ligamentous fulcra, and have rubbing contact with each other. The wave-form cannot run the gauntlet of all these transmissions without being to some extent knocked out of shape. It is much as if a very accurately drawn curve, representing the original wave-form, were copied and recopied, five times in succession, by five different pantagraphs not very firm in their connexions. The final copy so obtained would be sure to exhibit sensible departures from the original.

6. It appears likely that the chief seat of the disturbing actions in the ear is the junction of the hammer and anvil. "When the drumskin with the hammer is driven outwards, the anvil is not obliged to follow it. The interlocking teeth of the surfaces of the joint then separate, and the surfaces glide over each other with very little friction"*. Such action is likely to introduce derangement, increasing generally with the excursions of the drumskin, but not expressible as a definite function of the ordinates of the wave-curve. For a given pressure on the drumskin, the pressure communicated to the liquid in the cochlea will vary according to the relative position and relative motion of the two portions of this joint.

7. The principal resultant tone due to these actions is likely to be that which corresponds to the complete period of the actions, in other words the highest common fundamental of the two primaries, or what old writers called the "grave harmonic." This will not be the same as the "first difference-tone" unless the ratio of the two primaries is of the form

* Ellis's 'Helmholtz,' p. 133, 2nd edition.

$m : m+1$; and I have satisfied myself, both by my own trials and by a study of Kœnig's experimental results, that when the difference-tone and the common fundamental are not identical, the common fundamental is usually the predominant, and often the only audible resultant tone. (See Appendix.)

8. The common fundamental is, however, not the only resultant tone that can be thus accounted for. Similar reasoning to that employed in reference to A and B suffices to explain the introduction of any or all of the harmonics of the fundamental; but it is to be expected, from the analogy of ordinary experience in harmonic analysis, that the successive constituents will usually be smaller and smaller as we advance in the series. The octave is likely to be the largest of them; and Kœnig found, in several experiments with primaries in the ratio of 3 : 5, that both the fundamental 1 and its octave 2 were distinctly heard as resultant tones.

9. The following investigation bears on the relation between beats and resultant tones. The expression

$$a \cos m\theta + b \cos n\theta$$

can be reduced to the form

$$\mathrm{A} \cos\left(\frac{m+n}{2}\theta - \epsilon\right),$$

where A and ϵ are given by

$$\mathrm{A}^2 = a^2 + b^2 + 2ab \cos (n-m)\theta,$$

$$\tan \epsilon = \frac{a-b}{a+b} \tan \frac{n-m}{2}\theta,$$

and the beating together of two tones not differing much in pitch is explained by the fact, definitely expressed in these formulæ, that the whole effect may be regarded as a succession of waves with gradually varying amplitude. The frequency of the beats is the frequency of the maxima of A^2, and is the difference of m and n.

We have ascribed resultant tones to alterations made in the wave-form by the action of the ear, such alterations being in general largest at those points at which the excursions of the drumskin are largest. These excursions are measured by $\pm\mathrm{A}$, and the above investigation shows that their maxima have a frequency corresponding to the difference-tone. This is true whether m and n are commensurable or incommensurable. If they are commensurable, their greatest common

measure will be the frequency of the complete cycle of change. This cycle will not be conspicuous in the curve if the ratio of $n-m$ to $\frac{1}{2}(n+m)$ is very small, but will assert itself more and more as this ratio increases; and these remarks will apply to the comparison of the fundamental with the first difference-tone.

10. If the ear is able so to alter the form of waves impinging upon it as to generate resultant tones, it is natural to seek for some instance of a similar action in external bodies. A violin is very susceptible, like the ear, to vibrations of all frequencies between wide limits, and the sound-post serves, like the ossicles of the ear, to transmit vibrations from one portion to another. It is easy to produce resultant tones by bowing two strings of a violin together. For example, in the ordinary process of tuning, when the fourth and third strings with frequencies as 2 : 3 are combined, the resultant tone 1 is very observable if attention be directed towards it. But more striking effects are obtained when the resultant is at a larger interval from the primaries. The major sixth 3 : 5, the major second 8 : 9, and the minor seventh 5 : 9, are suitable intervals for calling out the fundamental 1, the strings employed being either the first and second or the second and third. The deep resultant tone thus obtained can not only be heard by the ear but felt as a tremor by the hand which holds the instrument. This is clear evidence of its objective existence, and I have succeeded in confirming the fact by means of a Helmholtz resonance-globe, the largest of the ordinary set, responding to C of 128 vibrations. When held with the edge of its mouth resting against the side of the violin, it responds to the combination C of 256 and G of 384 on the 4th and 3rd strings, or to the combination C of 512 and E of 640 on the 2nd and 1st; or, still better, to the 3rd and 2nd open strings each flattened one note, so as to be C of 256 and G of 384. Here, then, we have distinct evidence that the violin possesses the power which I have ascribed to the ear—the power of manufacturing the fundamental when the two primaries are supplied.

11. Sir John Herschel, in his treatise on Sound (Enc. Met. arts. 238, 239), mentions the fact that the common fundamental can be called out by sounding two or more of its harmonics on very accurately tuned strings or pipes, and says that the effect cannot be obtained from a pianoforte tuned in the ordinary way, because the intervals are tempered. I find, however, on trying the experiment with an upright Broadwood of date about 1860, that C of 64 is easily called out by simultaneously striking eight or ten of its harmonics;

and the effect is greatly enhanced if the key of C 64 is held down. In the latter case its note continues to be heard for a long time after the keys which were struck are released. From these experiments it appears probable that the sounding-board of a piano possesses the same property which we have proved to exist in the violin.

12. I now come to the explanation of the experiments of Professor Rücker and Mr. Edser (Proc. Phys. Soc. vol. xiii. p. 412, Phil. Mag. 1895, xxxix. p. 341). They were made with a Helmholtz siren, and in each instance the two primaries were produced in the same box, sometimes the upper and sometimes the lower box. The following explanation is a development of suggestions contained in Appendix xvi. of the *Tonempfindungen.*

The rate of escape of air from the box containing the two rows of holes which are employed may as a first approximation be assumed to be jointly proportional, at each instant, to the aperture for escape and the differential pressure which produces the escape. Again, this differential pressure may be regarded as the algebraic sum of two terms, one of them constant, and representing its average value, while the other represents the difference from the average due to the varying amount of the aperture from instant to instant. As a first approximation, equal increments of aperture must be regarded as producing equal decrements of pressure, so that the variable term will be proportional (with reversed sign) to the excess of the aperture above its mean value. This excess (defect being counted negative) will be a periodic function of the time, and if the ratio of the two primaries in lowest terms be $m:n$, the frequency for the complete period will be represented on the same scale by 1. In other words it will be the period of their common fundamental.

Let the aperture at time t be expressed in a Fourier series, θ being put for $2\pi t/\mathrm{T}$, where T is the complete period; and let the variable part of the expression be denoted by $f(\theta)$, while a_0 denotes the mean aperture, so that the aperture at time t is $a_0+f(\theta)$. We shall have

$$f(\theta)=\mathrm{A}\sin\theta+\ldots+a_1\sin(m\theta+\epsilon_1)+\ldots+b_1\sin(n\theta+\epsilon_2)+\ldots$$

The largest amplitudes will be a_1 and b_1 corresponding to the two primaries; but A, which corresponds to the fundamental, is likely to be sensible.

The pressure at time t is proportional to

$$\mathrm{C}-f(\theta),$$

C being a constant; and the aperture is

$$a_0+f(\theta).$$

Hence the rate of escape is proportional to

$$Ca_0 + (C - a_0)f(\theta) - \{f(\theta)\}^2.$$

a_0 is comparable with the maximum value of $f(\theta)$, and C is much greater; hence a_0 may be neglected in comparison with C.

Developing $\{(f(\theta)\}^2$, we shall obtain a term

$$2a_1b_1 \sin(m\theta + \epsilon_1) \sin(n\theta + \epsilon_2)$$

$$= a_1b_1[\cos\{(n-m)\theta + \epsilon_2 - \epsilon_1\} - \cos\{(n+m)\theta + \epsilon_2 + \epsilon_1\}],$$

representing a difference-tone and a summation-tone. From $(C - a_0)f(\theta)$ we have the common fundamental

$$(C - a_0)A \sin\theta, \text{ or } CA \sin\theta,$$

and the two primaries

$$Ca_1 \sin(m\theta + \epsilon_1), \quad Cb_1 \sin(n\theta + \epsilon_2).$$

Suppose for simplicity that $a_1 = b_1$, then, taking the amplitude of each of the primaries as 1, the amplitude of the common fundamental will be A/a_1, and the amplitudes of the summation-tone and difference-tone will each be a_1/C.

When $n - m = 1$, the difference-tone coincides with the fundamental, and their joint amplitude may be taken as the square root of the sum of the squares of a_1/C and A/a_1.

13. Professor Rücker and Mr. Edser in experiments i. and ii. obtained the difference-tone 64 from five distinct combinations of primaries,

256 & 320, 192 & 256, 320 & 384, 51·2 & 115·2, 96 & 160,

their ratios being

4 : 5, 3 : 4, 5 : 6, 4 : 9, 3 : 5.

The second combination appears to have given a stronger effect than either the first or the third; whence it would appear that low frequencies are favourable to strong effects. Nevertheless the fourth combination is mentioned as giving a rather feebler effect than any one of the first three. This confirms our conclusion that the difference-tone is weaker when it is distinct from the common fundamental than when it coincides with it.

Experiment iii. was directed to testing for the presence of the resultant 64 when the primaries were 256 and 576. which are as 4 : 9. Their common fundamental is 64, and it could not be detected. This may have been because the pitch 576 was too high to give a good effect. Or the failure may be an indication that A/a_1 is decidedly smaller than a_1/C. It would be interesting to repeat the experiment, employing 192 and 320 as the primaries.

14. Near the end of chapter vii. of the *Tonempfindungen* Helmholtz makes prominent mention of the slipping of the hammer on the anvil as an important cause of resultant tones, and appears to regard it as exemplifying his mathematical formula for the restoring force as a function of the displacement. But it is clear that if the hammer, which holds the drumskin, is liable to shift in its supports, the restoring force cannot be a mere function of the displacement, but must also depend on the relative position and relative velocity of the hammer and anvil at the moment considered. I accept all the consequences which Helmholtz deduces in the passage in question from this slipping, including its application to explain first difference-tones ; but I regard these consequences as lying outside the range of his general mathematical formulæ as given in Appendix xii.

15. To sum up my objections to the received mathematical theory of resultant tones :—

First. It assumes that the reaction of the drumskin against the air is a definite function of the displacement of the drumskin from a certain fixed position, whereas this reaction depends also on the position and motion of the further end of the hammer at the time.

Secondly. Even if the vibrations of the drumskin were in accordance with the received formulæ, there is plenty of scope for the introduction of additional constituents on the road from the drumskin to the liquid in the cochlea. The auditory ossicles, with their ligamentous supports and attachments, probably serve to protect the oval window of the cochlea against shocks and jars, and to smooth down asperities in the wave-form, thus mitigating the harshness of sounds and rendering them more musical. The changes thus introduced are very unlikely to fulfil the special conditions required for the vanishing of the common fundamental.

Thirdly. The received theory makes the common fundamental, when not coincident with the first difference-tone, depend on a term involving the cube or some higher power of the displacement. When the primaries are as 3 : 5, the fundamental 1 comes in as $2m-n$, and depends on the cube of the displacement. When they are as 4 : 11, the tone 1, which Kœnig found to be the loudest resultant, is $3m-n$, and depends on the fourth power. When they are in the ratio 4 : 15, as in Kœnig's experiment with the simple tones ut_5 and si_6, the common fundamental ut_3, which was the only resultant tone heard, is $4m-n$, and depends on the fifth power of the displacement ; the first difference-tone, which depends on the second power and should in theory be the

loudest, being inaudible. This is surely a *reductio ad absurdum* of the received theory.

I do not wish to be understood as denying that the theory has any basis of truth. My contention is that the actions to which it is truly applicable play only a subordinate part in the production of resultant tones.

Appendix.

Examples selected from Kœnig's *Expériences d'Acoustique*, pp. 103 and 104, illustrating the production of the common fundamental. The "single vibrations" of the original are here reduced to double or complete vibrations :—

ut_5 and si_5, which are as 8 : 15, gave only ut_2.

ut_5 and 2816, which are as 4 : 11, gave ut_3 corresponding to 1 louder than any other tone.

ut_5 and si_6, which are as 4 : 15, gave no audible tone but ut_3.

ut_5 and 3968, which are as 8 : 31, gave no audible tone but ut_2.

ut_6 and 3584, which are as 4 : 7, gave ut_4 more distinct than the difference-tone sol_5.

ut_6 and si_6, which are as 8 : 15, gave ut_3 distinct, the difference-tone 7 being inaudible.

ut_6 and 3968, which are 16 : 31, gave ut_2 only.

ut_6 and 4032, which are 32 : 63, gave ut_1 only.

XXV. *The Compound Law of Error.*

By Professor F. Y. Edgeworth, *M.A., D.C.L.**

THE compound law of error is an extension to the case of several dimensions of the simple law for the frequency with which a quantity of one dimension (x) tends to assume each particular value. A first approximation to the compound law has been obtained by several writers independently, —by Mr. De Forest, in the 'Analyst' for 1881; by the present writer, in the Philosophical Magazine for December 1892; and by Mr. S. H. Burbury, in the same Journal for January 1894. I propose here to employ the method of partial differential equations explained in a preceding paper † to verify the first approximation, and to discover a second approximation, to the compound law.

To begin with the case of two dimensions: let Q be the

* Communicated by the Author.

† "On the Asymmetrical Probability-Curve," Phil. Mag. February 1896.

sum (or more generally an expansible function *) of a number of elements ζ_1 ζ_2, &c., each of which, being a function of two variables x and y, assumes any particular system of values according to any law of frequency $\zeta_i = f_i(x, y)$; the functions f being in general different for different elements. If each of these functions is referred to its centre of gravity at origin, and expanded in powers of x and y, it appears, by parity of reasoning with that employed in the case of the simple law, that for a first approximation we need take account only of terms of the second order. Integrate between extreme limits of $x^2 f(xy) dx\, dy$ for each element; and let the sum of all these integrals be k. Also let

$$l = \Sigma \iint xy f_i(xy) dx\, dy,$$

$$m = \Sigma \iint y^2 f_i(xy) dx\, dy;$$

the integration extending between the extreme limits of each element, and the summation over all the elements. Then z, the sought function which is to express the frequency of Q, will be of the form

$$z = \Phi(x, y;\ k, l, m) \dagger.$$

This expression may be simplified by transforming the axes to new ones making an angle θ with the old ones, such that the new l vanishes. This will be effected if we put $\tan 2\theta = 2l \div (k - m)$ ‡. Thus we may write with sufficient generality:—

$$z = \Phi(x, y;\ k, m).$$

By superposing a new element after the analogy of the

* *Cf.* Phil. Mag. 1892, xxxiv. p. 431 *et seq.*

† I use a semicolon to separate the variables (x and y) from the constants (k, l, m).

‡ Put

$$x = X \cos\theta - Y \sin\theta,$$

$$y = X \sin\theta + Y \cos\theta.$$

The new l

$$= \Sigma \iint XY f_1 dX\, dY;$$

where f_1 is what $f_i(xy)$ becomes when for x and y are substituted their values in X and Y; each element is integrated between extreme limits, and all the integrals are summed. Transforming back to the old axes we have for the new l

$$\Sigma \iint f_i(xy) \left(\tfrac{1}{2}(y^2 - x^2) \sin 2\theta \pm xy \cos 2\theta\right) dx\, dy = \tfrac{1}{2}(m - k) \sin 2\theta + l \cos 2\theta;$$

which becomes null when

$$\tan 2\theta = 2l \div (k - m).$$

simple case* we obtain the differential equations

$$\frac{dz}{dk}=\frac{1}{2}\frac{d_2z}{dx_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{dz}{dm}=\frac{1}{2}\frac{d_2z}{dy^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Other differential equations are obtained by supposing the units of x and y altered; substituting for x and y, $x(1+\alpha)$ and $y(1+\beta)$ respectively. The expression for z thus transformed must be multiplied by $(1+\alpha)(1+\beta)$; since the measure of the solid contents of the parallelopiped intercepted between the surface, the plane of x_1y, and any two fixed adjacent points in that plane will be increased in that proportion. Thus

$$z=(1+\alpha)(1+\beta)\ \Phi(x, y;\ k, m).$$

Regarding α and β as infinitesimal, expanding and neglecting higher terms, we have

$$z+x\frac{dz}{dx}+\ 2k\frac{dz}{dk}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$z+y\frac{dz}{dy}+\ 2m\frac{dz}{dm}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From (3) and (4) we have

$$z=\frac{1}{\sqrt{k}}\phi\left(\frac{x}{\sqrt{k}}, \frac{l}{\sqrt{k}}\right)=\frac{1}{\sqrt{m}}\psi\left(\frac{y}{\sqrt{m}}, \frac{l}{\sqrt{m}}\right);$$

where ϕ and ψ are arbitrary functions. Whence

$$z=\frac{1}{\sqrt{km}}\chi\left(\frac{x}{\sqrt{m}}, \frac{y}{\sqrt{m}}\right),$$

where χ is an arbitrary function. The form of χ is restricted by the condition that its value is the same for positive and negative values of x and y, the surface being symmetrical about a vertical plane through each axis. For as we take account only of the second powers in the expansion of each element, we might replace the given system of elements by a new system of symmetrical functions having each the same centre of gravity and mean square of error as the old one†.

* See the preceding article, Phil. Mag. 1896, xli. p. 90.

† This does not mean that the given elements must be symmetrical, as is sometimes carelessly said with reference to the simple law of error. The given elements may have any degree of asymmetry, provided that their number is correspondingly great.

And a compound of symmetrical elements must itself be symmetrical. We have, therefore,

$$z=\frac{1}{\sqrt{km}}\chi\left(\frac{x^2}{k},\frac{y^2}{m}\right). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

To the five equations which have been stated there is to be added the condition that the integral of $z\,dx\,dy$ between extreme limits $=1$.

To solve this system: substitute in (3) and (4) the values of $\frac{dz}{dk}$ and $\frac{dz}{dm}$ given in (1) and (2) respectively. We have, then,

$$z+x\frac{dz}{dx}+k\frac{d_2z}{dx^2}=0, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$z+y\frac{dz}{dy}+m\frac{d_2z}{dy^2}=0. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Integrating (6) with respect to x, and (7) with respect to y, we have

$$zx+k\frac{dz}{dx}=\phi(y), \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$zy+m\frac{dz}{dy}=\psi(x)\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where ϕ and ψ are arbitrary functions.

Both these functions reduce to zero; as may thus be proved:—From (5) it appears that when $x=0$, $\frac{dz}{dx}$ also $=0$, whatever the value of y. If, then, we put $x=0$, the left side of equation (8) vanishes for all values of y. Therefore the right side of the equation vanishes for all values of y. Therefore $\phi(y)$ is identical with zero. By parity $\psi(y)$ is null.

By equation (8) thus reduced we have

$$z=\Phi(y)\times e^{-\frac{x^2}{2k}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$z=\Psi(x)e^{-\frac{y^2}{2m}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Identifying the right-hand members of (10) and (11) we have

$$z=Ce^{-\frac{x^2}{2k}-\frac{y^2}{2m}}, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where C is a constant; which is found to be $\frac{1}{2\pi\sqrt{km}}$ from the

condition that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\,dx\,dy = 1.$$

Transforming back from the *principal* axes which we have employed*, we find for the general expression

$$z = \frac{1}{2\pi\sqrt{km-l^2}} e^{\frac{-(mx^2-2lxy+ky^2)}{2(km-l^2)}}.$$

By parity of reasoning we obtain as the general form for the law of error relating to any number of variables x_1, x_2, x_3, &c.,

$$z = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\Delta}} e^{\frac{-K_1x_1^2+2L_{12}x_1x_2+2L_{13}x_1x_3+K_{22}x_2^2+\&c.}{2\Delta}};$$

where Δ is the determinant

$$\begin{vmatrix} k_1 & l_{12} & l_{13}\ldots \\ l_{21} & k_2 & l_{23}\ldots \\ l_{31} & l_{32} & k_3\ldots \\ \vdots & \vdots & \vdots \end{vmatrix}$$

$$k_1 = \Sigma\iiint\ldots\zeta_i x_1^2\,dx_1\,dx_2\,dx_3\ldots.$$

$$k_2 = \Sigma\iiint\ldots\zeta_i x_2^2\,dx_1\,dx_2\,dx_3\ldots.$$

$$\vdots$$

$$l_{12} = \Sigma\iiint\ldots\zeta_i x_1 x_2\,dx_1\,dx_2\,dx_3\ldots. = l_{21},$$

the limits of the integrals and extent of the summation being as before; K_1 is the first minor of the determinant formed by omitting the row and column containing k_1; L_{12} is the first minor formed by omitting the row and column containing l_{21}, or l_{12}; and so on.

* The values of the k and m which we have been employing with reference to *principal* axes are in terms of our original k, l, m referred to any axes respectively:

$$k\cos^2\theta - 2l\cos\theta\sin\theta + m\sin^2\theta,$$

and

$$k\sin^2\theta + 2l\cos\theta\sin\theta + m\cos^2\theta;$$

where

$$\tan 2\theta = (k-m)\div 2l.$$

See note on p. 208.

If the units of the variables be taken so that k_1, k_2, k_3, &c. each $=\frac{1}{2}$, then l_{12}, l_{13}, &c. will be replaced by $\frac{1}{2}\rho_{12}$, $\frac{1}{2}\rho_{13}$, &c., the coefficients of correlation which have been discussed in a former paper*.

To obtain a second approximation to the compound law of error by this method: beginning with the case of two variables, put as before

$$k=\Sigma\iint\zeta_i x^2\,dx\,dy,$$

$$m=\Sigma\iint\zeta_i y^2\,dx\,dy,$$

principal axes being employed. Also put

$$\begin{aligned} n &= \Sigma\iint\zeta_i x^3\,dx\,dy, \\ p &= \quad ,, \quad \zeta_i x^2 y \quad ,, \\ q &= \quad ,, \quad \zeta_i x y^2 \quad ,, \\ r &= \quad ,, \quad \zeta_i y^3 \quad ,, \end{aligned}$$

We have then, for z the law of the compound, the following system of equations:—

$$\frac{dz}{dk}=\frac{1}{2}\frac{d_2 z}{dx_2}; \quad \ldots\ldots \quad (1)$$

$$\frac{dz}{dm}=\frac{1}{2}\frac{d_2 z}{dy_2}; \quad \ldots\ldots \quad (2)$$

$$\frac{dz}{dn}=-\frac{1}{6}\frac{d_3 z}{dx^3}; \quad \ldots\ldots \quad (3)$$

$$\frac{dz}{dp}=-\frac{1}{2}\frac{d_3 z}{dx^2\,dy}; \quad \ldots\ldots \quad (4)$$

$$\frac{dz}{dq}=-\frac{1}{2}\frac{d_3 z}{dx\,dy^2}; \quad \ldots\ldots \quad (5)$$

$$\frac{dz}{dr}=-\frac{1}{6}\frac{d_3 z}{dy^3}; \quad \ldots\ldots \quad (6)$$

$$z+x\frac{dz}{dx}+2k\frac{dz}{dk}+3n\frac{dz}{dn}+2p\frac{dz}{dp}+q\frac{dz}{dq}=0, \quad . \quad (7)$$

$$z+y\frac{dz}{dy}+2m\frac{dz}{dm}+3r\frac{dz}{dr}+2q\frac{dz}{dq}+p\frac{dz}{dp}=0. \quad . \quad (8)$$

* Phil. Mag. 1892, xxxiv. p. 194 *et seqq.*

From (7) and (8) we obtain

$$z=\frac{1}{\sqrt{km}}\chi\left(\frac{x}{\sqrt{k}},\ \frac{y}{\sqrt{m}};\ \frac{n}{k^{\frac{3}{2}}},\ \frac{p}{km^{\frac{1}{2}}},\ \frac{q}{k^{\frac{1}{2}}m},\ \frac{r}{m^{\frac{3}{2}}}\right). \quad (9)$$

Put z_1 for the first approximation which has already been found, viz.:

$$z_1=\frac{1}{2\sqrt{km}}\,e^{-\left(\frac{x^2}{2k}+\frac{y^2}{2m}\right)}.$$

Put $z=z_1(1+\Theta)$. Then for Θ we may substitute with sufficient generality

$$\theta+\frac{n}{k^{\frac{3}{2}}}\theta_1+\frac{p}{km^{\frac{1}{2}}}\theta_2+\frac{q}{k^{\frac{1}{2}}m}\theta_3+\frac{r}{m^{\frac{3}{2}}}\theta_4,$$

where the θ's are functions of x and y; since, the coefficients in (9) being by parity of reasoning with that employed in the case of a single variable* small, second powers do not appear in the expression for z. Since z reduces to z_1 in the case of symmetry, θ must be null. For the other unknowns we have from equations (3), (4), (5), and (6), neglecting small quantities,

$$\frac{1}{k^{\frac{3}{2}}}\theta_1=-\frac{1}{6}\frac{d_3z_1}{dx^3};\quad \frac{1}{km^{\frac{1}{2}}}\theta_2=-\frac{1}{2}\frac{d_3z_1}{dx^2dy};$$

with corresponding equations for θ_4 and θ_3. Performing the work we have for the asymmetrical probability-surface,

$$z=\frac{1}{2\pi\sqrt{km}}\,e^{-\left(\frac{x^2}{2k}+\frac{y^2}{2m}\right)}$$
$$\times\left(1-\frac{1}{2}\frac{n}{k^{\frac{3}{2}}}\frac{x}{\sqrt{k}}\left(1-\frac{1}{3}\frac{x^2}{k}\right)-\frac{1}{2}\frac{p}{k\sqrt{m}}\frac{y}{\sqrt{m}}\left(1-\frac{x^2}{k}\right)\right.$$
$$\left.-\frac{1}{2}\frac{q}{\sqrt{km}}\frac{x}{\sqrt{k}}\left(1-\frac{y^2}{m}\right)-\frac{1}{2}\frac{r}{m^{\frac{3}{2}}}\frac{y}{\sqrt{m}}\left(1-\frac{1}{3}\frac{y^2}{m}\right)\right). \quad (10)$$

By construction, z satisfies equations (3), (4), (5), and (6).

From the form of the expression it is evident that it satisfies equation (9), and therefore equations (7) and (8). By actual trial it is found to satisfy equations (1) and (2); as may be seen by breaking up the expression into five terms, and observing that each term separately satisfies those equations.

We might also have proceeded by obtaining general solutions of equations (1) and (2) in the form of series, after the

* See the preceding article, Phil. Mag. 1896, xli. p. 90.

analogy of the case of a single variable (Philosophical Magazine, 1896, xli. p. 95 *et seqq.*), and then subjecting the general expression to the conditions that the quantities $\frac{n}{k^{\frac{3}{2}}}$ &c. are small. The form of the solution in series is such that the third differentials $\frac{d_3 z}{dx^3}$ $\frac{d_3 z}{dx^2\,dy}$ belong to the same form, which accounts for the circumstance that the functions θ_1 θ_2 &c., satisfy equations (1) and (2).

Such being the solution for principal axes, the solution for any axes x', y' is found by substituting in the above expression for x, $x' \cos\theta - y' \sin\theta$, and, for y, $-x' \sin\theta - y' \cos\theta$; where

$$\tan 2\theta = \frac{k' - m'}{2l'};$$

k', l', m' corresponding to our original k, l, m*; and by substituting in (10) for k, m, n, p, q, r the equivalents of those coefficients in terms of k', l', m', n', p', q', r' †.

It may be observed that to whatever axes the surface be referred, if we integrate between extreme limits with respect to y, the resulting curve in x is a probability-curve (of the asymmetric kind)‡. This theorem may be employed to test

* See p. 208. † See note to p. 208.

‡ This proposition may be deduced *a priori* from the reasoning employed on a former occasion to prove the symmetrical compound law of error (Phil. Mag. 1892, vol. xxxiv. p. 522). The proposition may be verified by integrating (10) with respect to y, between extreme limits; and observing that, of the five terms within the brackets, the first two remain unaltered because

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi m}} e^{-\frac{y^2}{2m}} dy = 1;$$

the third and fifth terms vanish because

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi m}} e^{-\frac{y^2}{2m}} \times y dy = 0;$$

and the third term vanishes because

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi m}} e^{-\frac{y^2}{2m}} \times y^2 dy = m.$$

Thus the integration with respect to y results in an asymmetrical probability-curve identical with that which has been given in the preceding article (z being substituted for y).

whether a given set of observations may be represented by a probability-surface.

A more summary test is afforded by observing that any strip (or slice) of the surface (or solid) ought to fulfil the condition that the mean-cube-of-error ÷ the mean-square-of-error raised to the power $\frac{3}{2}$ should be small. For it is zero for a strip of the symmetrical probability-surface; and the asymmetrical probability-surface differs from the symmetrical one only by small terms *.

The condition is often not fulfilled by actual observations. Take, for instance, the statistics of the frequency of marriages between men and women of different ages. I have elsewhere † constructed a table which may be translated into a surface such that z represents the probability that, out of the Italian marriageable population, a particular man aged x should marry a particular woman aged y within two years. Consider one strip of the surface, one row of the table, the one indicating the frequency with which women aged 22–23 marry men of different ages. Utilizing the entries in the table, and, for the extreme ages not represented in the table, the original materials, I find for the centre of gravity of the row 10·7, reckoned from the age of 16·5 as zero, and for the criterion $j \div k^{\frac{3}{2}}$ a figure between 2 and 3. It appears, therefore, that the number of the elements (in relation to their asymmetry) is not sufficiently great to generate a true probability-surface.

Analogous expressions for the compound asymmetrical function of many variables may be constructed by parity of reasoning.

* The theorem may be verified by putting $y=0$ in (10); expressing in terms of the coefficients the integrals $\int_{-\infty}^{\infty} x^2 z dx$ and $\int_{-\infty}^{\infty} x^3 dx$; and comparing the latter with the former (raised to the power $\frac{3}{2}$).

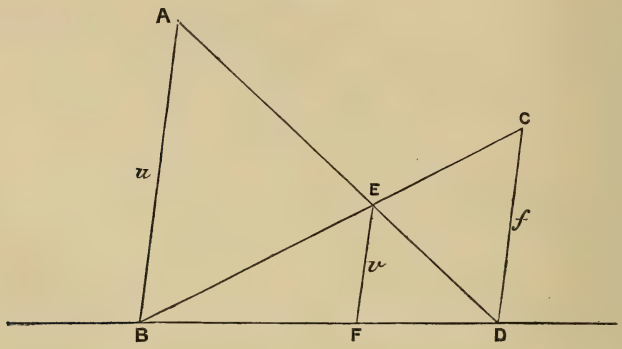
† Journal of the Royal Statistical Society, March 1894. The table there given does not correspond to the well-known stereogram constructed from the same materials by Signor Perozzo, but purports to be an improvement upon it.

XXVI. *Graphical Methods for Lenses.* *By* R. S. Cole, *M.A.**

THE following graphical methods, which I have not seen published, may be of interest.

They depend on the following geometrical construction:—

Fig. 1.



Let A B and C D be parallel straight lines terminated by B D; let A D and B C intersect in E, and draw E F parallel to A B or C D to meet B D in F; then it can be proved that

$$\frac{1}{EF} = \frac{1}{AB} + \frac{1}{CD}.$$

This furnishes a graphical method of compounding reciprocals.

Adopting the convention of signs which reckons lengths positive when measured from the surfaces (or with thick lenses from the nodal points) in a direction opposite to that of the incident light, the usual lens formula is

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}, \text{ or } \frac{1}{v} = \frac{1}{u} + \frac{1}{f}.$$

If u and f are given and v is required, we must draw A B and C D to scale to represent u and f respectively, and then E F represents v; and, on the other hand, if u and v are given, draw A B and E F to represent them, and from these complete the figure; C D will then give the focal length. Negative values of u, v, and f can be indicated by drawing the lines corresponding below B D.

The diagram exhibits the relative sizes of image and object,

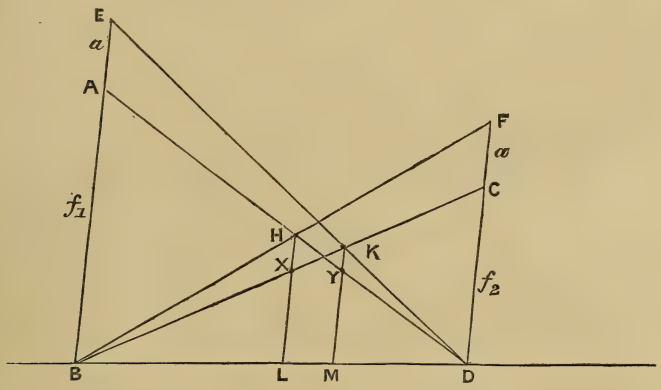
* Communicated by the Author.

for they are in the ratio $v : u$ or E F : A B; the image being direct or inverted according as E F and A B are or are not on the same side of B D.

The construction can also be used for finding the focal length of the lens equivalent to any number of lenses in contact, for this is done by adding the reciprocals of the focal lengths.

An extension can be made to the case of two lenses separated by a definite interval a, to find the focal length of the equivalent thick lens and the position of its nodal points.

Fig. 2.



Let A B and C D be parallel and let them represent f_1 and f_2; produce them to E and F so that

$$AE = CF = a.$$

Join A D, B F intersecting in H, and B C, E D intersecting in K.

Draw H L and K M parallel to meet B D in L and M, and let H L cut B C in X and K M cut A D in Y. Then it can be proved that

$$HX = \frac{af_1}{a+f_1+f_2}, \quad KY = \frac{af_2}{a+f_1+f_2}, \quad XL = YM = \frac{f_1 f_2}{a+f_1+f_2}.$$

Hence $-$H X and K Y represent the distances of the nodal points from the two component lenses, and X L or Y M represents the focal length of the equivalent thick lens.

Devonport.

XXVII. *Electro-optical Investigation of Polarized Light.* *By* J. Elster *and* H. Geitel*.

THE photo-electric current produced in an attenuated gas by illumination of the kathode has been shown to be dependent upon the inclination of the vibrations of light to the plane of the kathode, and to attain a maximum when the plane of polarization of the light is at right angles to the plane of incidence, and a minimum for the position at right angles to this†.

The further examination of this phenomenon proceeds in two directions:—We may ask, "According to what law does the photo-electric current vary when the plane of polarization of the incident light is made to rotate about the ray as its axis?" and we may inquire "how the intensity of this current depends upon the angle of incidence of the light."

We confine ourselves here for the most part to the first of these questions‡: towards the solution of the second (which offers greater difficulties) we can here only make some small contributions.

As we have previously remarked, we are led by the difficulties which present themselves in the production of polarized ultra-violet light, to choose the fluid alloy of sodium and potassium as the photo-electric sensitive surface in an atmosphere of a rarefied indifferent gas, which permits the use of light from the region of the visible spectrum. But this involves the necessity of enclosing the metallic surfaces subjected to experiment in glass vessels. It would be of advantage that the polarized beam of light should enter the vessel normally, through a glass plate with parallel plane surfaces. We should thus avoid all change of intensity in the exciting beam, which with oblique incidence against the glass wall is associated with change of azimuth. But the insertion of such plate-glass "windows" in the glass vessels involves the use of some cement, which must be of such a nature that in presence of vapour of the alkali metals in a vacuum it shall neither evaporate nor undergo any chemical change. We have not succeeded in finding a cement that will stand under these conditions. Organic substances such as resins are out of the question, since they contain volatile constituents which condense upon the kathode, forming a layer almost insensitive to light. And such inorganic substances as potassium and sodium

* Translated from *Ann. Phys. Chem.* Bd. lv. Communicated by the Authors.

† Elster and Geitel, *Sitzber. Berl. Akad. Wiss.* vi. p. 134 (1894), and Wied. *Ann.* lii. p. 440 (1894).

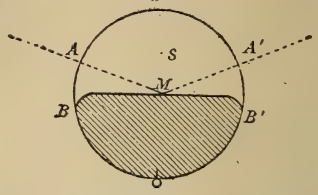
‡ These results have been published in part in *Sitzber. der Kgl. Acad. Berlin*, xi. p. 209 (1895).

silicates, and cements composed of phosphoric acid and metallic oxides produce a decrease in sensitiveness to light after a time, probably in consequence of the increase of gas-pressure due to the evolution of hydrogen by the action of metallic vapours on the water contained in the cement. A better result was obtained with molten glacial phosphoric acid if the precaution was taken of covering the joints, immediately after the plate had been cemented on, with zinc oxide which had been washed and ignited, and then covering the joint with a layer of a mixture of wax and resin. It is true that with the apparatus so constructed, the sensitiveness also decreased for the first few days after sealing-off from the air-pump; but this falling off soon ceased. We were thus able to construct vessels with windows of parallel glass, which might serve at least for control-experiments.

These difficulties induced us to use generally simple glass bulbs (cells) blown before the blowpipe, in which the alkali metal and its vapours were in contact only with the glass walls and the platinum electrodes. For most of the experiments here described we employed such bulbs of 50 millim. diameter which were half filled with the alloy of sodium and potassium. Each ray, therefore, that strikes the centre of this metallic mirror must cut the glass wall at right angles, and its intensity will not vary with the azimuth. The cross section of the beam was therefore made as small as the sensitiveness of the cell permitted.

In order to obtain a beam of light of small section and great intensity, we obtained a projection lantern (*sciopticon*) with a little disk of zirconia heated in the oxygen-coalgas-flame as source of light. After this had been so adjusted that the image of the piece of zirconia was obtained sharply at a considerable distance, a screen with a slit, about 3 millim. long and 1 millim. broad, was placed between the condenser and the projecting lens so that its outline was defined upon the wall of the cell. The sciopticon, which with the screen formed a rigid whole, was capable of rotation in a vertical plane, and in any position could be inclined to the horizontal through about 50° and held fast in that position. If it had such a position that the spot of light which marked the entrance of the ray of light was distinctly seen at A upon the glass wall, then the point of emergence at A′ could also be distinctly recognized. The distances AB and A′B′ of corresponding edges of these spots of light from the horizontal surfaces of the fluid metal were measured with a pair of compasses, and the sciopticon and cell were so placed that AB was equal to A′B′. We were thus sure that the ray of light struck the centre of the kathode, and cut the glass wall at a

Fig. 1.



right angle. The point of the anode *s* was about 10 millim. above M, consisting of a platinum wire (shown in the figure as a point) whose direction was at right angles to the plane A B A′ B′. The surface of the alkali-metal must be so bright that the place where the light strikes it must appear absolutely black to an eye not in the direction of the reflected ray. Sharper angles of incidence than could be obtained by inclining the sciopticon were obtained by the use of a silver mirror (the metallic side of a piece of plate-glass coated with silver), which rotated about a horizontal axis, and from which the horizontal ray was reflected downwards.

In order to determine the angle of incidence of the ray to the horizontal kathode we employed a simple instrument that is also useful in determining the altitude of the sun. From the centre of a quadrant of pasteboard, graduated into half degrees, hangs a plumb-line, the thread of which touches the graduations. In the prolongation of one of the bounding radii of the quadrant across the centre, is placed a pencil at right angles to the plane of the quadrant. The apparatus is so placed that the shadow of the pencil is thrown by the beam of light from the sciopticon in the direction of the radius. The angle of incidence is then equal to the angle between the radius and the thread, and can be read off upon the divided arc.

As in the previously described photo-electric measurements, the cell was placed with the galvanometer previously described in circuit with a battery of 100 to 400 Leclanché cells, of a total E.M.F. of about 450 volts, so that the surface of the alkali-metal was the kathode. The intensity of the photo-electric current was read off with mirror and scale; it is scarcely necessary to observe that the cell was shielded from the light of the lamp used to illuminate the scale, and that no stray light was allowed to escape from the sciopticon.

Immediately in front of the cell was placed a large Nicol's prism, provided with graduated circle and capable of rotation in horizontal and vertical planes. The cross-section of the beam of light was so cut down by the screen that it traversed the prism freely while it was rotated, as one could easily see by observing the path of the light within the calcspar.

If, then, the beam of light, polarized by passing through the Nicol's prism, fall upon the kathode-surface in the manner described at any incidence other than normal, the galvanometer shows a periodic change of intensity during the rotation of

the Nicol with two maxima and two minima. The maxima occur when the principal section is coincident with the plane of incidence, and the minima in the alternate positions at right angles to the former. If we reckon the angle of rotation α of the prism from the position of the maxima, for which the plane of polarization of the ray is at right angles to the plane of incidence, and if we denote by J the intensity of the current for the angle α as found from the reading of the galvanometer, then, within the limits of experimental error, the relation between these is expressed by the formula

$$J = A \cos^2 \alpha + B \sin^2 \alpha,$$

where A denotes the maximum current-intensity (for $\alpha = 0°$), and B the minimum intensity (for $\alpha = 90°$). In order that this connexion shall be verified without doubt, the positions of maximum and minimum position must be determined with as much sharpness as possible. The changeability of J is, as the nature of the function shows, least in the neighbourhood of $\alpha = 0°$ and $\alpha = 90°$: consequently, it is only possible to determine these principal positions by direct experiment with an uncertainty of more than a degree. We therefore preferred to determine the position of the greatest changeability of J, namely, when $\alpha = 45°$. Whilst one of us slowly rotated the Nicol's prism, the other observed at the galvanometer the intensity of the current, and read off the maximum and minimum values A and B. At the same time the corresponding positions of the Nicol were read off on the divided circle. For $\alpha = 45°$, the formula leads us to expect the value

$$J = \frac{A + B}{2}.$$

If the Nicol be turned from one of the observed positions through 45° we shall always obtain a value of J nearly equal to the calculated value $(A + B)/2$. By a slow rotation of the Nicol we brought it about that this number was actually read off, and regarded the position so obtained as that actually corresponding accurately to the azimuth 45°. Then, by turning on, or back, through 45°, we obtain the corrected principal positions, and obtain for these the old maximum and minimum values A and B again. We must further mention that on rotation of the prism the direction of the emergent beam was not absolutely constant, but there were slight displacements of the positions of entrance and emergence of the beam to be observed on the glass wall (see fig. 1). As there were but very slight changes of direction, and the displacements of the spot of light thus produced fell pretty well within the limits of accuracy of the method of measurement described, we have neglected them. The attempt to eliminate

this source of error by a new adjustment of the ray and cell would have so prolonged the duration of a series of measurements, that the change in the zircon-light in the time would, undoubtedly, have proved a source of more serious error.

The following Tables contain under I. A—D the values of current-intensity observed with the above-described cell for the angles of incidence 70°, 66°, 40°, 23° for each 15° of azimuth. Table II. gives a series for a cell in which the kathode surface was distant about the half radius from the centre of the glass bulb. III. refers to a receiver with windows of plate-glass.

I.

Cell I., half-filled with KNa alloy.

Azimuth a	0.	15.	30.	45.	60.	75.	90.
A. Angle of incidence = 70°. 27 January, 1895.							
Current-intensity (observed) .	149·6	138·0	111·0	74·6	38·9	12·7	3·2
Current-intensity (calculated).	147·3	137·6	111·3	75·2	39·3	12·9	3·2
Difference	+2·6	+0·4	−0·3	−0·6	−0·4	−0·2	...
B. Angle of incidence = 66°.							
Current-intensity (observed) .	144·0	132·5	107·0	72·3	38·3	12·5	4·0
Current-intensity (calculated)	141·6	132·4	107·2	72·8	38·4	13·2	4·0
Difference	+2·4	+0·1	−0·2	−0·5	−0·1	−0·7	...
C. Angle of incidence = 40°.							
Current-intensity (observed) .	161·3	149·5	122·0	85·9	47·0	19·0	7·1
Current-intensity (calculated)	161·7	151·3	123·1	84·5	45·8	17·5	7·1
Difference	−0·4	−1·8	−1·1	+1·4	+1·2	+1·5	...
D. Angle of incidence = 23°. 1 February, 1895.							
Current-intensity (observed) .	96·8	91·8	79·7	63·3	42·9	30·0	28·1
Current-intensity (calculated)	97·2	92·5	79·9	62·6	45·4	32·7	28·1
Difference	−0·4	−0·7	−0·2	+0·7	−2·5	−2·7	...

II.

Cell II., about $\frac{1}{4}$ filled with KNa alloy.
Angle of incidence = 65°.
31 December, 1894.

Azimuth a	0.	15.	30.	45.	60.	75.	90.
Current-intensity (observed) .	105·4	98·6	79·6	53·8	28·0	8·7	2·1
Current-intensity (calculated)	105·5	98·6	79·7	53·8	28·0	9·0	2·1
Difference	−0·1	0·0	−0·1	0·0	0·0	−0·3	...

III.

Cell III., with plate-glass windows.
Angle of incidence = 65°.
24 February, 1895.

Azimuth a	0.	15.	30.	45.	60.	75.	90.
Current-intensity (observed) .	63·7	60·1	48·5	33·7	17·3	6·0	1·3
Current-intensity (calculated)	64·7	60·5	48·9	33·0	17·2	5·5	1·3
Difference	−1·0	−0·4	−0·4	+0·7	+0·1	+0·5	...

The numbers given as calculated current-intensities are calculated from the formula

$$J = A \cos^2 a + B \sin^2 a$$

in the following manner. The equation may be written

$$J = (A - B) \cos^2 a + B$$

or

$$A - B = \frac{J - B}{\cos^2 a}.$$

If we subtract the minimum value of B (corresponding to $a = 90°$) from all the other values of J, and divide by the square of the cosine of the corresponding azimuths, we ought to obtain numbers nearly equal. Of these we take the arithmetic mean (M), and write down the numbers obtained from the formula

$$J = M \cos^2 a + B$$

for all azimuths as the calculated current-intensities.

As we see from the foregoing tables, the result found with

the cell with parallel glass windows confirms that found with the bulb-shaped ones; and even when the ray directed towards the centre of the kathode-surface cuts the glass wall at an acute angle, as was the case with the cell referred to in Table II., the regularity in the change in value of J is the same. The reason of this is to be found in the fact that the changes in intensity with change of azimuth, which a polarized ray suffers at acute passage through a single surface of glass, are only small even if the angle of incidence differs much from the polarizing angle.

For angles of incidence less than 40° (Table I. D) as already remarked, the ray was reflected by a silver mirror into the Nicol's prism. Strictly, we ought to take into account the amount of elliptical polarization due to the reflexion of the light from the silver, which itself would cause a change in the intensity of the light transmitted by the Nicol at different azimuths. But here also the error lies quite within the limits of accuracy of the measurements. Of this we convinced ourselves by measuring the brightness of the emergent beam with a sodium cell with solid kathode at right angles to the ray, during rotation of the Nicol through 90°: it remained almost constant.

In order to eliminate the effect of possible change in the zircon-light, the measurement with the first position of the Nicol was repeated after each series of measurements. Only those series were retained in which this control-measurement agreed with the first.

The relationship expressed in the above formula between the photo-electric current and the azimuth of the light may be deduced from the assumption, justified by previous experiments, that the current-strength is proportional to the intensity of the light, if we make the further assumption that the constant is not the same for light polarized at right angles to the plane of incidence as for light polarized in the plane of incidence. If a denote the amplitude of a polarized ray, whose plane of vibration makes an angle α with the plane of incidence, then the intensities of its components parallel and at right angles to the plane of incidence are respectively $a^2 \cos^2 \alpha$ and $a^2 \sin^2 \alpha$. The strength of the photo-electric current caused by this ray is therefore

$$J = a^2 x \cos^2 \alpha + a^2 y \sin^2 \alpha,$$

if we represent by x and y the two constants between the intensities of light and current. In this expression, $a^2 x$ and $a^2 y$ are the constants, independent of α, which were denoted above by A and B. A ray of light vibrating *in* the plane of

incidence, therefore, excites a photo-electric current stronger in the ratio $x:y$ or A : B than a ray of equal intensity vibrating in a plane at right angles to the plane of incidence, and therefore parallel to the surface of the kathode.

It is easily seen that the ratio A : B must depend upon the angle of incidence. For with normal incidence the position of the plane of incidence is undetermined, and therefore the difference between A and B must disappear. Experiment shows that the common value of the constants for this direction of the rays is comparatively small; whilst then with increasing angle of incidence A increases rapidly, attaining a maximum at about 60° and then decreasing, B becomes continually smaller and appears to become zero at a nearly grazing incidence. Thus between 60° and 70° the ratio A : B has the value of about 50 : 1 (*cf.* the tables). Experiments, still in progress, make it not improbable that the angle of incidence for which A attains its maximum coincides with the angle of polarization for the potassium-sodium alloy for the most electrically active rays, that is, for the blue rays.

It is an obvious suggestion that the different sensitiveness of the metallic kathode-surface to light polarized in and at right angles to the plane of incidence is connected with the greater depth to which, according to Quincke *, the latter penetrates into a metallic surface.

We have then the remarkable result that a ray of polarized light exerts a much smaller photo-electric effect at normal incidence than if it strikes the kathode at an acute angle with the plane of polarization at right angles to the plane of incidence. But it is to be observed that the like result must follow also with ordinary light, since we may regard this as consisting of two components polarized in planes at right angles, of which one vibrates in the plane of incidence. The one at right angles contributes little to the photo-electric effect at high angles of incidence on account of the smallness of the constant B.

In order that this prediction should be verified it is necessary that the surface of the kathode should be such a perfect plane as is only attainable by the use of the alkaline metals in the fluid condition. Kathodes of solid sodium or potassium have always a rough crystalline surface, and offer to the light elements of all possible positions. Consequently the increase of the photo-electric current with increasing angle of incidence can only be observed with cells containing fluid kathodes of NaK-alloy, whilst for those with solid kathodes the strength of the current is almost independent of the angle of incidence.

* Quincke, *Pogg. Ann.* cxxix. p. 117 (1866).

This independence is a consequence of the fact that with increasing angle of incidence the surface illuminated increases in the same ratio as the illumination of the unit-surface decreases.

In proof of what has been said, we give the two following series of observations :—

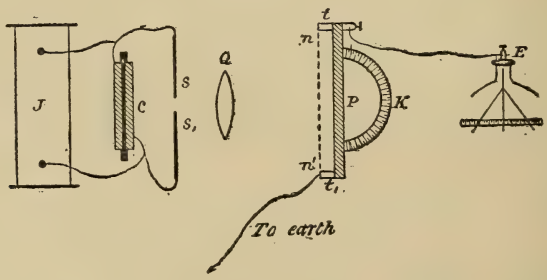
Natural Light.

18 Nov. 1894.

	I. Liquid Metal. (KNa-alloy.)	II. Solid Metal. (Surface rough, but as smooth as possible.)
Angle of Incidence 0°	10·2	34·0
" " 25	15·5	33·5
" " 45	44·2	33·0
" " 60	56·7	30·5
" " 0	8·4	33·1

Since, then, it is possible without the aid of polarization, simply by changing the angle of incidence, to show the increase in the photo-electric action of a ray of light which takes place when the vibrations of light have a component at right angles to the kathode, it was easy to make the like experiment with ultra-violet light; which, as we remarked at the outset, it is difficult,—possibly impossible with the usual means —to obtain in the condition of linear polarization. We have then the advantage that the kathode may be made of any metallic substance we like, and may be used in the open air.

Fig. 2.



The arrangement of the experiment was as follows (fig. 2):— A condenser C was connected with a large inductorium J, the brilliant discharge of which took place between two zinc points S and S′ in the focus of a lens Q of quartz. The rays thus made parallel fell upon a plate P of amalgamated zinc placed upright, which was capable of rotation over a graduated arc K about an axis at right angles to the direction of the rays of light. A wire frame n, n' connected to earth, and

attached to the plate by supports of sealing-wax t, t', carried a fine copper wire stretched across the space between t and t'. When the sparks had passed for a certain time between S and S′, the ultra-violet light from it withdrew from the plate a greater or smaller quantity—according to the intensity of the light—of the negative charge communicated to it. The fall of potential was observed with an Exner's electroscope connected to it. From the values V and V′ of the potential before and after illumination, the intensity of the light was calculated by means of the formula $J = \log V/V'$, previously established *. At the beginning of each series of experiments the brightly polished zinc plate stood at right angles to the incident rays ($\alpha = 0°$), then the plate was turned through 50° right and then left ($\alpha = \pm 50°$), and finally the initial position was restored. At the beginning the potential was always 258 volts (25 divisions of the scale of the electroscope); the potentials V′ after 10 seconds' exposure are given in the following table :—

13 April, 1895.

Angle of incidence		0°	+50	−50	0	+50	−50	0	+50	−50	0
V′ in divisions of the scale.	Series I.	9·1	7·5	7·5	8·8	7·5	7·9	9·0	7·9	7·8	9·1
	Series II.	8·9	7·9	7·4	8·5	8·0	8·0	8·9	7·5	7·5	9·0

The mean of the observations gives :—

For normal incidence $V'_0 = 8·9$ scale-divisions $= 117$ volts.

For oblique incidence $V'_{\pm 50} = 7·7$ scale-divisions $= 107$ volts.

Hence for the ratio of the photo-electric action we have

$$\frac{J'_{\pm 50}}{J'_0} = \frac{\log 258 - \log 107}{\log 258 - \log 117} = 1·11.$$

Hence the photo-electric activity of obliquely incident ultra-violet light in the open air is also greater than that of normally incident light, although the differences are much smaller than those found for visible light with alkali-metal surfaces *in vacuo*. We suspected that this was to be accounted for by the imperfection of the surface of the amalgamated zinc, and repeated the same experiment with a surface of mercury, altering the arrangement of the apparatus so as to suit the condition that the kathode-surface must remain fixed and that the light-ray must now be rotated. But here also the differences were found to lie within the same limits. Consequently either the ratio of photo-electric activity of light polarized parallel and at right angles to the plane of incidence is dependent on the wave-length, so that it approaches to unity for small wave-lengths, or other conditions

* Elster and Geitel, Wied. *Ann.* xlviii. p. 347 (1893).

(for example, the difference in pressure) play a part in the phenomenon.

Further, induced thereto by the attempts of Herr Wanka* to recognize in the well-known Hertzian experiment of the production of an electric spark by the light of another, the influence of the direction of the vibrations of light to the illuminated kathode, we have looked for a dependence of the same phenomenon on the angle of incidence. The experimental arrangement was essentially the same as that already described, only that the action of the light was observed not by the fall of potential of an illuminated surface, but by the discharge of the synchronous, so-called passive spark of a second inductorium in similar phase with the first. The condenser used with the first is not now required, but it is advisable to connect the second with a Leyden jar in order to give greater intensity to the passive spark, and thus to facilitate its observation. We have not found any perceptible difference in the action which was clearly recognizable at different angles of incidence (0° and 50°), either with the zinc plate as kathode or with the mercury surface.

It is, however, to be remarked that the Hertzian phenomenon, the resolution of a spark by the ultra-violet illumination of the kathode, is not to be regarded as altogether the same as the scattering of negative electricity in ultra-violet light observed by Herr Hallwachs. This latter is within wide limits proportional to the intensity of the light, and therefore first disappears completely on excluding the light, whilst with the first, for a given distance apart of the electrodes, a finite strength of light is necessary and sufficient. On this account the above described experiment would perhaps have given a positive result if the striking distance of the passive spark had each time been adjusted by a micrometer-screw to its maximum value.

In regard to the nature of the photo-electric process, the results of the experiments with polarized light on the whole indicate that we have here to do with an immediate action which the light-rays exert by exciting electrical vibrations. In this or a similar sense Herrn Wiedemann and Ebert †, we ourselves ‡, and Herr Jaumann § have already declared. Yet,

* J. Wanka, *Mitth. d. deutsch. math. Ges. in Prag.* p. 63 (1892).

† E. Wiedemann and H. Ebert, Wied. *Ann.* xxxiii. p. 263, (1888); and xxxv. p. 259 (1888).

‡ J. Elster and H. Geitel, Wied. *Ann.* xli. p. 175 (1890), and xliv. p. 736 (1891).

§ In the above-cited paper of Wanka, p. 58 (1892); also G. Jaumann, *Wien. Ber.* cxiv., ii *a.* p. 9 (Jan. 1895).

if we do not with the first named investigators connect the photo-electric process with the production of kathode-rays, it remains to be explained why it is limited just to the kathode-surface. To supply this deficiency a further assumption is necessary which has recently been made by Prof. J. J. Thomson *, namely, that the kathode-surface in contact with air, or whatever the gas may be, has a double electric charge, the positive side formed by the molecules of the kathode, the negative by molecules of the gas.

If we now imagine that a ray of light so strikes the kathode-surface of metal that the electric displacements in the ray have a component at right angles to the surface, then electric vibrations will be induced in the molecules of the metal in which such a component will also be present. This will cause the place of contact between the molecules of metal and gas, so far as it belongs to the first, to take alternately positive and negative charges in rapid succession. It is perhaps possible that in *that* phase of the vibrations in which the electric density is negative, the connexion of the metallic molecule with the equally negative gas-molecule may be dissolved in consequence of electrostatic repulsion, and the latter may be driven off into the free gas-space, whilst another that now takes its place communicates positive electricity to the metallic molecule by contact with it, and itself takes a negative charge.

This view is supported by the circumstance that alkali-metal cells, which, instead of rarefied hydrogen, contain the much more strongly electro-negative gases oxygen or carbon dioxide, are particularly sensitive to light, and that with similar gaseous atmospheres the sensitiveness to light rises with the electro-positive character of the kathode-metal. We hope to return to these phenomena.

It may certainly be urged against the view here expressed, that the photo-electric action does not entirely cease when the electric displacements take place parallel to the kathode-surface, and when consequently the normal component is zero. It appears to us to be of great importance to our knowledge of the photo-electric process to make further investigations into its dependence upon the angle of incidence, using, of course, polarized light.

* J. J. Thomson, Phil. Mag. xxxvii. p. 356 (1894).

XXVIII. *On the Production of Electrical Phenomena by the* Röntgen *Rays.* *By* Aug. Righi*.

1. It is known that the most important properties of the X-rays, that is to say the power of exciting fluorescence and of acting on photographic preparations (which Röntgen thinks is a secondary effect of the former), belong more especially to the more refrangible of the ordinary rays. Since these, in like manner, are eminently suited to produce photo-electrical properties, I was desirous of examining whether phenomena of this kind would also be produced by the X-rays.

As my investigation, though only recently begun, has already given results, I will communicate them to the Academy. For the sake of clearness it is necessary to give a brief account of photo-electric phenomena.

2. A body negatively electrified rapidly loses its charge if exposed to radiations. The rapidity with which the charge is dispersed is greater or less according to the nature of the body and the wave-length of the radiations used. So that, while for most bodies the phenomenon is produced only by the rays of smallest wave-length emitted by the voltaic arc (and especially by that produced between carbon and zinc or aluminium), or by that of burning magnesium, for some bodies, like amalgamated zinc, the sun's light is sufficient, and for others, like the alkaline metals, the ordinary artificial lights.

I ought more especially to mention here an experimental arrangement, that of the *photo-electric couple* †. The active radiations fall on a metal disk after passing through wire gauze parallel and close to it, and in connexion with the earth. The radiations dispersing the negative charge which, owing to the difference of potential by contact, the gauze or the disk possesses (they having been placed in contact just before the experiment) produce a deflexion, positive or negative according to circumstances, in an electrometer in connexion with the disk: this deflexion measures sensibly the difference of potential in question ‡.

If, in repeating the experiment, the disk is put further and

* *Rendiconti dell'Accademia di Bologna*, February 9, 1896. From a separate copy communicated by the Author.

† *Mem. della R. Acc. di Bologna*, series 4, vol. xiv. p. 369; *Nuovo Cimento*, 1889, vol. xxv. p. 20.

‡ *Ibid.* page 351.

further away from the gauze, the deflexion increases in absolute value if it is positive, and diminishes to zero, or is reversed, if it is negative.

The direct cause of this is a phenomenon brought out from my earliest researches on the electrical phenomena of radiations: namely, that a discharged body becomes positively charged when it is struck by radiations *.

The above phenomenon is expressed by a law which I have established by many experiments, and which can be thus enunciated: the positive charge of the body on which the radiations fall ceases to increase when the electrical density has attained a certain value, which is constant for a given substance.

It follows from this law, that the positive deflexion produced by the radiations becomes smaller if the conductor in connexion with the earth is brought gradually nearer the body on which the radiations fall, the capacity of which is thereby increased. Hence to show the positive charge produced by radiations, the body on which they fall should not be too near the uninsulated conductor. In the opposite case, which is that of photo-electric couples, the formation of this final positive charge has almost no influence on the deflexions obtained.

3. In order to examine whether the X-rays disperse the charge of a body electrified negatively, and charge positively an uncharged body, I worked with methods similar to those now described:—

(1) I charged a conductor in any given way, and then examined whether its potential, as shown by a quadrant-electrometer in connexion with it, underwent a more rapid diminution than that arising from the usual dispersion of the electric charge when the X-rays fall on it; or

(2) I caused the X-rays to fall on a photo-electric couple and observed whether they produced a deflexion in the electrometer communicating with the disk, the gauze being put to earth; lastly,

(3) I tried whether those rays falling on an uncharged body produce a positive charge.

In combining the most convenient experimental arrangements, it was necessary to do so in such a manner that there could be no possibility of any kind of electrical action being produced directly on the electrometer, or on bodies in connexion with it, by the apparatus generating the X-rays.

I therefore arranged the Crookes' tubes, together with the coil, the contact-breaker, &c., inside a large metal box in con-

* *Ibid.* page 387; *Nuovo Cimento*, vol. xxv. p. 128 (1889).

nexion with the earth. One of the sides of the box is made in chief part of a large lead plate provided with a circular window, in front of which on the inside and at a small distance is that part of a Crookes' tube which is struck by the kathodic rays. The window may be closed either by a large lead plate or by a thin plate of aluminium. In either case, when the apparatus in the box are at work, no electrical force proceeding from the internal charge of the case is manifested on the outside.

When conductors, electrified or not, are placed in front of the window closed with aluminium, I observed the following phenomena :—

4. If a conductor charged negatively is placed in front of the window, no sooner is the Crookes' tube at work than the charge rapidly disperses.

If the conductor is uncharged, it becomes positively electrified ; and in the same conditions the final charge is different according to the nature of the substance : thus, with gas-graphite it is greater than with copper, and with copper greater than with zinc.

With the latter metal the deflexion obtained is negative when the distance between it and the aluminium is small enough. This arises from the difference of potential of contact; but at a greater distance the positive charge produced by the X-rays preponderates.

In fine, if a photo-electrical couple is placed in front of the window, a deflexion of the electrometer is obtained, as if with ultra-violet radiations. I have not yet examined whether this deflexion is exactly equal to that which the latter radiations would produce.

Hence the X-rays have, in common with the ultra-violet ones, the property of dispersing the negative charge, and of giving rise to positive charges in unelectrified bodies.

The electrical action of the X-rays diminishes, as was to be expected, if the bodies on which they act are moved away from the window from which the rays proceed.

If a board of pine-wood, or a thick plate of aluminium or of glass, or the hand, is placed so as entirely to cover the aperture, it more or less diminishes the effect produced by the X-rays, but in general does not obscure them completely. A piece of glass-mirror less than a centimetre in thickness absorbs more than pine-wood 6 centim. in thickness.

5. Ultra-violet radiations produce no appreciable action on bodies positively electrified, for if in any case they seem to disperse positive electricity, it may be ascertained that the effect observed was due to the dispersion of negative electricity by the surrounding bodies.

Having given a positive charge to the conductor (gas-graphite, copper, zinc, &c.) on which the X-rays fall, I observed the same action as in the case of the negative charge.

Accordingly, *the* X-*rays, unlike the ultra-violet ones, produce the dispersion also of bodies electrified positively.*

Of course, while in the case of an initial positive charge the dispersion ceases when there is on the conductor a charge equal to that which, starting from the neutral state, the rays would impart to it: on the other hand, with an initial negative charge the dispersion continues until the body is discharged, and then the same final result is obtained as it has when the original charge is null or positive. This result may be expressed by saying that the conductor communicating with the electrometer behaves like an electrode in a conducting medium which acquires the potential of the region in which it is placed, whatever was the initial potential.

6. I desire to call attention to the fact that the new property here mentioned has the advantage of furnishing a means of measuring the X-rays. I believe that to compare the absorption produced by various bodies, it will be better to express this absorption by the velocity of dispersion, rather than by an estimation based on a comparison of the intensity of the shadow projected on a fluorescent screen or on a photographic plate.

The new property will evidently have to be taken into account when the time comes for discussing the various hypotheses proposed to explain the nature of the X-rays.

7. The following is a lecture-experiment for showing the dispersion of electricity produced by the Röntgen rays. The metal box is dispensed with, retaining only the large lead plate, with the window closed by aluminium. A disk of any metal is placed at a few centimetres from the window on one side of the lead plate, while on the other side is a Crookes' tube with the apparatus for working it. The disk is connected with a gold-leaf electroscope. The disk being well insulated, a charge of either sign is given to the system —the leaves remain divergent and motionless, but no sooner is the Crookes' tube at work than they fall and become vertical. If the aperture is closed with a large leaden plate, the phenomena are no longer produced. If various other substances are placed against the window, such as the hand, wood, &c., the rate at which the leaves fall is slower.

XXIX. *Notices respecting New Books.*

Dynamo-Electric Machinery. By S. P. Thompson, *D.Sc., F.R.S.* Fifth Edition. London: E. & F. N. Spon, 1896.

THE rapid increase in the number of technical applications of Electricity has occasioned the development of many new types of machinery in recent years. Any treatise which aims at giving a complete account of the principles involved in the construction of these machines must therefore grow visibly larger with each new edition. Prof. Thompson's work has reached the stage at which a single volume will not contain all the new matter without omitting or condensing part of the text of previous editions; consequently he has been obliged to transfer a portion of it to another treatise ('The Electromagnet') where it can be more adequately discussed. Even with this overflow some curtailment of the more theoretical chapters has been found necessary, and it seems evident that the sixth edition, whenever it appears, will be in two volumes.

The new matter in the present edition relates mostly to alternate-current machinery; rotary-field motors have a chapter to themselves; and the methods of synchronizing alternators to run as two dynamos or as dynamo and motor are described in a separate short section. The volume is enriched by many plates giving details of the construction of typical machines; and the student whose thirst for knowledge on any point cannot be satisfied by the text of the volume may often derive help from the plates, and will find throughout the book copious references to other works.

J. L. H.

Elementary Treatise on Electricity and Magnetism, founded on Joubert's 'Traité Élémentaire d'Électricité.' By G. C. Foster, *F.R.S., and* E. Atkinson, *Ph.D.* London: Longmans, 1896.

That the French physicists excel in the writing of elementary text-books is evidenced by the great popularity of the treatises by Ganot and Deschanel, the translations of which have passed through very many editions in this country. Although not nearly so well known in England, M. Joubert's *Traité élémentaire d'Electricité* has a deservedly high reputation in France: its scope is somewhat wider than that of the other two works, but the subject is treated logically and in a progressive manner throughout. In common with many other text-books, both English and continental, it has, however, the disadvantage of presenting too exclusively the action-at-a-distance character of electric and magnetic attractions, and it does not sufficiently emphasize the importance of the medium which, according to more modern ideas, is the seat of electric force and energy. Undoubtedly, the older view furnishes a beginner with an easy explanation of the simpler phenomena of electrostatics and magnetism, and, once adopted, it can only be discarded with difficulty. Prof. Carey Foster and Dr. Atkinson, taking M. Joubert's work as a basis, have found much re-writing necessary in their endeavour to present such explanations of all phenomena

as are consistent with the ideas of Faraday and Maxwell. Familiar terms like *charge* and *current* have been retained, but their relation to the ethereal medium is made clear as each arises; by this means the authors have been able to adhere to the order of treatment followed by Joubert. The result is a very satisfactory text-book, with all the advantages that can be imparted to it by teachers of skill and experience. J. L. H.

Computation Rules and Logarithms, with tables of other useful functions. By SILAS W. HOLMAN. New York: Macmillan, 1896; pp. xlv + 73.

THIS is not a Manual such as the 'Manual of Logarithms' by G. F. Matthews and the 'Examples for practice in the use of seven-figure Logarithms' by Wolstenholme, nor is it conterminous with the 'Logarithmic Tables' by Prof. George W. Jones, but it treats the subject more from the point of view of the engineering and scientific student. This is to be expected from the author's position of Professor of Physics at the Massachusetts Institute of Technology.

The preliminary matter is very clearly put and several practical exercises are worked out. It treats of Rules of Computation (up to date), Logarithms, Antilogs and Cologs, Squares and Square Roots, Reciprocals, the ordinary Trigonometrical Natural and Log Sines, &c., and Slide Wire Ratios (this will be of use to students of physical chemistry). For those who are unacquainted with the terms employed in the previous introduction, definitions and explanations of them are appended.

The tables are in the main four-place ones, but space is also devoted to five-place logarithms of numbers and of the Trigonometrical functions.

Prof. Holman has done his work well, and the many contrivances in working and in the printing of the tables evince him to be a thoroughly practical teacher. We can accord him no higher praise.

XXX. *Intelligence and Miscellaneous Articles.*

BLACK LIGHT. BY M. GUSTAVE LE BON.

THE recent publication of photographic experiments with light of kathodic origin has decided me to make known, though they are still very incomplete, some researches which I have carried on for the last two years in photography through bodies opaque to *ordinary light.* The two subjects are very different; only in their results are there some analogies.

The following experiments prove that ordinary light, or at any rate some of its radiations, traverses the most opaque bodies without difficulty. Opacity is a phenomenon which only exists for an eye like our own; if it were somewhat differently constructed, we could see through walls.

In an ordinary positive photograph-frame place a sensitive plate and above this any ordinary photograph, then above the photograph,

and in close contact with it, a plate of iron entirely covering the interior of the frame. The plate thus masked by the metal plate is exposed to the light of a petroleum-lamp for about three hours. An energetic and very prolonged development of the sensitive plate carried on to almost complete blackness will give a very pale image of the photograph, but one which is very marked by transmitted light.

By a slight modification of the preceding experiment, images may be obtained almost as sharp as if no obstacle had been interposed between the light and the sensitive plate. Without altering anything in the preceding arrangement, let a plate of lead of any thickness be placed behind the sensitive plate, and its edges bent over until they slightly cover the edges of the iron plate. The sensitive plate is thus enclosed in a sort of metal box, the front part of which consists of an iron plate, the back and the sides being formed of lead. After three hours of exposure to petroleum light, we obtain as before a vigorous image after development.

What part does the lead plate play in this second experiment. *Provisionally* I imagine that the contact of two different metals gives rise to very feeble thermoelectric currents, the action of which adds itself to that of the luminous radiations which have passed through the iron plate.

I hope to be able soon to determine the part of the various factors which come into play in producing the preceding results. I hope thus to determine the properties of light after its passage through opaque bodies. The action which might be exerted on the clichés by heat, or that of light stored up, have been entirely eliminated by my experiments.

The action of the sun's light gives the same result as that of petroleum, and does not appear to be much more active.

Cardboard and metals, particularly iron and copper, are easily traversed by light. This passage of light through the bodies is only a question of time.

If the experiments are repeated in the photographic camera, that is to say, if a metal plate is placed in front of a sensitive plate, and therefore between the latter and the object to be photographed, on exposure for two hours to the sunlight, an intense blackening is obtained on development, proving the passage of light through the opaque plate, but images are only exceptionally obtained and in conditions which I have not yet been able to determine.

As they are invisible to the eye, I give the name of *black light* to those radiations of unknown origin which pass in this way through opaque bodies. Considering the divergences between the numbers of vibrations producing various kinds of energy, such as electricity and light, we may imagine that there are intermediate numbers corresponding to natural forces still unknown. The latter must be connected by imperceptible transitions to the forces we know. The possible forms of energy, although we know very little about them at present, must be infinite in number. *Black light* perhaps represents one of the forces of which we are ignorant.—*Comptes Rendus*, Jan. 27, 1896.

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[FIFTH SERIES.]

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XXXI. *On the Influence of Carbonic Acid in the Air upon the Temperature of the Ground.* *By* Prof. SVANTE ARRHENIUS *.

I. *Introduction : Observations of* Langley *on Atmospherical Absorption.*

A GREAT deal has been written on the influence of the absorption of the atmosphere upon the climate. Tyndall † in particular has pointed out the enormous importance of this question. To him it was chiefly the diurnal and annual variations of the temperature that were lessened by this circumstance. Another side of the question, that has long attracted the attention of physicists, is this : Is the mean temperature of the ground in any way influenced by the presence of heat-absorbing gases in the atmosphere ? Fourier ‡ maintained that the atmosphere acts like the glass of a hot-house, because it lets through the light rays of the sun but retains the dark rays from the ground. This idea was elaborated by Pouillet § ; and Langley was by some of his researches led to the view, that " the temperature of the earth under direct sunshine, even though our atmosphere were present as now, would probably fall to −200° C., if that atmosphere did not possess the quality of selective

* Extract from a paper presented to the Royal Swedish Academy of Sciences, 11th December, 1895. Communicated by the Author.

† 'Heat a Mode of Motion,' 2nd ed. p. 405 (Lond., 1865).

‡ *Mém. de l'Ac. R. d. Sci. de l'Inst. de France*, t. vii. 1827.

§ *Comptes rendus*, t. vii. p. 41 (1838).

absorption" *. This view, which was founded on too wide a use of Newton's law of cooling, must be abandoned, as Langley himself in a later memoir showed that the full moon, which certainly does not possess any sensible heat-absorbing atmosphere, has a "mean effective temperature" of about 45° C.†

The air retains heat (light or dark) in two different ways. On the one hand, the heat suffers a selective diffusion on its passage through the air; on the other hand, some of the atmospheric gases absorb considerable quantities of heat. These two actions are very different. The selective diffusion is extraordinarily great for the ultra-violet rays, and diminishes continuously with increasing wave-length of the light, so that it is insensible for the rays that form the chief part of the radiation from a body of the mean temperature of the earth ‡.

* Langley, 'Professional Papers of the Signal Service,' No. 15. "Researches on Solar Heat," p. 123 (Washington, 1884).

† Langley, "The Temperature of the Moon." Mem. of the National Academy of Sciences, vol. iv. 9th mem. p. 193 (1890).

‡ Langley, 'Prof. Papers,' No. 15, p. 151. I have tried to calculate a formula for the value of the absorption due to the selective reflexion, as determined by Langley. Among the different formulæ examined, the following agrees best with the experimental results:—

$$\log a = b\,(1/\lambda) + c\,(1/\lambda)^3.$$

I have determined the coefficients of this formula by aid of the method of least squares, and have found—

$$b = -0{\cdot}0463, \qquad c = -0{\cdot}008204.$$

a represents the strength of a ray of the wave-length λ (expressed in μ) after it has entered with the strength 1 and passed through the air-mass 1. The close agreement with experiment will be seen from the following table:—

λ.	$a^{1/7\cdot6}$ (obs.).	$a^{1/7\cdot6}$ (calc.).	Prob. error.
0·358 μ	0·904	0·911	
0·383	0·920	0·923	0·0047
0·416	0·935	0·934	
0·440	0·942	0·941	
0·468	0·950	0·947	0·0028
0·550	0·960	0·960	
0·615	0·968	0·967	
0·781	0·978	0·977	
0·870	0·982	0·980	0·0017
1·01	0·985	0·984	
1·20	0·987	0·987	
1·50	0·989	0·990	0·0011
2·59	0·990	0·993	0·0018

The selective absorption of the atmosphere is, according to the researches of Tyndall, Lecher and Pernter, Röntgen, Heine, Langley, Ångström, Paschen, and others *, of a wholly different kind. It is not exerted by the chief mass of the air, but in a high degree by aqueous vapour and carbonic acid, which are present in the air in small quantities. Further, this absorption is not continuous over the whole spectrum, but nearly insensible in the light part of it, and chiefly limited to the long-waved part, where it manifests itself in very well-defined absorption-bands, which fall off rapidly on both sides †. The influence of this absorption is comparatively small on the heat from the sun, but must be of great importance in the transmission of rays from the earth. Tyndall held the opinion that the water-vapour has the greatest influence, whilst other authors, for instance Lecher and Pernter, are inclined to think that the carbonic acid plays the more important part. The researches of Paschen show that these gases are both very effective, so that probably sometimes the one, sometimes the other, may have the greater effect according to the circumstances.

In order to get an idea of how strongly the radiation of the earth (or any other body of the temperature $+15°$ C.) is absorbed by quantities of water-vapour or carbonic acid in the proportions in which these gases are present in our atmosphere, one should, strictly speaking, arrange experiments on the absorption of heat from a body at 15° by means of appropriate quantities of both gases. But such experiments have not been made as yet, and, as they would require very expensive apparatus beyond that at my disposal, I have not been in a position to execute them. Fortunately there are other researches by Langley in his work on 'The Temperature

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For ultra-violet rays the absorption becomes extremely great in accordance with facts.

As one may see from the probable errors which I have placed alongside for the least concordant values and also for one value ($1{\cdot}50\ \mu$), where the probable error is extremely small, the differences are just of the magnitude that one might expect in an exactly fitting formula. The curves for the formula and for the experimental values cut each other at four points ($1/\lambda = 2{\cdot}43$, $1{\cdot}88$, $1{\cdot}28$, and $0{\cdot}82$ respectively). From the formula we may estimate the value of the selective reflexion for those parts of the spectrum that prevail in the heat from the moon and the earth (angle of deviation $= 38 - 36°$, $\lambda = 10{\cdot}4 - 24{\cdot}4\ \mu$). We find that the absorption from this cause varies betweeen 0·5 and 1 p. c. for air-mass 1. This insensible action, which is wholly covered by the experimental errors, I have neglected in the following calculations.

* Vide Winkelmann, *Handbuch der Physik.*

† Cf., e.g., Trabert, *Meteorologische Zeitschrift*, Bd. ii. p. 238 (1894).

of the Moon,' with the aid of which it seems not impossible to determine the absorption of heat by aqueous vapour and by carbonic acid in precisely the conditions which occur in our atmosphere. He has measured the radiation of the full moon (if the moon was not full, the necessary correction relative to this point was applied) at different heights and seasons of the year. This radiation was moreover dispersed in a spectrum, so that in his memoir we find the figures for the radiant heat from the moon for 21 different groups of rays, which are defined by the angle of deviation with a rocksalt prism having a refracting angle of 60 degrees. The groups lie between the angles 40° and 35°, and each group is separated from its neighbours by an interval of 15 minutes. Now the temperature of the moon is nearly the same as that of the earth, and the moon-rays have, as they arrive at the measuring-instruments, passed through layers of carbonic acid and of aqueous vapour of different thickness according to the height of the moon and the humidity of the air. If, then, these observations were wholly comparable with one another, three of them would suffice for calculating the absorption coefficient relatively to aqueous vapour and carbonic acid for any one of the 21 different groups of rays. But, as an inspection of the 24 different series of observations will readily show, this is not the case. The intensity of radiation for any group of rays should always diminish with increasing quantity of aqueous vapour or carbonic acid traversed. Now the quantity of carbonic acid is proportional to the path of the ray through the atmosphere, that is, to the quantity called "Air-mass" in Langley's figures. As unit for the carbonic acid we therefore take air-mass = 1, *i. e.* the quantity of carbonic acid that is traversed in the air by a vertical ray. The quantity of aqueous vapour traversed is proportional partly to the "air-mass," partly to the humidity, expressed in grammes of water per cubic metre. As unit for the aqueous vapour I have taken the quantity of aqueous vapour that is traversed by a vertical ray, if the air contains 10 grammes per cubic metre at the earth's surface *. If we tabulate the 24 series of observations published by Langley in the work cited with respect to the quantities of carbonic acid and aqueous vapour, we immediately detect that his figures run very irregularly, so that very many exceptions are found to the rule that the transmitted heat should continuously decrease when both these quantities increase.

* This unit nearly corresponds to the mean humidity of the air (see Table VI. p. 264).

And it seems as if periodic alterations with the time of observation occurred in his series. On what circumstance these alterations with the time depend one can only make vague conjectures: probably the clearness of the sky may have altered within a long period of observation, although this could not be detected by the eye. In order to eliminate this irregular variation, I have divided the observations into four groups, for which the mean quantities of carbonic acid (K) and of water-vapour (W) were 1·21 and 0·36, 2·21 and 0·86, 1·33 and 1·18, and 2·22 and 2·34 respectively. With the help of the mean values of the heat-radiation for every group of rays in these four groups of observations, I have roughly calculated the absorption coefficients (x and y) for both gases, and by means of these reduced the value for each observation to the value that it would have possessed if K and W had been 1·5 and 0·88 respectively. The 21 values for the different rays were then summed up, so that I obtained the total heat-radiation for every series of observations, reduced to K = 1·5 and W = 0·88. If the materials of observation were very regular, the figures for this total radiation should not differ very much from one another. In fact, one sees that observations that are made at nearly the same time give also nearly equal values, but if the observations were made at very different times, the values differ also generally very much. For the following periods I have found the corresponding mean values of the total radiation :—

Period.	Mean value.	Reduction factor.
1885. Feb. 21–June 24	4850	1·3
1885. July 29–1886. Feb. 16.	6344	1·00
1886. Sept. 13–Sept. 18	2748	2·31
1886. Oct. 11–Nov. 8	5535	1·15
1887. Jan. 8–Feb. 9	3725	1·70

In order to reduce the figures of Langley to comparability with one another, I have applied the reduction factors tabulated above to the observations made in the respective periods. I have convinced myself that by this mode of working no systematic error is introduced into the following calculations.

After this had been done, I rearranged the figures of Langley's groups according to the values of K and W in the following table. (For further details see my original memoir.)

TABLE I.—*Radiation (i) of the Full Moon for*

	40°.	39°·45.	39°·30.	39°·15.	39°.	38°·45.	38°·30.	38°·15.	38°.	37°·45.	37°·30.
K	1·16	1·12	1·16	1·13	1·16	1·13	1·16	1·13	1·16	1·13	1·16
W	0·32	0·269	0·32	0·271	0·32	0·271	0·32	0·271	0·32	0·271	0·32
i obs. ...	28·7	26·6	27·0	26·4	24·8	24·8	12·6	20·1	43·8	65·9	74·4
i calc....	27·0	34·5	29·0	25·7	24·4	23·5	12·5	19·4	40·8	58·0	68·8
G	79	27	75	56	69	53	35	43	121	140	206
K	1·28	1·27	1·29	1·29	1·29	1·29	1·27	1·26	1·29	1·27	1·27
W	0·81	1·07	0·86	1·04	0·86	1·04	0·90	0·96	0·86	1·07	1·00
i obs. ...	22·9	31·2	26·7	21·3	18·2	11·0	5·8	3·7	14·0	32·0	52·3
i calc...	23·1	27·9	25·4	21·2	21·8	12·5	8·6	12·8	26·1	42·1	52·7
G	76	135	109	73	74	38	24	13	57	139	261
K	1·46	1·40	1·39	1·49	1·49	1·49	1·50	1·49	1·50	1·49	1·50
W	0·75	0·823	0·78	0·87	0·89	0·89	0·82	0·89	0·82	0·87	0·84
i obs. ..	11·9	28·2	23·0	18·9	18·0	9·2	9·9	14·4	24·6	34·8	46·6
i calc....	23·6	29·4	25·4	20·9	18·6	12·7	7·8	10·8	24·4	43·2	55·2
G	28	28	25	38	37	17	33	28	81	70	151
K	1·48	1·52	1·48	1·51	1·48	1·51	1·48	1·51	1·48	1·52	1·48
W	1·80	2·03	1·78	1·64	1·78	1·95	1·80	1·95	1·80	2·03	1·67
i obs. ..	25·2	27·6	24·6	18·3	27·6	4·8	3·7	3·6	17·6	45·5	43·9
i calc....	16·9	21·4	20·2	17·9	18·5	5·9	4·7	6·6	12·0	28·2	40·2
G	30	22	51	31	37	5	4	3	21	37	119
K	2·26	2·26	2·26	2·26	2·26	2·26	2·26	2·26	2·27	2·26	2·27
W	1·08	1·08	1·08	1·08	1·08	1·08	1·08	1·08	1·06	1·08	1·06
i obs. ...	21·3	23·4	20·8	16·4	11·1	8·2	4·5	3·5	17·3	36·1	47·1
i calc....	21·2	25·9	21·3	16·6	10·1	7·7	4·5	5·1	14·7	33·9	48·3
G	44	49	43	34	23	17	9	7	37	75	112
K	2·05	1·92	1·92	1·93	1·92	1·92	1·92	2·45	2·37	1·92	2·05
W	1·93	2·30	2·24	2·16	2·24	2·30	2·24	2·25	2·20	2·30	1·93
i obs. ...	13·4	12·8	14·8	15·1	10·3	6·6	3·4	3·4	7·9	20·8	31·5
i calc....	16·2	19·4	17·3	14·5	13·0	3·8	2·9	2·6	6·1	23·4	35·1
G	55	29	35	47	25	15	8	10	26	47	129

different Values of K *and* W.

	37°·15.	37°.	36°·45.	36°·30.	36°·15.	36°.	35°·45.	35°·30.	35°·15.	35°.
K	1·16	1·16	1·18	1·18	1·27	1·16	1·27	1·27	1·27	1·16
W	0·32	0·32	0·34	0·34	0·48	0·32	0·48	0·48	0·48	0·32
i obs. ...	68·6	59	56·2	48·3	43·4	40·7	39·0	32·6	31·5	19·7
i calc....	73·7	57·1	50·9	46·0	34·9	36·4	31·3	27·7	27·3	19·3
G	190	163	118	102	28	112	25	21	20	54
K	1·27	1·27	1·31	1·32	1·32	1·28	1·33	1·33	1·33	1·25
W	1·00	1·00	1·05	1·00	1·00	0·81	0·51	0·51	1·07	0·60
i obs. ...	58·9	50·3	47·9	41·2	31·7	29·7	25·7	18·8	27·5	16·6
i calc....	53·0	51·2	47·1	39·2	34·2	31·1	30·3	26·8	21·3	17·2
G	294	251	205	140	108	98	16	12	39	22
K	1·49	1·48	1·48	1·48	1·41	1·45	1·41	1·41	1·41	1·41
W	0·87	0·85	0·85	0·85	0·97	0·89	0·97	0·98	0·98	0·98
i obs. ...	43·1	36·4	35·4	31·2	28·3	24·9	16·6	15·4	10·3	9·2
i calc....	55·2	47·1	42·5	36·3	33·0	29·3	27·3	22·3	22·0	14·7
G	87	149	146	127	54	78	32	29	19	17
K	1·48	1·48	1·48	1·48	1·48	1·48	1·48	1·48	1·48	1·48
W	1·66	1·58	1·66	1·66	1·83	1·66	1·83	1·58	1·83	1·66
i obs. ...	47·5	48·7	45·8	34·5	35·0	27·5	28·7	21·4	17·4	15·4
i calc....	38·2	43·4	42·5	33·0	32·0	23·6	23·4	17·8	15·4	11·6
G	136	176	131	99	82	79	67	81	41	43
K	2·26	2·12	1·91	1·90	1·91	2·09	1·91	1·90	1·90	2·12
W	1·08	1·15	1·10	1·11	1·10	1·18	1·10	1·11	1·11	1·15
i obs ...	44·6	32·0	27·8	24·7	26·6	24·5	19·0	16·0	13·9	10·1
i calc....	47·1	33·5	32·8	27·4	26·8	23·6	21·3	17·5	20·4	12·2
G	93	98	66	58	63	72	45	37	32	31
K	1·92	2·05	2·45	2·37	2·45	2·37	1·97	1·97	1·97	1·97
W	2·30	1·93	2·25	2·20	2·25	2·20	2·33	2·33	2·33	2·33
i obs. ...	24·7	33·2	26·7	19·4	22·6	18·8	16·4	10·9	12·1	7·9
i calc....	27·1	31·8	23·7	18·4	21·4	16·8	17·4	11·5	12·2	8·4
G	56	137	77	63	65	61	32	22	24	16

In this table the angle of deviation is taken as head-title. After K and W stand the quantities of carbonic acid and water-vapour traversed by the ray in the above-mentioned units. Under this comes after *i* obs. the intensity of radiation (reduced) observed by Langley on the bolometer, and after this the corresponding value *i* calc., calculated by means of the absorption-coefficients given in Table II. below. G is the "weight" given to the corresponding *i* obs. in the calculation, using the method of least squares.

For the absorption-coefficients, calculated in this manner, I give the following table. (The common logarithms of the absorption-coefficients are tabulated.)

TABLE II.—*Absorption-Coefficients of Carbonic Acid* (x) *and Aqueous Vapour* (y).

Angle of Deviation.	log x.	log y.	A.
40°	+0·0286 0·0000	−0·1506 −0·1455	27·2
39·45	−0·0296	−0·1105	34·5
39·30	−0·0559	−0·0952	29·6
39·15	−0·1070	−0·0862	26·4
39·0	−0·3412	−0·0068	27·5
38·45	−0·2035	−0·3114	24·5
38·30	−0·2438	−0·2362	13·5
38·15	−0·3760	−0·1933	21·4
38·0	−0·1877	−0·3198	44·4
37·45	−0·0931	−0·1576	59·0
37·30	−0·0280	−0·1661	70·0
37·15	−0·0416	−0·2036	75·5
37·0	−0·2067	−0·0484	62·9
36·45	−0·2465 −0·2466	+0·0008 −0·0000	56·4
36·30	−0·2571	−0·0507	51·4
36·15	−0·1708 −0·1652	+0·0065 −0·0000	39·1
36·0	−0·0940	−0·1184	37·9
35·45	−0·1992	−0·0628	36·3
35·30	−0·1742	−0·1408	32·7
35·15	−0·0188	−0·1817	29·8
35·0	−0·0891	−0·1444	21·9

The signification of these figures may be illustrated by an example. If a ray of heat, corresponding to the angle of deviation 39°·45, passes through the unit of carbonic acid, it de-

creases in intensity in the proportion 1 : 0·934 (log = −0·0296), the corresponding value for the unit of water-vapour is 1 : 0·775 (log = −0·1105). These figures are of course only valid for the circumstances in which the observations were made, viz., that the ray should have traversed a quantity of carbonic acid K = 1·1 and a quantity of water-vapour W = 0·3 before the absorption in the next quantities of carbonic acid and water-vapour was observed. And these second quantities should not exceed K = 1·1 and W = 1·8, for the observations are not extended over a greater interval than between K = 1·1 and K = 2·2, and W = 0·3 and W = 2·1 (the numbers for K and W are a little different for rays of different kind). Below A is written the relative value of the intensity of radiation for a given kind of ray in the moonlight after it has traversed K = 1 and W = 0·3. In some cases the calculation gives positive values for log x or log y. As this is a physical absurdity (it would signify that the ray should be strengthened by its passage through the absorbing gas), I have in these cases, which must depend on errors of observation, assumed the absorption equal to zero for the corresponding gas, and by means of this value calculated the absorption-coefficient of the other gas, and thereafter also A.

As will be seen from an inspection of Table I., the values of i obs. agree in most cases pretty well with the calculated values i calc. But in some cases the agreement is not so good as one could wish. These cases are mostly characterized by a small "weight" G, that is in other words, the material of observation is in these cases relatively insufficient. These cases occur also chiefly for such rays as are strongly absorbed by water-vapour. This effect is probably owing to the circumstance that the aqueous vapour in the atmosphere, which is assumed to have varied proportionally to the humidity at the earth's surface, has not always had the assumed ideal and uniform distribution with the height. From observations made during balloon voyages, we know also that the distribution of the aqueous vapour may be very irregular, and different from the mean ideal distribution. It is also a marked feature that in some groups, for instance the third, nearly all the observed numbers are less than the calculated ones, while in other groups, for instance the fourth, the contrary is the case. This circumstance shows that the division of the statistic material is carried a little too far; and a combination of these two groups would have shown a close agreement between the calculated and the observed figures. As, however, such a combination is without influence on the correctness of the calculated absorption-coefficients, I have omitted

a rearrangement of the figures in greater groups, with consequent recalculation.

A circumstance that argues very greatly in favour of the opinion that the absorption-coefficient given in Table II. cannot contain great errors, is that so very few logarithms have a positive value. If the observations of Langley had been wholly insufficient, one would have expected to find nearly as many positive as negative logarithms. Now there are only three such cases, viz., for carbonic acid at an angle of 40°, and for water-vapour at the angles 36°·45 and 36°·15. The observations for 40° are not very accurate, because they were of little interest to Langley, the corresponding rays not belonging to the moon's spectrum but only to the diffused sunlight from the moon. As these rays also do not occur to any sensible degree in the heat from a body of 15° C., this non-agreement is without importance for our problem. The two positive values for the logarithms belonging to aqueous vapour are quite insignificant. They correspond only to errors of 0·2 and 1·5 per cent. for the absorption of the quantity $W=1$, and fall wholly in the range of experimental errors.

It is certainly not devoid of interest to compare these absorption-coefficients with the results of the direct observations by Paschen and Ångström*. In making this comparison, we must bear in mind that an exact agreement cannot be expected, for the signification of the above coefficients is rather unlike that of the coefficients that are or may be calculated from the observations of these two authors. The above coefficients give the rate of absorption of a ray that has traversed quantities of carbonic acid ($K=1·1$) and water-vapour ($W=0·3$); whilst the coefficients of Paschen and Ångström represent the absorption experienced by a ray on the passage through the first layers of these gases. In some cases we may expect a great difference between these two quantities, so that only a general agreement can be looked for.

According to Paschen's figures there seems to exist no sensible emission or absorption by the aqueous vapour at wave-lengths between $0·9\mu$ and $1·2\mu$ (corresponding to the angle of deviation 40°). On the other hand, the representation of the sun's spectrum by Langley shows a great many

* Paschen, Wied. *Ann.* l. p. 409, 1893; li. p. 1, lii. p. 209, and liii. p. 334, 1894, especially vol. l. tab. ix. fig. 5, curve 1 for carbonic acid, curve 2 for aqueous vapour. Ångström, *Bihang till K. Vet.-Ak. Handlingar*, Bd. xv. Afd. 1, No. 9, p. 15, 1889; *Öfversigt af K. Vet.-Ak. Förhandl.* 1889, No. 9, p. 553.

strong absorption-bands in this interval, among which those marked ρ, σ, τ, and ϕ are the most prominent*, and these absorption-bands belong most probably to the aqueous vapour, That Paschen has not observed any emission by water-vapour in this interval may very well be accounted for by the fact that his heat-spectrum had a very small intensity for these short-waved rays. But it may be conceded that the absorption-coefficient for aqueous vapour at this angle in Table II. is not very accurate (probably too great), in consequence of the little importance that Langley attached to the corresponding observations. After this occurs in Langley's spectrum the great absorption-band ψ at the angle 39·45 ($\lambda = 1{\cdot}4\,\mu$), where in Paschen's curve the emission first becomes sensible ($\log y = -0{\cdot}1105$ in Table II.). At wave-lengths of greater value we find according to Paschen strong absorption-bands at $\lambda = 1{\cdot}83\,\mu$ (Ω in Langley's spectrum), *i. e.* in the neighbourhood of 39°·30 and at $\lambda = 2{\cdot}64\,\mu$ (Langley's X) a little above the angle 39°·15. In accordance with this I have found rather large absorption-coefficients for aqueous vapour at these angles ($\log y = -0{\cdot}0952$ and $-0{\cdot}0862$ resp.). From $\lambda = 3{\cdot}0\,\mu$ to $\lambda = 4{\cdot}7\,\mu$ thereafter, according to Paschen the absorption is very small, in agreement with my calculation ($\log y = -0{\cdot}0068$ at 39°, corresponding to $\lambda = 4{\cdot}3\,\mu$). From this point the absorption increases again and presents new maxima at $\lambda = 5{\cdot}5\,\mu$, $\lambda = 6{\cdot}6\,\mu$, and $\lambda = 7{\cdot}7\,\mu$, *i. e.* in the vicinity of the angles 38°·45 ($\lambda = 5{\cdot}6\,\mu$) and 38°·30 ($\lambda = 7{\cdot}1\,\mu$). In this region the absorption of the water-vapour is continuous over the whole interval, in consequence of which the great absorption-coefficient in this part ($\log y = -0{\cdot}3114$ and $-0{\cdot}2362$) becomes intelligible. In consequence of the decreasing intensity of the emission-spectrum of aqueous vapour in Paschen's curve we cannot pursue the details of it closely, but it seems as if the emission of the water-vapour would also be considerable at $\lambda = 8{\cdot}7\,\mu$ (39°·15), which corresponds with the great absorption-coefficient ($\log y = -0{\cdot}1933$) at this place. The observations of Paschen are not extended further, ending at $\lambda = 9{\cdot}5\,\mu$, which corresponds to an angle of 39°·08.

For carbonic acid we find at first the value zero at 40°, in agreement with the figures of Paschen and Ångström†. The absorption of carbonic acid first assumes a sensible value at

* Langley, *Ann. Ch. et Phys.* sér. 6, t. xvii. pp. 323 and 326, 1889, Prof. Papers, No. 15, plate 12. Lamansky attributed his absorption-bands, which probably had this place, to the absorbing power of aqueous vapour (Pogg. *Ann.* cxlvi. p. 200, 1872).

† It must be remembered that at this point the spectrum of Paschen was very weak, so that the coincidence with his figure may be accidental.

$\lambda = 1{\cdot}5\,\mu$, after which it increases rapidly to a maximum at $\lambda = 2{\cdot}6\,\mu$, and attains a new extraordinarily strong maximum at $\lambda = 4{\cdot}6$ (Langley's Y). According to Ångström the absorption of carbonic acid is zero at $\lambda = 0{\cdot}9\,\mu$, and very weak at $\lambda = 1{\cdot}69\,\mu$, after which it increases continuously to $\lambda = 4{\cdot}6\,\mu$ and decreases again to $\lambda = 6{\cdot}0\,\mu$. This behaviour is entirely in agreement with the values of $\log x$ in Table II. From the value zero at 40° ($\lambda = 1{\cdot}0\,\mu$) it attains a sensible value ($-0{\cdot}0296$) at 39°·45 ($\lambda = 1{\cdot}4\,\mu$), and thereafter greater and greater values ($-0{\cdot}0559$ at 39°·30, and $-0{\cdot}1070$ at 39°·15) till it reaches a considerable maximum ($-0{\cdot}3412$ at 39°, $\lambda = 4{\cdot}3\,\mu$). After this point the absorption decreases (at 38°·45 $= 5{\cdot}6\,\mu$, $\log x = -0{\cdot}2035$). According to Table II. the absorption of carbonic acid at 38°·30 and 38°·15 ($\lambda = 7{\cdot}1\,\mu$ and $8{\cdot}7\,\mu$) has very great values ($\log x = -0{\cdot}2438$ and $-0{\cdot}3730$), whilst according to Ångström it should be insensible. This behaviour may be connected with the fact that Ångström's spectrum had a very small intensity for the larger wave-lengths. In Paschen's curve there are traces of a continuous absorption by the carbonic acid in this whole region with weak maxima at $\lambda = 5{\cdot}2\,\mu$, $\lambda = 5{\cdot}9\,\mu$, $\lambda = 6{\cdot}6\,\mu$ (possibly due to traces of water-vapour), $\lambda = 8{\cdot}4\,\mu$, and $\lambda = 8{\cdot}9\,\mu$. In consequence of the strong absorption of water-vapour in this region of the spectrum, the intensity of radiation was very small in Langley's observations, so that the calculated absorption-coefficients are there not very exact (*cf.* above, pp. 242–243). Possibly the calculated absorption of the carbonic acid may have come out too great, and that of the water-vapour too small in this part (between 38°·30 and 38°·0). This can happen the more easily, as in Table I. K and W in general increase together because they are both proportional to the "air-mass." It may be pointed out that this also occurs in the problems that are treated below, so that the error from this cause is not of so great importance as one might think at the first view.

For angles greater than 38° ($\lambda > 9{\cdot}5\,\mu$) we possess no direct observations of the emission or absorption of the two gases. The sun's spectrum, according to Langley, exhibits very great absorption-bands at about 37°·50, 37°·25, 37°, and 36°·40°. According to my calculations the aqueous vapour has its greatest absorbing power in the spectrum from 38° to 35° at angles between 37°·15 and 37°·45 (the figures for 35°·45, 35°·30, and 35°·15 are very uncertain, as they depend upon very few measurements), and the carbonic acid between 36°·30 and 37°·0. This seems to indicate that the first two absorption-bands are due to the action of water-

vapour, the last two to that of carbonic acid. It should be emphasized that Langley has applied the greatest diligence in the measurement of the intensity of the moon's radiation at angles between 36° and 38°, where this radiation possesses its maximum intensity. It may, therefore, be assumed that the calculated absorption-coefficients for this part of the spectrum are the most exact. This is of great importance for the following calculations, for the radiation from the earth * has by far the greatest intensity (about two thirds, *cf.* p. 250) in this portion of the spectrum.

II. *The Total Absorption by Atmospheres of Varying Composition.*

As we have now determined, in the manner described, the values of the absorption-coefficients for all kinds of rays, it will with the help of Langley's figures † be possible to calculate the fraction of the heat from a body at 15° C. (the earth) which is absorbed by an atmosphere that contains specified quantities of carbonic acid and water-vapour. To begin with, we will execute this calculation with the values $K=1$ and $W=0{\cdot}3$. We take that kind of ray for which the best determinations have been made by Langley, and this lies in the midst of the most important part of the radiation (37°). For this pencil of rays we find the intensity of radiation at $K=1$ and $W=0{\cdot}3$ equal to 62·9; and with the help of the absorption-coefficients we calculate the intensity for $K=0$ and $W=0$, and find it equal to 105. Then we use Langley's experiments on the spectral distribution of the radiation from a body of 15° C., and calculate the intensity for all other angles of deviation. These intensities are given under the heading M. After this we have to calculate the values for $K=1$ and $W=0{\cdot}3$. For the angle 37° we know it to be 62·9. For any other angle we could take the values A from Table II. if the moon were a body of 15° C. But a calculation of the figures of Very ‡ shows that the full moon has a higher temperature, about 100° C. Now the spectral distribution is nearly, but not quite, the same for the heat from a body of 15° C. and for that from one of 100° C. With the help of Langley's figures it is, however, easy to reduce the intensities for the hot body at 100° (the moon) to be valid for a body at 15°

* After having been sifted through an atmosphere of $K=1{\cdot}1$ and $W=0{\cdot}3$.

† 'Temperature of the Moon,' plate 5.

‡ "The Distribution of the Moon's Heat," Utrecht Society of Arts and Sc. The Hague, 1891.

(the earth). The values of A reduced in this manner are tabulated below under the heading N.

Angle...	40°.	39·45.	39·30.	39·15.	39·0.	38·45.	38·30.	38·15.	38·0.	37·45.	37·30.
M	3·4	11·6	24·8	45·9	84·0	121·7	161	189	210	210	188
N	3·1	10·1	11·3	13·7	18·0	18·1	11·2	19·6	44·4	59	70

Angle...	37°·15.	37·0.	36·45.	36·30.	36·15.	36·0.	35·45.	35·30.	35·15.	35·0.	Sum.	P.c.
M	147	105	103	99	60	51	65	62	43	39	2023	100
N	75·5	62·9	56·4	51·4	39·1	37·9	39·2	37·6	36·0	28·7	743·2	37·2

For angles less than 37° one finds, in the manner above described, numbers that are a little inferior to the tabulated ones, which are found by means of the absorption-coefficients of Table II. and the values of N. In this way the sum of the M's is a little greater (6·8 per cent.) than it would be according to the calculation given above. This non-agreement results probably from the circumstance that the spectrum in the observations was not quite pure.

The value 37·2 may possibly be affected with a relatively great error in consequence of the uncertainty of the M-values. In the following calculations it is not so much the value 37·2 that plays the important part, but rather the diminution of the value caused by increasing the quantities K and W. For comparison, it may be mentioned that Langley has estimated the quantity of heat from the moon that passed through the atmosphere (of mean composition) in his researches to be 38 per cent.* As the mean atmosphere in Langley's observations corresponded with higher values of K and W than K = 1 and W = 0·3, it will be seen that he attributed to the atmosphere a greater transparence for opaque rays than I have done. In accordance with Langley's estimation, we should expect for K = 1 and W = 0·3 a value of about 44 instead of 37·2. How great an influence this difference may exert will be investigated in what follows.

The absorption-coefficients quoted in Table II. are valid for an interval of K between about 1·1 and 2·25, and for W between 0·3 and 2·22. In this interval one may, with the help of those coefficients and the values of N given above, calculate the value of N for another value of K and W, and so in this way obtain by means of summation the total heat that passes through an atmosphere of given condition. For further calculations I have also computed values of N for atmospheres that contain greater quantities of carbonic acid and aqueous vapour. These values must be considered as extrapolated. In the following table (Table III.) I have given these values of N. The numbers printed in italics are found directly in the manner

* Langley, 'Temperature of the Moon,' p. 197.

described, those in ordinary type are interpolated from them with the help of Pouillet's exponential formula. The table has two headings, one which runs horizontally and represents the quantity of aqueous vapour (W), and another that runs vertically and represents the quantity of carbonic acid (K) in the atmosphere.

TABLE III.—*The Transparency of a given Atmosphere for Heat from a body of* 15° C.

→ H_2O ↓ CO_2.	0·3.	0·5.	1·0:	1·5.	2·0.	3·0.	4·0.	6·0.	10·0.
1	*37·2*	*35·0*	*30·7*	*26·9*	*23·9*	*19·3*	*16·0*	*10·7*	*8·9*
1·2	*34·7*	32·7	28·6	*25·1*	22·2	17·8	14·7	9·7	8·0
1·5	*31·5*	29·6	25·9	*22·6*	19·9	*15·9*	13·0	8·4	6·9
2	*27·0*	*25·3*	*21·9*	*19·1*	16·7	13·1	10·5	6·6	*5·3*
2·5	*23·5*	22·0	19·0	16·6	14·4	*11·0*	8·7	5·3	4·2
3	*20·1*	18·8	16·3	*14·2*	12·3	9·3	*7·4*	4·2	*3·3*
4	*15·8*	14·7	12·7	10·8	9·3	*7·1*	5·6	3·1	2·0
6	*10·9*	10·2	*8·7*	*7·3*	6·3	4·8	3·7	*1·9*	*0·93*
10	*6·6*	6·1	5·2	4·3	3·5	*2·4*	1·8	1·0	*0·26*
20	*2·9*	2·5	2·2	*1·8*	1·5	*1·0*	0·75	0·39	*0·07*
40	*0·88*	0·81	0·67	*0·56*	0·46	*0·32*	0·24	*0·12*	*0·02*

Quite different from this dark heat is the behaviour of the heat from the sun on passing through new parts of the earth's atmosphere. The first parts of the atmosphere exert without doubt a selective absorption of some ultra-red rays, but as soon as these are extinguished the heat seems not to diminish as it traverses new quantities of the gases under discussion. This can easily be shown for aqueous vapour with the help of Langley's actinometric observations from Mountain Camp and Lone Pine in Colorado*. These observations were executed at Lone Pine from the 18th of August to the 6th of September 1882 at $7^h\ 15^m$ and $7^h\ 45^m$ A.M., at $11^h\ 45^m$ A.M. and $12^h\ 15^m$ P.M., and at $4^h\ 15^m$ and $4^h\ 45^m$ P.M. At Mountain Camp the observations were carried out from the 22nd to the 25th of August at the same times of the day, except that only one observation was performed in the morning (at $8^h\ 0^m$). I have divided these observations into two groups for each station according to the humidity of the air. In the following little table are quoted, first the place of observation, and after this under D the mean date of the observations (August 1882), under W the quantity of water, under I the radiation observed by means of the actinometer, under I_1 the second observation of the same quantity.

* Langley, 'Researches on Solar Heat,' pp. 94, 98, and 177.

	Morning.				Noon.				Evening,			
	D.	W.	I.	I_1.	D.	W.	I.	I_1.	D.	W.	I.	I_1.
Lone Pine.	29·3	0·61	1·424	1·554	23·6	0·46	1·692	1·715	26·6	0·51	1·417	1·351
	21·1	0·84	1·458	1·583	26·9	0·59	1·699	1·721	23·2	0·74	1·428	1·359
Mountain Camp.	23·5	0·088	1·790		22·5	0·182	1·904	1·873	24·5	0·205	1·701	1·641
	23·5	0·153	1·749		24·5	0·245	1·890	1·917	22·5	0·32	1·601	1·527

At a very low humidity (Mountain Camp) it is evident that the absorbing power of the aqueous vapour has an influence, for the figures for greater humidity are (with an insignificant exception) inferior to those for less humidity. But for the observations from Lone Pine the contrary seems to be true. It is not permissible to assume that the radiation can be strengthened by its passage through aqueous vapour, but the observed effect must be caused by some secondary circumstance. Probably the air is in general more pure if there is more water-vapour in it than if there is less. The selective diffusion diminishes in consequence of this greater purity, and this secondary effect more than counterbalances the insignificant absorption that the radiation suffers from the increase of the water-vapour. It is noteworthy that Elster and Geitel have proved that invisible actinic rays of very high refrangibility traverse the air much more easily if it is humid than if it is dry. Langley's figures demonstrate meanwhile that the influence of aqueous vapour on the radiation from the sun is insensible as soon as it has exceeded a value of about 0·4.

Probably the same reasoning will hold good for carbonic acid, for the absorption spectrum of both gases is of the same general character. Moreover, the absorption by carbonic acid occurs at considerably greater wave-lengths, and consequently for much less important parts of the sun's spectrum than the absorption by water-vapour*. It is, therefore, justifiable to assume that the radiation from the sun suffers no appreciable diminution if K and W increase from a rather insignificant value ($K=1$, $W=0·4$) to higher ones.

Before we proceed further we need to examine another question. Let the carbonic acid in the air be, for instance, the same as now ($K=1$ for vertical rays), and the quantity of water-vapour be 10 grammes per cubic metre ($W=1$ for

* *Cf.* above, pages 246–248, and Langley's curve for the solar spectrum, *Ann. d. Ch. et d. Phys.* sér. 6, t. xvii. pp. 323 and 326 (1889); 'Prof. Papers,' No. 15, plate 12.

vertical rays). Then the vertical rays from the earth traverse the quantities K = 1 and W = 1; rays that escape under an angle of 30° with the horizon (air-mass = 2) traverse the quantities K = 2, W = 2; and so forth. The different rays that emanate from a point of the earth's surface suffer, therefore, a different absorption—the greater, the more the path of the ray declines from the vertical line. It may then be asked how long a path must the total radiation make, that the absorbed fraction of it is the same as the absorbed fraction of the total mass of rays that emanate to space in different directions. For the emitted rays we will suppose that the cosine law of Lambert holds good. With the aid of Table III. we may calculate the absorbed fraction of any ray, and then sum up the total absorbed heat and determine how great a fraction it is of the total radiation. In this way we find for our example the path (air-mass) 1·61. In other words, the total absorbed part of the whole radiation is just as great as if the total radiation traversed the quantities 1·61 of aqueous vapour and of carbonic acid. This number depends upon the composition of the atmosphere, so that it becomes less the greater the quantity of aqueous vapour and carbonic acid in the air. In the following table (IV.) we find this number for different quantities of both gases.

TABLE IV.—*Mean path of the Earth's rays.*

→ H_2O ↓ CO_2	0·3.	0·5.	1.	2.	3.
0·67	1·69	1·68	1·64	1·57	1·53
1	1·66	1·65	1·61	1·55	1·51
1·5	1·62	1·61	1·57	1·51	1·47
2	1·58	1·57	1·52	1·46	1·43
2·5	1·56	1·54	1·50	1·45	1·41
3	1·52	1·51	1·47	1·44	1·40
3·5	1·48	1·48	1·45	1·42	

If the absorption of the atmosphere approaches zero, this number approaches the value 2.

III. *Thermal Equilibrium on the Surface and in the Atmosphere of the Earth.*

As we now have a sufficient knowledge of the absorption of heat by the atmosphere, it remains to examine how the temperature of the ground depends on the absorptive power of the air. Such an investigation has been already performed by Pouillet*, but it must be made anew, for Pouillet used hypotheses that are not in agreement with our present knowledge.

In our deductions we will assume that the heat that is conducted from the interior of the earth to its surface may be wholly neglected. If a change occurs in the temperature of the earth's surface, the upper layers of the earth's crust will evidently also change their temperature; but this later process will pass away in a very short time in comparison with the time that is necessary for the alteration of the surface temperature, so that at any time the heat that is transported from the interior to the surface (positive in the winter, negative in the summer) must remain independent of the small secular variations of the surface temperature, and in the course of a year be very nearly equal to zero.

Likewise we will suppose that the heat that is conducted to a given place on the earth's surface or in the atmosphere in consequence of atmospheric or oceanic currents, horizontal or vertical, remains the same in the course of the time considered, and we will also suppose that the clouded part of the sky remains unchanged. It is only the variation of the temperature with the transparency of the air that we shall examine.

All authors agree in the view that there prevails an equilibrium in the temperature of the earth and of its atmosphere. The atmosphere must, therefore, radiate as much heat to space as it gains partly through the absorption of the sun's rays, partly through the radiation from the hotter surface of the earth and by means of ascending currents of air heated by contact with the ground. On the other hand, the earth loses just as much heat by radiation to space and to the atmosphere as it gains by absorption of the sun's rays. If we consider a given place in the atmosphere or on the ground, we must also take into consideration the quantities of heat that are carried to this place by means of oceanic or atmospheric currents. For the radiation we will suppose that

* Pouillet, *Comptes rendus*, t. vii. p. 41 (1838).

Stefan's law of radiation, which is now generally accepted, holds good, or in other words that the quantity of heat (W) that radiates from a body of the albedo $(1-\nu)$ and temperature T (absolute) to another body of the absorption-coefficient β and absolute temperature θ is

$$W=\nu\beta\gamma(T^4-\theta^4),$$

where γ is the so-called radiation constant ($1{\cdot}21 . 10^{-12}$ per sec. and cm.^{2}). Empty space may be regarded as having the absolute temperature 0*.

Provisionally we regard the air as a uniform envelope of the temperature θ and the absorption-coefficient α for solar heat; so that if A calories arrive from the sun in a column of 1 cm.^{2} cross-section, αA are absorbed by the atmosphere and $(1-\alpha)$A reach the earth's surface. In the A calories there is, therefore, not included that part of the sun's heat which by means of selective reflexion in the atmosphere is thrown out towards space. Further, let β designate the absorption-coefficient of the air for the heat that radiates from the earth's surface; β is also the emission-coefficient of the air for radiation of low temperature—strictly 15°; but as the spectral distribution of the heat varies rather slowly with the temperature, β may be looked on as the emission-coefficient also at the temperature of the air. Let the albedo of the earth's crust be designated by $(1-\nu)$, and the quantities of heat that are conveyed to the air and to the earth's surface at the point considered be M and N respectively. As unit of time we may take any period: the best choice in the following calculation is perhaps to take three months for this purpose. As unit of surface we may take 1 cm.^{2}, and for the heat in the air that contained in a column of 1 cm.^{2} cross-section and the height of the atmosphere. The heat that is reflected from the ground is not appreciably absorbed by the air (see p. 252), for it has previously traversed great quantities of water-vapour and carbonic acid, but a part of it may be returned to the ground by means of diffuse reflexion. Let this part not be included in the albedo $(1-\nu)$. γ, A, ν, M, N, and α are to be considered as constants, β as the independent, and θ and T as the dependent variables.

Then we find for the column of air

$$\beta\gamma\theta^4=\beta\gamma\nu(T^4-\theta^4)+\alpha A+M. \quad . \quad . \quad . \quad (1)$$

The first member of this equation represents the heat

* Langley, 'Prof. Papers,' No. 15, p. 122. "The Temperature of the Moon," p. 206.

radiated from the air (emission-coefficient β, temperature θ) to space (temperature 0). The second one gives the heat radiated from the soil (1 cm.^{2}, temperature T, albedo $1-\nu$) to the air; the third and fourth give the amount of the sun's radiation absorbed by the air, and the quantity of heat obtained by conduction (air-currents) from other parts of the air or from the ground. In the same manner we find for the earth's surface

$$\beta\gamma\nu(T^4-\theta^4)+(1-\beta)\gamma\nu T^4=(1-\alpha)\nu A+N. \quad . \quad (2)$$

The first and second members represent the radiated quantities of heat that go to the air and to space respectively, $(1-\alpha)\nu A$ is the part of the sun's radiation absorbed, and N the heat conducted to the point considered from other parts of the soil or from the air by means of water- or air-currents.

Combining both these equations for the elimination of θ, which has no considerable interest, we find for T^4

$$T^4=\frac{\alpha A+M+(1-\alpha)A(1+\nu)+N(1+1/\nu)}{\gamma(1+\nu-\beta\nu)}=\frac{K}{1+\nu(1-\beta)}. \quad (3)$$

For the earth's solid crust we may, without sensible error, put ν equal to 1, if we except the snowfields, for which we assume $\nu=0{\cdot}5$. For the water-covered parts of the earth I have calculated the mean value of ν to be 0·925 by aid of the figures of Zenker*. We have, also, in the following to make use of the albedo of the clouds. I do not know if this has ever been measured, but it probably does not differ very much from that of fresh fallen snow, which Zöllner has determined to be 0·78, *i. e.* $\nu=0{\cdot}22$. For old snow the albedo is much less or ν much greater; therefore we have assumed 0·5 as a mean value.

The last formula shows that the temperature of the earth augments with β, and the more rapidly the greater ν is. For an increase of 1° if $\nu=1$ we find the following increases for the values of $\nu=0{\cdot}925$, 0·5, and 0·22 respectively:—

β.	$\nu=0{\cdot}925$.	$\nu=0{\cdot}5$.	$\nu=0{\cdot}22$.
0·65	0·944	0·575	0·275
0·75	0·940	0·556	0·261
0·85	0·934	0 535	0·245
0·95	0·928	0·512	0·228
1·00	0·925	0·500	0·220

This reasoning holds good if the part of the earth's surface

* Zenker, *Die Vertheilung der Wärme auf der Erdoberfläche*, p. 54 (Berlin, 1888).

considered does not alter its albedo as a consequence of the altered temperature. In that case entirely different circumstances enter. If, for instance, an element of the surface which is not now snow-covered, in consequence of falling temperature becomes clothed with snow, we must in the last formula not only alter β but also ν. In this case we must remember that α is very small compared to β. For α we will choose the value 0·40 in accordance with Langley's* estimate. Certainly a great part of this value depends upon the diffusely reflected part of the sun's heat, which is absorbed by the earth's atmosphere, and therefore should not be included in α, as we have defined it above. On the other hand, the sun may in general stand a little lower than in Langley's measurements, which were executed with a relatively high sun, and in consequence of this α may be a little greater, so that these circumstances may compensate each other. For β we will choose the value 0·70, which corresponds when $K=1$ and $W=0{\cdot}3$ (a little below the freezing-point) with the factor 1·66 (see p. 253). In this case we find the relation between T (uncovered) and T_1 (snow-covered surface) to be

$$T^4 : T_1^4 = \frac{A(1+1-0{\cdot}40)+M}{\gamma(1+1-0{\cdot}70)} : \frac{A(1+0{\cdot}50-0{\cdot}20)+M}{\gamma(1+0{\cdot}50-0{\cdot}35)}$$
$$= \frac{1{\cdot}60+\phi}{1{\cdot}30} : \frac{1{\cdot}30+\phi}{1{\cdot}15},$$

if $M=\phi A$. We have to bear in mind that the mean M for the whole earth is zero, for the equatorial regions negative and for the polar regions positive. For a mean latitude $M=0$, and in this case T_1 becomes 267·3 if $T=273$, that is the temperature decreases in consequence of the snow-covering by 5°·7 C.† The decrease of temperature from this cause will be valid until $\phi=1$, *i. e.* till the heat delivered by convection to the air exceeds the whole radiation of the sun. This can only occur in the winter and in polar regions.

But this is a secondary phenomenon. The chief effect that we examine is the direct influence of an alteration of β upon the temperature T of the earth's surface. If we start from a value $T=273$ and $\beta=0{\cdot}70$, we find the alteration (t) in the

* Langley, "Temperature of the Moon," p. 189. On p. 197 he estimates a to be only 0·33.

† According to the correction introduced in the sequel for the different heights of the absorbing and radiating layers of the atmosphere, the number 5°·7 is reduced to 4°·0. But as about half the sky is cloud-covered, the effect will be only half as great as for cloudless sky, *i. e.* the mean effect will be a lowering of about 2° C.

temperature which is caused by the variation of β to the following values to be

$\beta = 0{\cdot}60$	$t = -\ 5°$ C.
0·80	+ 5·6
0·90	+11·7
1·00	+18·6.

These values are calculated for $\nu = 1$, *i. e.* for the solid crust of the earth's surface, except the snowfields. For surfaces with another value of ν, as for instance the ocean or the snowfields, we have to multiply this value t by a fraction given above.

We have now shortly to consider the influence of the clouds. A great part of the earth's surface receives no heat directly from the sun, because the sun's rays are stopped by clouds. How great a part of the earth's surface is covered by clouds we may find from Teisserenc de Bort's work* on Nebulosity. From tab. 17 of this publication I have determined the mean nebulosity for different latitudes, and found:—

Latitude..	60.	45.	30.	15.	0.	−15.	−30.	−45.	−60.
Nebulosity.	0·603	0·48	0·402	0·511	0·581	0·463	0·53	0·701	

For the part of the earth between 60° S. and 60° N. we find the mean value 0·525, *i. e.* 52·5 per cent. of the sky is clouded. The heat-effect of these clouds may be estimated in the following manner. Suppose a cloud lies over a part of the earth's surface and that no connexion exists between this shadowed part and the neighbouring parts, then a thermal equilibrium will exist between the temperature of the cloud and of the underlying ground. They will radiate to each other and the cloud also to the upper air and to space, and the radiation between cloud and earth may, on account of the slight difference of temperature, be taken as proportional to this difference. Other exchanges of heat by means of air-currents are also, as a first approximation, proportional to this difference. If we therefore suppose the temperature of the cloud to alter (other circumstances, as its height and composition, remaining unchanged), the temperature of the ground under it must also alter in the same manner if the same supply of heat to both subsists—if there were no supply to the ground from neighbouring parts, the cloud and the ground would finally assume the same mean temperature. If, therefore, the temperature of the clouds varies in a determined manner

* Teisserenc de Bort, "Distribution moyenne de la nébulosité," *Ann. du bureau central météorologique de France*, Année 1884, t. iv. 2de partie, p. 27.

(without alteration of their other properties, as height, compactness, &c.), the ground will undergo the same variations of temperature. Now it will be shown in the sequel that a variation of the carbonic acid of the atmosphere in the same proportion produces nearly the same thermal effect independently of its absolute magnitude (see p. 265). Therefore we may calculate the temperature-variation in this case as if the clouds covered the ground with a thin film of the albedo 0·78 ($\nu=0{\cdot}22$, see p. 256). As now on the average $K=1$ and $W=1$ nearly, and in this case β is about 0·79, the effect on the clouded part will be only 0·25 of the effect on parts that have $\nu=1$. If a like correction is introduced for the ocean ($\nu=0{\cdot}925$) on the supposition that the unclouded part of the earth consists of as much water as of solid ground (which is approximately true, for the clouds are by preference stored up over the ocean), we find a mean effect of, in round numbers, 60 p. c. of that which would exist if the whole earth's surface had $\nu=1$. The snow-covered parts are not considered, for, on the one hand, these parts are mostly clouded to about 65 p. c.; further, they constitute only a very small part of the earth (for the whole year on the average only about 4 p. c.), so that the correction for this case would not exceed 0·5 p. c. in the last number 60. And further, on the border countries between snowfields and free soil secondary effects come into play (see p. 257) which compensate, and perhaps overcome, the moderating effect of the snow.

In the foregoing we have supposed that the air is to be regarded as an envelope of perfectly uniform temperature. This is of course not true, and we now proceed to an examination of the probable corrections that must be introduced for eliminating the errors caused by this inexactness. It is evident that the parts of the air which radiate to space are chiefly the external ones, and on the other hand the layers of air which absorb the greatest part of the earth's radiation do not lie very high. From this cause both the radiation from air to space ($\beta\gamma\theta^4$ in eq. 1) and also the radiation of the earth to the air ($\beta\gamma\nu(T^4-\theta^4)$ in eq. 2), are greatly reduced, and the air has a much greater effect as protecting against the loss of heat to space than is assumed in these equations, and consequently also in eq. (3). If we knew the difference of temperature between the two layers of the air that radiate to space and absorb the earth's radiation, it would be easy to introduce the necessary correction in formulæ (1), (2), and (3). For this purpose I have adduced the following consideration.

As at the mean composition of the atmosphere ($K=1$,

W = 1) about 80 p. c. of the earth's radiation is absorbed in the air, we may as mean temperature of the absorbing layer choose the temperature at the height where 40 p. c. of the heat is absorbed. Since emission and absorption follow the same quantitative laws, we may as mean temperature of the emitting layer choose the temperature at the height where radiation entering from space in the opposite direction to the actual emission is absorbed to the extent of 40 p. c.

Langley has made four measurements of the absorptive power of water-vapour for radiation from a hot Leslie cube of 100° C.* These give nearly the same absorption-coefficient if Pouillet's formula is used for the calculation. From these numbers we calculate that for the absorption of 40 p. c. of the radiation it would be necessary to intercalate so much water-vapour between radiator and bolometer that, when condensed, it would form a layer of water 3·05 millimetres thick. If we now suppose as mean for the whole earth K = 1 and W = 1 (see Table VI.), we find that vertical rays from the earth, if it were at 100°, must traverse 305 metres of air to lose 40 p. c. Now the earth is only at 15° C., but this cannot make any great difference. Since the radiation emanates in all directions, we have to divide 305 by 1·61 and get in this way 209 metres. In consequence of the lowering of the quantity of water-vapour with the height † we must apply a slight correction, so that the final result is 233 metres. Of course this number is a mean value, and higher values will hold good for colder, lower for warmer parts of the earth. In so small a distance from the earth, then, 40 p. c. of the earth's radiation should be stopped. Now it is not wholly correct to calculate with Pouillet's formula (it is rather strange that Langley's figures agree so well with it), which gives necessarily too low values. But, on the other hand, we have not at all considered the absorption by the carbonic acid in this part, and this may compensate for the error mentioned. In the highest layers of the atmosphere there is very little water-vapour, so that we must calculate with carbonic acid as the chief absorbent. From a measurement by Ångström ‡, we learn that the absorption-coefficients of water-vapour and of carbonic acid in equal quantities (equal number of molecules) are in the proportion 81 : 62. This ratio is valid for the least hot radiator that Ångström used, and there is no doubt

* Langley, "Temperature of the Moon," p. 186.

† Hann, *Meteorologische Zeitschrift*, xi. p. 196 (1894).

‡ Ångström, *Bihang till K. Vet.-Ak. Handl.* Bd. xv. Afd. 1, No. 9, pp. 11 and 18 (1889).

that the radiation of the earth is much less refrangible. But in the absence of a more appropriate determination we may use this for our purpose ; it is probable that for a less hot radiator the absorptive power of the carbonic acid would come out a little greater compared with that of water-vapour, for the absorption-bands of CO_2 are, on the whole, less refrangible than those of H_2O (see pp. 246–248). Using the number 0·03 vol. p. c. for the quantity of carbonic acid in the atmosphere, we find that rays which emanate from the upper part of the air are derived to the extent of 40 p. c. from a layer that constitutes 0·145 part of the atmosphere. This corresponds to a height of about 15,000 metres. Concerning this value we may make the same remark as on the foregoing value. In this case we have neglected the absorption by the small quantities of water-vapour in the higher atmosphere. The temperature-difference of these two layers—the one absorbing, the other radiating—is, according to Glaisher's measurements* (with a little extrapolation), about 42° C.

For the clouds we get naturally slightly modified numbers. We ought to take the mean height of the clouds that are illuminated by the sun. As such clouds I have chosen the summits of the cumuli that lie at an average height of 1855 metres, with a maximum height of 3611 metres and a minimum of 900 metres †. I have made calculations for mean values of 2000 and 4000 metres (corresponding to differences of temperature of 30° C. and 20° C. instead of 42° C. for the earth's surface).

If we now wish to adjust our formulæ (1) to (3), we have in (1) and (2) to introduce θ as the mean temperature of the radiating layer and $(\theta+42)$, $(\theta+30)$, or $(\theta+20)$ respectively for the mean temperature of the absorbing layer. In the first case we should use $\nu=1$ and $\nu=0{\cdot}925$ respectively, in the second and the third case $\nu=0{\cdot}22$.

We then find instead of the formula (3)

$$T^4=\frac{K}{1+\nu(1-\beta)},$$

another very similar formula

$$T^4=\frac{K}{1+c\nu(1-\beta)}, \quad \ldots \ldots \ldots \quad (4)$$

* Joh. Müller's *Lehrbuch d. kosmischen Physik*, 5te–Aufl. p. 539 (Braunschweig, 1894).

† According to the measurements of Ekholm and Hagström, *Bihang till K. Sv. Vet-Ak. Handlingar*, Bd. xii. Afd. 1, No. 10, p. 11 (1886).

where c is a constant with the values 1·88, 1·58, and 1·37 respectively for the three cases *. In this way we find the following corrected values which represent the variation of temperature, if the solid ground changes its temperature 1° C. in consequence of a variation of β as calculated by means of formula (3).

TABLE V.—*Correction Factors for the Radiation.*

$\beta=$	Solid ground, $\nu=1$.	Water, $\nu=0{\cdot}925$.	Snow, $\nu=0{\cdot}5$.	Clouds ($\nu=0{\cdot}22$) at a height of		
				0 m.	2000 m.	4000 m.
0·65	1·53	1·46	0·95	0·49	0·42	0·37
0·75	1·60	1·52	0·95	0·47	0·40	0·35
0·85	1·69	1·59	0·95	0·46	0·38	0·33
0·95	1·81	1·68	0·94	0·43	0·36	0·31
1·00	1·88	1·74	0·94	0·41	0·35	0·30

If we now assume as a mean for the whole earth $K=1$ and $W=1$, we get $\beta=0{\cdot}785$, and taking the clouded part to be 52·5 p. c. and the clouds to have a height of 2000 metres, further assuming the unclouded remainder of the earth's surface to consist equally of land and water, we find as average variation of temperature

$$1{\cdot}63\times0{\cdot}2385+1{\cdot}54\times0{\cdot}2385+0{\cdot}39\times0{\cdot}525=0{\cdot}979,$$

or very nearly the same effect as we may calculate directly from the formula (3). On this ground I have used the simpler formula.

In the foregoing I have remarked that according to my estimation the air is less transparent for dark heat than on Langley's estimate and nearly in the proportion 37·2 : 44. How great an influence this difference may exercise is very easily calculated with the help of formula (3) or (4). According to Langley's valuation, the effect should be nearly 15 p. c. greater than according to mine. Now I think that my estimate agrees better with the great absorption that Langley has found for heat from terrestrial radiating bodies (see p. 260), and in all circumstances I have preferred to slightly underestimate than to overrate the effect in question.

* $1{\cdot}88=\left(\frac{288}{246}\right)^4$, $1{\cdot}58=\left(\frac{276}{246}\right)^4$, and $1{\cdot}37=\left(\frac{266}{246}\right)^4$. 246° is the mean absolute temperature of the higher radiating layer of the air.

IV. *Calculation of the Variation of Temperature that would ensue in consequence of a given Variation of the Carbonic Acid in the Air.*

We now possess all the necessary data for an estimation of the effect on the earth's temperature which would be the result of a given variation of the aërial carbonic acid. We only need to determine the absorption-coefficient for a certain place with the help of Table III. if we know the quantity of carbonic acid ($K=1$ now) and water-vapour (W) of this place. By the aid of Table IV. we at first determine the factor ρ that gives the mean path of the radiation from the earth through the air and multiply the given K- and W-values by this factor. Then we determine the value of β which corresponds to ρK and ρW. Suppose now that the carbonic acid had another concentration K_1 (*e.g.* $K_1 = 1{\cdot}5$). Then we at first suppose W unaltered and seek the new value of ρ, say ρ_1, that is valid on this supposition. Next we have to seek β, which corresponds to $\rho_1 K_1$ ($1{\cdot}5\rho_1$) and $\rho_1 W$. From formula (3) we can then easily calculate the alteration (t) (here increase) in the temperature at the given place which will accompany the variation of β from β to β_1. In consequence of the variation (t) in the temperature, W must also undergo a variation. As the relative humidity does not vary much, unless the distribution of land and water changes (see table 8 of my original memoir), I have supposed that this quantity remains constant, and thereby determined the new value W_1 of W. A fresh approximation gives in most cases values of W_1 and β_1 which may be regarded as definitive. In this way, therefore, we get the variation of temperature as soon as we know the actual temperature and humidity at the given place.

In order to obtain values for the temperature for the whole earth, I have calculated from Dr. Buchan's charts of the mean temperature at different places in every month * the mean temperature in every district that is contained between two parallels differing by 10 and two meridians differing by 20 degrees, (*e. g.*, between 0° and 10° N. and 160° and 180° W.). The humidity has not as yet been sufficiently examined for the whole earth; and I have therefore collected a great many measurements of the relative humidity at different places (about 780) on the earth and marked them down in maps of the world, and thereafter estimated the mean values for every district. These quantities I have tabulated for the four seasons, Dec.-Feb., March–May, June–Aug., and Sept.–Nov. The detailed table and the observations used are to be found in my original memoir: here I reproduce only the mean values for every tenth parallel (Table VI.).

* Buchan: Report on the Scientific Results of the Voyage of H.M.S. 'Challenger,' Physics and Chemistry, vol. ii., 1889.

TABLE VI.—*Mean Temperature, Relative and Absolute Humidity**.

Latitude.	Mean Temperature.					Mean Relative Humidity.					Mean Absolute Humidity.				
	Dec.–Feb.	March–May.	June–Aug.	Sept.–Nov.	Mean of the year.	Dec.–Feb.	March–May.	June–Aug.	Sept.–Nov.	Mean of the year.	Dec.–Feb.	March–May.	June–Aug.	Sept.–Nov.	Mean of the year.
70	−21·1	− 8·3	+ 7·5	− 6·0	− 7·0	86	81	77	84	82	1·15	2·14	6·22	2·84	3·09
60	−11·2	+ 0·2	+13·5	+ 2·2	+ 1·2	83	74	76	80	78·2	2·22	3·82	8·82	4·7	4·9
50	− 1·4	+ 7·8	+18·7	+ 9·7	+ 8·7	78	73	69	76	74	3·86	5·98	10·8	7·16	6·95
40	+ 8·4	+14·5	+21·8	+16·6	+15·3	73	68	67	71	69·7	6·53	8·63	13·4	10·13	9·7
30	+17·0	+21·5	+26·0	+23·0	+21·9	71	68	70	73	70·5	10·36	12·63	17·1	15·0	13·8
20	+23·2	+25·5	+26·8	+25·9	+25·4	74	73	78	77	75·5	15·3	17·0	19·6	16·8	17·2
10	+25·5	+25·8	+25·4	+25·5	+25·5	77	78	82	81	79·5	17·7	18·9	19·9	19·3	18·9
0	+25·7	+25·5	+24·0	+25·0	+25·1	81	81	82	80	81	19·4	19·0	17·9	18·3	18·7
−10	+24·9	+24·0	+20·8	+23·1	+23·2	79	78	80	77	78·5	18·0	17·1	14·6	16·0	16·4
−20	+22·4	+20·5	+16·4	+19·3	+19·7	75	79	80	75	77·2	14·8	14·0	11·1	13·0	13·2
−30	+17·5	+15·2	+11·3	+14·2	+14·5	75	80	80	79	78·5	11·1	10·4	8·1	9·6	9·8
−40	+11·6	+ 9·5	+ 5·9	+ 8·2	+ 8·7	81	81	83	79	81	8·34	7·08	5·94	6·63	6·99
−50	+ 5·3	+ 2·0	− 0·4	+ 1·6	+ 2·1	83	79	—	—	—	5·74	4·46	—	—	—
−60															

* From the figures for temperature and relative humidity I have calculated the absolute humidity in grams per cubic metre.

By means of these values, I have calculated the mean alteration of temperature that would follow if the quantity of carbonic acid varied from its present mean value (K=1) to another, viz. to K=0·67, 1·5, 2, 2·5, and 3 respectively. This calculation is made for every tenth parallel, and separately for the four seasons of the year. The variation is given in Table VII.

A glance at this Table shows that the influence is nearly the same over the whole earth. The influence has a minimum near the equator, and increases from this to a flat maximum that lies the further from the equator the higher the quantity of carbonic acid in the air. For K=0·67 the maximum effect lies about the 40th parallel, for K=1·5 on the 50th, for K=2 on the 60th, and for higher K-values above the 70th parallel. The influence is in general greater in the winter than in the summer, except in the case of the parts that lie between the maximum and the pole. The influence will also be greater the higher the value of ν, that is in general somewhat greater for land than for ocean. On account of the nebulosity of the Southern hemisphere, the effect will be less there than in the Northern hemisphere. An increase in the quantity of carbonic acid will of course diminish the difference in temperature between day and night. A very important secondary elevation of the effect will be produced in those places that alter their albedo by the extension or regression of the snow-covering (see p. 257), and this secondary effect will probably remove the maximum effect from lower parallels to the neighbourhood of the poles *.

It must be remembered that the above calculations are found by interpolation from Langley's numbers for the values K=0·67 and K=1·5, and that the other numbers must be regarded as extrapolated. The use of Pouillet's formula makes the values for K=0·67 probably a little too small, those for K=1·5 a little too great. This is also without doubt the case for the extrapolated values, which correspond to higher values of K.

We may now inquire how great must the variation of the carbonic acid in the atmosphere be to cause a given change of the temperature. The answer may be found by interpolation in Table VII. To facilitate such an inquiry, we may make a simple observation. If the quantity of carbonic acid decreases from 1 to 0·67, the fall of temperature is nearly the same as the increase of temperature if this quantity augments to 1·5. And to get a new increase of this order of magnitude (3°·4), it will be necessary to alter the quantity of carbonic acid till it reaches a value nearly midway between 2 and 2·5.

* See Addendum, p. 275.

TABLE VII.—*Variation of Temperature caused by a given Variation of Carbonic Acid.*

Latitude.	Carbonic Acid = 0·67.					Carbonic Acid = 1·5.					Carbonic Acid = 2·0.					Carbonic Acid = 2·5.					Carbonic Acid = 3·0.				
	Dec.–Feb.	March–May.	June–Aug.	Sept.–Nov.	Mean of the year.	Dec.–Feb.	March–May.	June–Aug.	Sept.–Nov.	Mean of the year.	Dec.–Feb.	March–May.	June–Aug.	Sept.–Nov.	Mean of the year.	Dec.–Feb.	March–May.	June–Aug.	Sept.–Nov.	Mean of the year.	Dec.–Feb.	March–May.	June–Aug.	Sept.–Nov.	Mean of the year.
70																									
	−2·9	−3·0	−3·4	−3·1	−3·1	3·3	3·4	3·8	3·6	3·52	6·0	6·1	6·0	6·1	6·05	7·9	8·0	7·9	8·0	7·95	9·1	9·3	9·4	9·4	9·3
60																									
	−3·0	−3·2	−3·4	−3·3	−3·22	3·4	3·7	3·6	3·8	3·62	6·1	6·1	5·8	6·1	6·02	8·0	8·0	7·6	7·9	7·87	9·3	9·5	8·9	9·5	9·3
50																									
	−3·2	−3·3	−3·3	−3·4	−3·3	3·7	3·8	3·4	3·7	3·65	6·1	6·1	5·5	6·0	5·92	8·0	7·9	7·0	7·9	7·7	9·5	9·4	8·6	9·2	9·17
40																									
	−3·4	−3·4	−3·2	−3·3	−3·32	3·7	3·6	3·3	3·5	3·52	6·0	5·8	5·4	5·6	5·7	7·9	7·6	6·9	7·3	7·42	9·3	9·0	8·2	8·8	8·82
30																									
	−3·3	−3·2	−3·1	−3·1	−3·17	3·5	3·3	3·2	3·5	3·47	5·6	5·4	5·0	5·2	5·3	7·2	7·0	6·6	6·7	6·87	8·7	8·3	7·5	7·9	8·1
20																									
	−3·1	−3·1	−3·0	−3·1	−3·07	3·5	3·2	3·1	3·2	3·25	5·2	5·0	4·9	5·0	5·02	6·7	6·6	6·3	6·6	6·52	7·9	7·5	7·2	7·5	7·52
10																									
	−3·1	−3·0	−3·0	−3·0	−3·02	3·2	3·2	3·1	3·1	3·15	5·0	5·0	4·9	4·9	4·95	6·6	6·4	6·3	6·4	6·42	7·4	7·3	7·2	7·3	7·3
0																									
	−3·0	−3·0	−3·1	−3·0	−3·02	3·1	3·1	3·2	3·2	3·15	4·9	4·9	5·0	5·0	4·95	6·4	6·4	6·6	6·6	6·5	7·3	7·3	7·4	7·4	7·35
−10																									
	−3·1	−3·1	−3·2	−3·1	−3·12	3·2	3·2	3·2	3·2	3·2	5·0	5·0	5·2	5·1	5·07	6·6	6·6	6·7	6·7	6·65	7·4	7·5	8·0	7·6	7·62
−20																									
	−3·1	−3·2	−3·3	−3·2	−3·2	3·2	3·2	3·4	3·3	3·27	5·2	5·3	5·5	5·4	5·35	6·7	6·8	7·0	7·0	6·87	7·9	8·1	8·6	8·3	8·22
−30																									
	−3·3	−3·3	−3·4	−3·4	−3·35	3·4	3·5	3·7	3·5	3·52	5·5	5·6	5·8	5·6	5·62	7·0	7·2	7·7	7·4	7·32	8·6	8·7	9·1	8·8	8·8
−40																									
	−3·4	−3·4	−3·3	−3·4	−3·37	3·6	3·7	3·8	3·7	3·7	5·8	6·0	6·0	6·0	5·95	7·7	7·9	7·9	7·9	7·85	9·1	9·2	9·4	9·3	9·25
−50																									
	−3·2	−3·3	—	—	—	3·8	3·7	—	—	—	6·0	6·1	—	—	—	7·9	8·0	—	—	—	9·4	9·5	—	—	—
−60																									

Thus if the quantity of carbonic acid increases in geometric progression, the augmentation of the temperature will increase nearly in arithmetic progression. This rule—which naturally holds good only in the part investigated—will be useful for the following summary estimations.

5. *Geological Consequences.*

I should certainly not have undertaken these tedious calculations if an extraordinary interest had not been connected with them. In the Physical Society of Stockholm there have been occasionally very lively discussions on the probable causes of the Ice Age; and these discussions have, in my opinion, led to the conclusion that there exists as yet no satisfactory hypothesis that could explain how the climatic conditions for an ice age could be realized in so short a time as that which has elapsed from the days of the glacial epoch. The common view hitherto has been that the earth has cooled in the lapse of time; and if one did not know that the reverse has been the case, one would certainly assert that this cooling must go on continuously. Conversations with my friend and colleague Professor Högbom, together with the discussions above referred to, led me to make a preliminary estimate of the probable effect of a variation of the atmospheric carbonic acid on the temperature of the earth. As this estimation led to the belief that one might in this way probably find an explanation for temperature variations of 5°–10° C., I worked out the calculation more in detail, and lay it now before the public and the critics.

From geological researches the fact is well established that in Tertiary times there existed a vegetation and an animal life in the temperate and arctic zones that must have been conditioned by a much higher temperature than the present in the same regions *. The temperature in the arctic zones appears to have exceeded the present temperature by about 8 or 9 degrees. To this genial time the ice age succeeded, and this was one or more times interrupted by interglacial periods with a climate of about the same character as the present, sometimes even milder. When the ice age had its greatest extent, the countries that now enjoy the highest civilization were covered with ice. This was the case with Ireland, Britain (except a small part in the south), Holland, Denmark, Sweden and Norway, Russia (to Kiev,

* For details *cf.* Neumayr, *Erdgeschichte*, Bd. 2, Leipzig, 1887; and Geikie, 'The Great Ice-Age,' 3rd ed. London, 1894; Nathorst, *Jordens historia*, p. 989, Stockholm, 1894.

Orel, and Nijni Novgorod), Germany and Austria (to the Harz, Erz-Gebirge, Dresden, and Cracow). At the same time an ice-cap from the Alps covered Switzerland, parts of France, Bavaria south of the Danube, the Tyrol, Styria, and other Austrian countries, and descended into the northern part of Italy. Simultaneously, too, North America was covered with ice on the west coast to the 47th parallel, on the east coast to the 40th, and in the central part to the 37th (confluence of the Mississippi and Ohio rivers). In the most different parts of the world, too, we have found traces of a great ice age, as in the Caucasus, Asia Minor, Syria, the Himalayas, India, Thian Shan, Altai, Atlas, on Mount Kenia and Kilimandjaro (both very near to the equator), in South Africa, Australia, New Zealand, Kerguelen, Falkland Islands, Patagonia and other parts of South America. The geologists in general are inclined to think that these glaciations were simultaneous on the whole earth *; and this most natural view would probably have been generally accepted, if the theory of Croll, which demands a genial age on the Southern hemisphere at the same time as an ice age on the Northern and *vice versâ*, had not influenced opinion. By measurements of the displacement of the snow-line we arrive at the result,—and this is very concordant for different places—that the temperature at that time must have been 4°–5° C. lower than at present. The last glaciation must have taken place in rather recent times, geologically speaking, so that the human race certainly had appeared at that period. Certain American geologists hold the opinion that since the close of the ice age only some 7000 to 10,000 years have elapsed, but this most probably is greatly underestimated.

One may now ask, How much must the carbonic acid vary according to our figures, in order that the temperature should attain the same values as in the Tertiary and Ice ages respectively? A simple calculation shows that the temperature in the arctic regions would rise about 8° to 9° C., if the carbonic acid increased to 2·5 or 3 times its present value. In order to get the temperature of the ice age between the 40th and 50th parallels, the carbonic acid in the air should sink to 0·62—0·55 of its present value (lowering of temperature 4°–5° C.). The demands of the geologists, that at the genial epochs the climate should be more uniform than now, accords very well with our theory. The geographical annual and diurnal ranges of temperature would be partly smoothed away, if the quantity of carbonic acid was augmented. The

* Neumayr, *Erdgeschichte*, p. 648; Nathorst, *l. c.* p. 992.

reverse would be the case (at least to a latitude of 50° from the equator), if the carbonic acid diminished in amount. But in both these cases I incline to think that the secondary action (see p. 257) due to the regress or the progress of the snow-covering would play the most important rôle. The theory demands also that, roughly speaking, the whole earth should have undergone about the same variations of temperature, so that according to it genial or glacial epochs must have occurred simultaneously on the whole earth. Because of the greater nebulosity of the Southern hemisphere, the variations must there have been a little less (about 15 per cent.) than in the Northern hemisphere. The ocean currents, too, must there, as at the present time, have effaced the differences in temperature at different latitudes to a greater extent than in the Northern hemisphere. This effect also results from the greater nebulosity in the arctic zones than in the neighbourhood of the equator.

There is now an important question which should be answered, namely :—Is it probable that such great variations in the quantity of carbonic acid as our theory requires have occurred in relatively short geological times? The answer to this question is given by Prof. Högbom. As his memoir on this question may not be accessible to most readers of these pages, I have summed up and translated his utterances which are of most importance to our subject *:—

"Although it is not possible to obtain exact quantitative expressions for the reactions in nature by which carbonic acid is developed or consumed, nevertheless there are some factors, of which one may get an approximately true estimate, and from which certain conclusions that throw light on the question may be drawn. In the first place, it seems to be of importance to compare the quantity of carbonic acid now present in the air with the quantities that are being transformed. If the former is insignificant in comparison with the latter, then the probability for variations is wholly other than in the opposite case.

"On the supposition that the mean quantity of carbonic acid in the air reaches 0·03 vol. per cent., this number represents 0·045 per cent. by weight, or 0·342 millim. partial pressure, or 0·466 gramme of carbonic acid for every cm.^{2} of the earth's surface. Reduced to carbon this quantity would give a layer of about 1 millim. thickness over the earth's surface. The quantity of carbon that is fixed in the living organic world can certainly not be estimated with the

* Högbom, *Svensk kemisk Tidskrift*, Bd. vi. p. 169 (1894).

same degree of exactness; but it is evident that the numbers that might express this quantity ought to be of the same order of magnitude, so that the carbon in the air can neither be conceived of as very great nor as very little, in comparison with the quantity of carbon occurring in organisms. With regard to the great rapidity with which the transformation in organic nature proceeds, the disposable quantity of carbonic acid is not so excessive that changes caused by climatological or other reasons in the velocity and value of that transformation might be not able to cause displacements of the equilibrium.

"The following calculation is also very instructive for the appreciation of the relation between the quantity of carbonic acid in the air and the quantities that are transformed. The world's present production of coal reaches in round numbers 500 millions of tons per annum, or 1 ton per km.^{2} of the earth's surface. Transformed into carbonic acid, this quantity would correspond to about a thousandth part of the carbonic acid in the atmosphere. It represents a layer of limestone of 0·003 millim. thickness over the whole globe, or 1·5 km.^{3} in cubic measure. This quantity of carbonic acid, which is supplied to the atmosphere chiefly by modern industry, may be regarded as completely compensating the quantity of carbonic acid that is consumed in the formation of limestone (or other mineral carbonates) by the weathering or decomposition of silicates. From the determination of the amounts of dissolved substances, especially carbonates, in a number of rivers in different countries and climates, and of the quantity of water flowing in these rivers and of their drainage-surface compared with the land-surface of the globe, it is estimated that the quantities of dissolved carbonates that are supplied to the ocean in the course of a year reach at most the bulk of 3 km.^{3} As it is also proved that the rivers the drainage regions of which consist of silicates convey very unimportant quantities of carbonates compared with those that flow through limestone regions, it is permissible to draw the conclusion, which is also strengthened by other reasons, that only an insignificant part of these 3 km.^{3} of carbonates is formed directly by decomposition of silicates. In other words, only an unimportant part of this quantity of carbonate of lime can be derived from the process of weathering in a year. Even though the number given were on account of inexact or uncertain assumptions erroneous to the extent of 50 per cent. or more, the comparison instituted is of very great interest, as it proves that the most important of all the processes by means of which carbonic acid has been

removed from the atmosphere in all times, namely the chemical weathering of siliceous minerals, is of the same order of magnitude as a process of contrary effect, which is caused by the industrial development of our time, and which must be conceived of as being of a temporary nature.

"In comparison with the quantity of carbonic acid which is fixed in limestone (and other carbonates), the carbonic acid of the air vanishes. With regard to the thickness of sedimentary formations and the great part of them that is formed by limestone and other carbonates, it seems not improbable that the total quantity of carbonates would cover the whole earth's surface to a height of hundreds of metres. If we assume 100 metres,—a number that may be inexact in a high degree, but probably is underestimated,—we find that about 25,000 times as much carbonic acid is fixed to lime in the sedimentary formations as exists free in the air. Every molecule of carbonic acid in this mass of limestone has, however, existed in and passed through the atmosphere in the course of time. Although we neglect all other factors which may have influenced the quantity of carbonic acid in the air, this number lends but very slight probability to the hypothesis, that this quantity should in former geological epochs have changed within limits which do not differ much from the present amount. As the process of weathering has consumed quantities of carbonic acid many thousand times greater than the amount now disposable in the air, and as this process from different geographical, climatological and other causes has in all likelihood proceeded with very different intensity at different epochs, the probability of important variations in the quantity of carbonic acid seems to be very great, even if we take into account the compensating processes which, as we shall see in what follows, are called forth as soon as, for one reason or another, the production or consumption of carbonic acid tends to displace the equilibrium to any considerable degree. One often hears the opinion expressed, that the quantity of carbonic acid in the air ought to have been very much greater formerly than now, and that the diminution should arise from the circumstance that carbonic acid has been taken from the air and stored in the earth's crust in the form of coal and carbonates. In many cases this hypothetical diminution is ascribed only to the formation of coal, whilst the much more important formation of carbonates is wholly overlooked. This whole method of reasoning on a continuous diminution of the carbonic acid in the air loses all foundation in fact, notwithstanding that enormous quantities of carbonic

acid in the course of time have been fixed in carbonates, if we consider more closely the processes by means of which carbonic acid has in all times been supplied to the atmosphere. From these we may well conclude that enormous variations have occurred, but not that the variation has always proceeded in the same direction.

"Carbonic acid is supplied to the atmosphere by the following processes :—(1) volcanic exhalations and geological phenomena connected therewith; (2) combustion of carbonaceous meteorites in the higher regions of the atmosphere ; (3) combustion and decay of organic bodies ; (4) decomposition of carbonates ; (5) liberation of carbonic acid mechanically inclosed in minerals on their fracture or decomposition. The carbonic acid of the air is consumed chiefly by the following processes :—(6) formation of carbonates from silicates on weathering ; and (7) the consumption of carbonic acid by vegetative processes. The ocean, too, plays an important rôle as a regulator of the quantity of carbonic acid in the air by means of the absorptive power of its water, which gives off carbonic acid as its temperature rises and absorbs it as it cools. The processes named under (4) and (5) are of little significance, so that they may be omitted. So too the processes (3) and (7), for the circulation of matter in the organic world goes on so rapidly that their variations cannot have any sensible influence. From this we must except periods in which great quantities of organisms were stored up in sedimentary formations and thus subtracted from the circulation, or in which such stored-up products were, as now, introduced anew into the circulation. The source of carbonic acid named in (2) is wholly incalculable.

"Thus the processes (1), (2), and (6) chiefly remain as balancing each other. As the enormous quantities of carbonic acid (representing a pressure of many atmospheres) that are now fixed in the limestone of the earth's crust cannot be conceived to have existed in the air but as an insignificant fraction of the whole at any one time since organic life appeared on the globe, and since therefore the consumption through weathering and formation of carbonates must have been compensated by means of continuous supply, we must regard volcanic exhalations as the chief source of carbonic acid for the atmosphere.

"But this source has not flowed regularly and uniformly. Just as single volcanoes have their periods of variation with alternating relative rest and intense activity, in the same manner the globe as a whole seems in certain geological epochs to have exhibited a more violent and general volcanic

activity, whilst other epochs have been marked by a comparative quiescence of the volcanic forces. It seems therefore probable that the quantity of carbonic acid in the air has undergone nearly simultaneous variations, or at least that this factor has had an important influence.

"If we pass the above-mentioned processes for consuming and producing carbonic acid under review, we find that they evidently do not stand in such a relation to or dependence on one another that any probability exists for the permanence of an equilibrium of the carbonic acid in the atmosphere. An increase or decrease of the supply continued during geological periods must, although it may not be important, conduce to remarkable alterations of the quantity of carbonic acid in the air, and there is no conceivable hindrance to imagining that this might in a certain geological period have been several times greater, or on the other hand considerably less, than now."

As the question of the probability of quantitative variation of the carbonic acid in the atmosphere is in the most decided manner answered by Prof. Högbom, there remains only one other point to which I wish to draw attention in a few words, namely: Has no one hitherto proposed any acceptable explanation for the occurrence of genial and glacial periods? Fortunately, during the progress of the foregoing calculations, a memoir was published by the distinguished Italian meteorologist L. De Marchi which relieves me from answering the last question*. He examined in detail the different theories hitherto proposed—astronomical, physical, or geographical, and of these I here give a short *résumé*. These theories assert that the occurrence of genial or glacial epochs should depend on one or other change in the following circumstances:—

(1) The temperature of the earth's place in space.
(2) The sun's radiation to the earth (solar constant).
(3) The obliquity of the earth's axis to the ecliptic.
(4) The position of the poles on the earth's surface.
(5) The form of the earth's orbit, especially its eccentricity (Croll).
(6) The shape and extension of continents and oceans.
(7) The covering of the earth's surface (vegetation).
(8) The direction of the oceanic and aërial currents.
(9) The position of the equinoxes.

De Marchi arrives at the conclusion that all these hypotheses must be rejected (p. 207). On the other hand, he is of the

* Luigi De Marchi: *Le cause dell' era glaciale*, premiato dal R. Istituto Lombardo, Pavia, 1895.

opinion that a change in the transparency of the atmosphere would possibly give the desired effect. According to his calculations, "a lowering of this transparency would effect a lowering of the temperature on the whole earth, slight in the equatorial regions, and increasing with the latitude into the 70th parallel, nearer the poles again a little less. Further, this lowering would, in non-tropical regions, be less on the continents than on the ocean and would diminish the annual variations of the temperature. This diminution of the air's transparency ought chiefly to be attributed to a greater quantity of aqueous vapour in the air, which would cause not only a direct cooling but also copious precipitation of water and snow on the continents. The origin of this greater quantity of water-vapour is not easy to explain." De Marchi has arrived at wholly other results than myself, because he has not sufficiently considered the important quality of selective absorption which is possessed by aqueous vapour. And, further, he has forgotten that if aqueous vapour is supplied to the atmosphere, it will be condensed till the former condition is reached, if no other change has taken place. As we have seen, the mean relative humidity between the 40th and 60th parallels on the northern hemisphere is 76 per cent. If, then, the mean temperature sank from its actual value $+5{\cdot}3$ by 4°–5° C., *i. e.* to $+1{\cdot}3$ or $+0{\cdot}3$, and the aqueous vapour remained in the air, the relative humidity would increase to 101 or 105 per cent. This is of course impossible, for the relative humidity cannot exceed 100 per cent. in the free air. *A fortiori* it is impossible to assume that the absolute humidity could have been greater than now in the glacial epoch.

As the hypothesis of Croll still seems to enjoy a certain favour with English geologists, it may not be without interest to cite the utterance of De Marchi on this theory, which he, in accordance with its importance, has examined more in detail than the others. He says, and I entirely agree with him on this point:—"Now I think I may conclude that from the point of view of climatology or meteorology, in the present state of these sciences, the hypothesis of Croll seems to be wholly untenable as well in its principles as in its consequences"*.

It seems that the great advantage which Croll's hypothesis promised to geologists, viz. of giving them a natural chronology, predisposed them in favour of its acceptance. But this circumstance, which at first appeared advantageous, seems with the advance of investigation rather to militate

* De Marchi, *l. c.* p. 166.

against the theory, because it becomes more and more impossible to reconcile the chronology demanded by Croll's hypothesis with the facts of observation.

I trust that after what has been said the theory proposed in the foregoing pages will prove useful in explaining some points in geological climatology which have hitherto proved most difficult to interpret.

Addendum*.

As the nebulosity is very different in different latitudes, and also different over the sea and over the continents, it is evident that the influence of a variation in the carbonic acid of the air will be somewhat different from that calculated above, where it is assumed that the nebulosity is the same over the whole globe. I have therefore estimated the nebulosity at different latitudes with the help of the chart published by Teisserenc de Bort, and calculated the following table for

Latitude.	Nebulosity.		Continent per cent.	Reduction factor.			K=0·67.		K=1·5.	
	Continent.	Ocean.		Continent.	Ocean.	Mean.	Continent.	Ocean.	Continent.	Ocean.
70										
	58·1	66·7	72·1	0·899	0·775	0·864	−2·8	−2·4	3·1	2·7
60										
	56·3	67·6	55·8	0·924	0·763	0·853	−3·0	−2·4	3·3	2·7
50										
	45·7	63·3	52·9	1·057	0·813	0·942	−3·5	−2·7	3·8	2·9
40										
	36·5	52·5	42·9	1·177	0·939	1·041	−3·9	−3·1	4·1	3·3
30										
	28·5	47·2	38·8	1·296	1·009	1·120	−4·1	−3·2	4·5	3·5
20										
	28·5	47·0	24·2	1·308	1·017	1·087	−4·1	−3·2	4·3	3·4
10										
	50·1	56·7	23·3	1·031	0·903	0·933	−3·1	−2·7	3·3	2·9
0										
	54·8	59·7	24·2	0·97	0·867	0·892	−2·9	−2·6	3·1	2·8
−10										
	47·8	54·0	22·5	1·056	0·932	0·96	−3·3	−2·9	3·4	3·0
−20										
	29·6	49·6	23·3	1·279	0·979	0·972	−4·1	−3·1	4·2	3·2
−30										
	38·9	51·0	12·5	1·152	0·958	0·982	−3·8	−3·2	4·0	3·4
−40										
	62·0	61·1	2·5	0 86	0·837	0·838	−2·9	−2·8	3·2	3·1
−50										
	71·0	71·5	0·9	0·749	0·719	0·719				
−60										

* *Cf.* p. 265.

the value of the variation of temperature, if the carbonic acid decreases to 0·67 or increases to 1·5 times the present quantity. In the first column is printed the latitude; in the second and third the nebulosity over the continent and over the ocean; in the fourth the extension of the continent in hundredths of the whole area. After this comes, in the fifth and sixth columns, the reduction factor with which the figures in the table are to be multiplied for getting the true variation of temperature over continents and over oceans, and, in the seventh column, the mean of both these correction factors. In the eighth and ninth columns the temperature variations for K=0·67, and in the tenth and eleventh the corresponding values for K=1·5 are tabulated.

The mean value of the reduction factor N. of equator is for the continent (to 70° N. lat.) 1·098 and for the ocean 0·927, in mean 0·996. For the southern hemisphere (to 60° S. lat.) it is found to be for the continent 1·095, for the ocean 0·871, in mean 0·907. The influence in the southern hemisphere will, therefore, be about 9 per cent. less than in the northern. In consequence of the minimum of nebulosity between 20° and 30° latitude in both hemispheres, the maximum effect of the variation of carbonic acid is displaced towards the equator, so that it falls at about 25° latitude in the two cases of K=0·67 and K=1·5.

XXXII. *On the Calculation of the Conductivity of Mixtures of Electrolytes.* *By* Prof. J. G. MacGregor, *Dalhousie College, Halifax, N.S.**

ARRHENIUS has deduced†, as one of the consequences of the dissociation theory of electrolytic conduction, that the condition which must be fulfilled in order that two aqueous solutions of single electrolytes, which have one ion in common and which undergo no change of volume on being mixed, may be isohydric, *i. e.* may on being mixed undergo no change in their state of dissociation or ionisation, is that the concentration of ions, *i. e.* the number of dissociated gramme-molecules per unit of volume, shall be the same for both solutions. He obtained this result by combining the equations of kinetic equilibrium for the constituent electrolytes before and after mixture.

According to the above theory, the specific conductivity of a mixture of two solutions of electrolytes 1 and 2, whose

* Abstract of a paper read before the Nova Scotian Institute of Science on the 9th of December, 1895. Communicated by the Author.

† *Zschr. f. physikalische Chemie,* ii. p. 284 (1888).

volumes before the mixture were v_1' and v_2' respectively, which contained n_1 and n_2 gramme-molecules of the electrolytes per unit of volume, whose combined volume after mixture is $p(v_1'+v_2')$, whose coefficients of ionisation after mixture are α_1 and α_2, and whose specific molecular conductivities at infinite dilution under the circumstances in which they exist in the mixture are μ_{∞_1} and μ_{∞_2}, is given by the expression

$$k=\frac{1}{p(v_1'+v_2')}(\alpha_1 n_1 v_1' \mu_{\infty_1}+\alpha_2 n_2 v_2' \mu_{\infty_2}).$$

Since in any case in which isohydric solutions are mixed without change of volume, n_1, v_1', n_2, v_2' are known, α_1 and α_2 readily determinable, and p equal to unity, the specific conductivity can be calculated provided we assume that μ_{∞_1} and μ_{∞_2} have the same values for solutions in a mixture as for simple solutions. In the case in which equal volumes of the constituents are mixed without change of volume, the specific conductivity of the mixture becomes the mean of the specific conductivities of the constituent solutions.

Arrhenius has subjected the above result to a number of tests. In one he determined by experiment several series of dilute aqueous solutions of different single acids, such that if any two of the members of the same series were mixed in equal volumes the mixture was found to have a conductivity equal to the mean of the conductivities of the constituents. Regarding the solutions of each series as shown thereby to be isohydric among one another, he calculated the concentrations of the ions in the various solutions by the aid of data due to Ostwald. The following table gives the result, the numbers specifying the concentration of dissociated hydrogen (in mgr. per litre) in the constituent solutions, and those in each row applying to solutions found as above to be isohydric with one another :—

HCl.	$(COOH)_2$.	$C_4H_6O_6$.	HCOOH.	CH_3COOH.
151·5	152·6			
42·3	35·1			
22·03	21·37	19·07		
4·48	4·09	4·17	4·42	3·96
1·33	1·24	1·25	1·44	1·33
0·379	0·397	0·381	.....	0·402

It will be observed that while the numbers in the various horizontal rows show a general agreement, they differ very considerably from one another, the extreme differences ranging from 0·7 to 20·5 per cent.

He found also that two solutions of Ammonium Acetate and Acetic Acid respectively, which were determined as above to be isohydric, contained, according to Kohlrausch, amounts of the ion CH_3COO which were in the ratio 1 : 0·79, a ratio which is only very roughly equal to unity.

So far as result is concerned these tests are not satisfactory. But the lack of agreement may have been due to various causes: (1) the data for calculation may have been defective; (2) the change of volume which would doubtless occur on mixing, even with very dilute solutions, may have been too great for the application of Arrhenius's deduction; and (3) the difference between the values of μ_∞ in simple solution and in a mixture may be too great to admit of the identification of isohydric solutions by the method employed.

On the other hand, Arrhenius has calculated* the conductivities of two dilute solutions containing in each case given quantities of two acids, employing for this purpose a series of approximations based on his own observations of isohydric solutions of the acids; and the calculated values were found to agree with those observed to within 0·5 and 0·2 per cent. respectively. So far as result is concerned this forms a much more satisfactory test than those mentioned above. But the number of calculations is too small to exclude the possibility of accidental agreement.

The calculation of the conductivity of a mixture of electrolytes is so severe a test of the ionisation theory of electrolysis that I have thought it well to test its possibility on a more extensive scale, especially as a considerable body of material is available for this purpose in the observations of the conductivity of mixtures of solutions of Potassium and Sodium Chlorides made by Bender†. The present paper contains the results of calculations of the conductivities of mixtures determined experimentally by him.

Method of Calculation.

In order to make such calculations by Arrhenius's method, it would be necessary to make a preliminary determination of a number of isohydric solutions of the two salts, and to restrict the calculations to very dilute solutions. They may be made, however, without such preliminary experiments and without

* Wiedemann's *Annalen*, xxx. p. 73 (1887).

† *Ibid.* xxii. p. 197 (1884).

such restriction by employing a more general form of Arrhenius's deduction. Two electrolytes having a common ion and in a state of equilibrium in the same solution, may be regarded as occupying definite portions of the volume of the solution. If we apply the equilibrium conditions to the parts of the solution occupied by the respective electrolytes as well as to the whole solution, we obtain equations which, *mutatis mutandis*, are identical with those obtained by Arrhenius for isohydric solutions and their mixture, and which give a similar result, viz., that for equilibrium the concentrations of the ions of the respective electrolytes per unit volume of the portion of the complex solution or mixture occupied by them must be the same.

With the aid of this result, we can find the ionisation-coefficients of the constituents of a mixture. For if, in addition to the symbols used above, v_1 and v_2 be taken to represent the volumes of the portions of the mixture occupied by the respective electrolytes, it gives us the equation

$$\frac{\alpha_1 n_1 v_1'}{v_1} = \frac{\alpha_2 n_2 v_2'}{v_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We have also

$$v_1 + v_2 = p(v_1' + v_2') \; ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and as the coefficients of ionisation are functions of the dilution only, at constant temperature, we have

$$\alpha_1 = f_1\left(\frac{v_1}{n_1 v_1'}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\alpha_2 = f_2\left(\frac{v_2}{n_2 v_2'}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Of the quantities involved in these equations, n_1, n_2, v_1', v_2' are known, and p may be determined by density measurements before and after mixture. The form of the functions in (3) and (4) may be determined if measurements of the conductivities of sufficiently extended series of simple solutions of the constituent electrolytes are made. We have thus four equations with but four unknown quantities.

If we employ the symbol V to represent the dilution (v/nv'), we may write the above equations as follows :—

$$\frac{\alpha_1}{V_1} = \frac{\alpha_2}{V_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$V_1 + \frac{n_2 v_2'}{n_1 v_1'} V_2 = p\left(V_1' + \frac{n_2 v_2'}{n_1 v_1'} V_2'\right); \quad . \quad . \quad . \quad (2)$$

which, in the case of mixtures of equal volumes, becomes

$$V_1 + \frac{n_2}{n_1} V_2 = p\left(V_1' + \frac{n_2}{n_1} V_2'\right),$$

$$\frac{\alpha_1}{V_1} = \phi_1(V_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{\alpha_2}{V_2} = \phi_2(V_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

I determined α_1 and α_2 from these equations by the following graphical process:—Equation (3) was employed by drawing, from experimental data, for simple solutions of electrolyte 1, a curve with values of the concentration of the ions (α/V) as abscissæ and corresponding values of the dilution (V) as ordinates. This curve was drawn once for all and was used in all determinations. The curve embodying equation (4) had to be drawn anew for each mixture examined. If this mixture was formed of solutions containing n_1 and n_2 gramme-molecules per unit volume of electrolytes 1 and 2 respectively, the curve had as abscissæ the concentrations of ions of a series of simple solutions of electrolyte 2, and as ordinates, since Bender's mixtures were mixtures of equal volumes, n_2/n_1 times the corresponding values of the dilutions. Equations (1) and (2) were applied by finding, by inspection, two points, one in each curve, having a common abscissa ($\alpha_1/V_1 = \alpha_2/V_2$), and having ordinates (V_1 and $\frac{n_2}{n_1} V_2$ respectively) of such magnitude as to have a sum equal to p times the sum of the ordinates of the points on the curves determined by the dilutions (V_1' and V_2' respectively) before mixing. The value of the abscissa common to the two points thus determined gives the concentration of ions of both constituents in the mixture. The corresponding ordinate of the first curve, and that of the second curve multiplied by n_1/n_2, give the dilutions (V_1 and V_2) of the constituents in the mixture. The products of the common value of α/V into V_1 and V_2 are the required values of α_1 and α_2 respectively.

It will be obvious that the values of α_1 and α_2 for a solution containing two electrolytes with a common ion may be determined in this way, whether it has been formed by the mixing of two simple solutions or not. It may always be imagined to have been formed in this way; and if data are not available for the determination of p, special density measurements may be made.

Data for the Calculations.

Bender's paper contains all the data required for the calculation of the conductivities of mixtures of solutions of Potassium and Sodium Chlorides, with the single exception of the specific molecular conductivity of the simple solutions at infinite dilution. Owing to the want of this datum, I have drawn the curves $\alpha/V = \phi(V)$ by means of data based on Kohlrausch and Grotrian's and Kohlrausch's observations* of the conductivity of solutions of KCl and NaCl. They are are as follows :—

NaCl Solutions.

Grm.-molecules per litre.	Specific Molecular Conductivity.	Litres per grm.-molecule.	Concentration of Ions.
0·5	757	2	0·3682
0·884	710·42	1·1312	0·6109
1	695	1	0·6761
1·830	618·59	0·5465	1·1012
2·843	539·93	0·3517	1·4932
3	528	0·3333	1·5418
3·924	466·35	0·2548	1·7802
5	398	0·2	1·936
5·085	392·53	0·1967	1·9416
5·325	377·65	0·1878	1·9562
5·421	371·95	0·1845	1·9611

KCl Solutions.

Grm.-molecules per litre.	Specific Molecular Conductivity.	Litres per grm.-molecule.	Concentration of Ions.
0·5	958	2	0·3939
0·691	933·43	1·4472	0·5304
1	919	1	0·7558
1·427	890·70	0·7008	1·0452
2·208	855·52	0·4529	1·5535
3	827	0·3333	2·0409
3·039	823·95	0·3291	2·0592
3·213	817·94	0·3112	2·1612

These data are quite sufficient for drawing the curves representing α/V as $\phi(V)$ in the parts corresponding to small dilutions, but they are few for the parts corresponding to the

* Wiedemann's *Annalen*, vi. p. 37 (1879), and xxvi. p. 195 (1885).

greater dilutions, where the curvature is most rapid. I therefore obtained interpolation formulæ, by means of which I drew in the latter parts of the curves, expressing α/V in the case of each salt in terms of the reciprocals of powers of V. These formulæ, having no permanent value, need not be given here. The table of results below shows that they were accurate enough for the purpose in hand.

As Bender measured the specific gravities of both his simple solutions and his mixtures, his paper affords the necessary data for determining the change of volume on mixing. Such change will have a double effect on the calculated conductivity: (1) it will affect the value of α as determined from the curves, and (2) it introduces the factor p in the final computation. In the case of Bender's solutions, though in some cases they were nearly or quite saturated, the first effect was so small as to be much less than the error incidental to the graphical process, and I did not therefore take it into account. The second effect was also very small; but as in some cases it was nearly as great as Bender's estimated error, I took it into account in all cases.

While Kohlrausch's solutions had at 18° C. both the constitution and the conductivity specified in his tables, Bender's solutions had at 15° the constitution and at 18° the conductivity ascribed to them. I found that it did not appreciably affect the values found for α_1 and α_2 to take the concentrations at 15° as being the concentrations at 18°; but that this approximation was inadmissible in calculating the conductivity, as in some cases it made a difference of about the same magnitude as Bender's estimated error. Hence in the calculation I took the values of n_1 and n_2 to be Bender's values multiplied by the ratio of the volume of the solution at 15° to its volume at 18°. As Bender measured the thermal expansion of his solutions, his paper affords the necessary data for this correction.

The conductivities given by Bender as the results of his observations are the actual results of measurements, and are thus affected by accidental errors, which in some cases are considerable. In order that his measurements may be rendered comparable with the results of calculations, these accidental errors must, as far as possible, be removed. I therefore plotted all his series of observations on coordinate paper, drew smooth curves through them, and estimated as well as I could in this way the accidental errors of the single measurements. The correction thus determined is referred to in the table on p. 285 as correction a.

Bender himself draws attention to certain differences between his observations of the conductivity of simple solutions

of KCl and NaCl and those for solutions of the same strength contained in Kohlrausch's tables of interpolated values, ascribing them (1) to his own observations being the results of actual measurement, and (2) to the different temperatures at which their respective solutions had the specified strengths. These differences are shown in the following table :—

Salt in Solution.	Conductivity.		Difference.
	Bender.	Kohlrausch.	
NaCl	388	380	+ 8
KCl	478	471	+ 7
NaCl	702	698	+ 4
KCl	916	911	+ 5
NaCl	977	974	+ 3
NaCl	1217	1209	+ 8
KCl	1362	1328	+34
NaCl	1425	1412	+13
NaCl	1594	1584	+10
KCl	1741	1728	+13
NaCl	1745	1728	+17
NaCl	1845	1846	− 1
KCl	2106	2112	− 6
KCl	2484	2480	+ 4
KCl	2820	2822	− 2

Again, it will be noticed that the differences are all of the same sign up to conductivities of about 1800, and nearly all of the opposite sign for higher conductivities: also that for any given conductivity the difference is of the same sign and order of magnitude for solutions of both salts. If they were due to the first of the above assigned causes, since Kohlrausch's interpolated values agree well with his observations, we should expect much more alternation of sign : if to the second, there should be no change of sign : if to both, there should be greater and more irregular variation in magnitude. The fact that the differences are practically the same for both electrolytes at any given value of the conductivity, would seem to show that the cause of the differences—a defect in the apparatus possibly or in the distilled water—was operative in the measurements of both sets of simple solutions, and therefore probably in the measurements of the mixtures. Hence, to render the results of calculations based on Kohlrausch's data for the simple solutions comparable with Bender's results for mixtures, we must determine what the conductivities of Bender's mixtures would have been found to be if Kohlrausch had prepared and measured them. To find this out as nearly as possible, I have plotted the data of the above table with Bender's conductivities as abscissæ and the differences between

them and Kohlrausch's corresponding values as ordinates, and drawn a smooth curve through the points. By the aid of this curve I determined the correction b of the table given below. The correction is, of course, a more or less doubtful one; for it is not certain that the observations on mixtures suffered from the same unknown source of error as the observations on simple solutions. It seems probable, however, that they did; and the results of the table given below would appear to render it almost certain.

It may be well in one case to give an example of the mode of calculation. We may take for this purpose the mixture of solutions containing 1 grm.-molecule of salt each. It is found by the graphical process that the value of α/V for this mixture is 0·718 grm.-molecule per litre, and that the dilutions of the mixture are 0·937 and 1·063 litre per grm.-molecule for the NaCl and the KCl respectively. The densities of the constituent solutions were 1·0444 and 1·0401 respectively, and that of the mixture 1·0422. The expansions per unit volume between 15° and 20° C. were 0·0013569 and 0·0012489 respectively. The values of the conductivity at infinite dilution I took to be 1028 and 1216, according to Kohlrausch's observations. Hence the conductivity of the mixture,

$$k=\frac{1}{2}\cdot\frac{2\times1{\cdot}0422}{2{\cdot}0845}\left(\frac{1\times0{\cdot}718\times0{\cdot}937\times1028}{1+0{\cdot}6\times0{\cdot}00136}+\frac{1\times0{\cdot}718\times1{\cdot}063\times1216}{1+0{\cdot}6\times0{\cdot}00125}\right)=809{\cdot}2.$$

Bender's observed value (he used the same standard as Kohlrausch) was 814. To this a correction of about -3 must be applied to make the observation agree with the others of the same series (correction a), and a correction of about -3 to make it comparable with a calculated value based on Kohlrausch's data (correction b). Bender's reduced result is thus 808, which differs from the calculated value by 1·2 or 0·15 per cent.

Results of the Calculations.

The following Table gives the results of the calculations; the second and third columns containing the numbers of grm.-molecules per litre in the simple solutions at 15° C.; the fourth column, Bender's observed values of the conductivities of the mixtures; the fifth and sixth, corrections a and b referred to above; the seventh, Bender's reduced values; the eighth, the calculated values; and the ninth, the excess of the calculated values over those observed, expressed as percentages of the latter.

No.	Constituent Solutions (grm.-molecules per litre).		Conductivity of Mixture.					Difference (per cent.).
	NaCl.	KCl.	Bender (observed).	Corrections. *a.*	Corrections. *b.*	Bender (reduced.)	Calculated.	
1.	0·5	0·1875	291	0	.....	291	289·5	−0·52
2.	„	0·375	377	0	− 7	370	373·1	+0·84
3.	„	0·5	436	0	− 6	430	426·1	−0·90
4.	„	0·75	545	0	− 5	540	537·6	−0·44
5.	„	1·5	866	0	− 3	863	858·3	−0·54
6.	1·0	0·1875	442	+23	− 6	459	461·4	+0·52
7.	„	0·375	546	0	− 5	541	540·6	−0·07
8.	„	0·75	707	0	− 4	703	701·1	−0·27
9.	„	1·0	814	− 3	− 3	808	809·2	+0·15
10.	„	1·5	1014	+ 6	− 5	1015	1015·2	+0·02
11.	„	2·0	1224	− 6	− 9	1209	1200·6	−0·69
12.	2·0	0·1875	776	0	− 3	773	773·9	+0·12
13.	„	1·0	1085	0	− 6	1079	1086·3	+0·68
14.	„	2·0	1458	0	−13	1445	1458	+0·90
15.	„	3·0	1832	− 9	0	1823	1808·6	−0·79
16.	3·0	1·0	1332	0	−11	1321	1324	+0·23
17.	„	2·0	1674	0	−10	1664	1660	−0·24
18.	„	3·0	2003	0	+ 4	2007	1988·7	−0·91
19.	4·0	0·375	1367	−10	−12	1345	1350·4	+0·40
20.	„	2·0	1857	0	+ 1	1858	1849·3	−0·47
21.	„	3·5	2300	0	+ 3	2303	2239·2	−2·77
22.	„	4·0	2428	+ 6	− 2	2432	2345·3	−3·56

It will be seen that in the case of the more dilute solutions Nos. 1–17 and 19, the differences, which are in all cases less than 1 per cent. and for the most part considerably less, are one half positive and one half negative; and that whether the solutions are arranged in the order of conductivity or in the order of mean concentration, they exhibit quite a sufficient alternation of sign to warrant the conclusion that they are due, chiefly at least, to errors in the observations and the graphical portion of the calculations.

In the case of the stronger solutions, Nos. 16–18 and 19–22, the alternation of sign has disappeared. In the weakest solutions of these two series the differences are positive and small; but as the concentration increases, the differences become negative and take increasing negative values, the negative difference having its greatest value in No. 22, which is a mixture of a strong solution of NaCl with a saturated solution of KCl. The tendency towards a negative difference as the concentration increases may be recognized also in Nos. 11 and 15; and it is perhaps worth noting that, while the mean value of the positive differences is slightly greater than that of the negative differences up to a concentration of 1 gramme-molecule of salt per litre, the mean negative difference is the greater for higher concentrations.

It is manifest from these results that for solutions of these chlorides containing less than, say, 2 gramme-molecules per litre, it is possible to calculate the conductivity very exactly, but that for stronger solutions the calculated value is less than the observed.

This excess of the observed over the calculated conductivities shows one or more of the assumptions implied in the mode of calculation to be erroneous. It would seem to be probable that the error is at any rate largely due to the assumption that the molecular conductivity of an electrolyte at infinite dilution is the same whether it exists in a simple solution or in a mixture, and that the discrepancy is thus due to the effect of mixing on the velocities of the ions. The mode of calculation assumes that in the mixture the constituents are not really mixed, but lie side by side so that the ions of each electrolyte in their passage from electrode to electrode travel through the solution to which they belong only. They must rather be regarded, however, as passing in rapid alternation, now through a region occupied by one solution and now through a region occupied by the other. The actual mean velocities of the ions in the mixture will therefore probably differ from their values in a solution of

their own electrolyte only. In the case of dilute solutions the difference will be small, in sufficiently dilute solutions inappreciable; but in the case of the stronger solutions it may account in large part for the discrepancy observed above. We have, however, so far as I am aware, no data for calculating the effect of mixture on the ionic velocities, or the extent to which the discrepancy is due to this effect.

To obtain some rough conception of its magnitude, I have calculated the conductivity of the mixture No. 18 on two assumptions, which seemed more or less probable,—viz. (1) that the velocities of the ions of each electrolyte in the mixture were the same as they would be in a simple solution of their own electrolyte of a concentration (in gramme-molecules per litre) equal to the mean concentration of the mixture; and (2) that the velocities of the ions of each electrolyte, when passing through a region occupied by the other electrolyte, were the same as they would be in a simple solution of the former of a dilution equal to that of the latter. The expression used for the conductivity was

$$k=\frac{1}{2p}\left(\alpha_1 n_1 \mu_{\infty_1}\frac{u_1'}{u_1}+\alpha_2 n_2 \mu_{\infty_2}\frac{u_2'}{u_2}\right),$$

where u_1 and u_2 are the sums of the velocities of the ions of electrolytes 1 and 2 respectively in simple solutions of the dilutions which they have in the mixture, while u_1' and u_2' are the values these ionic velocities would have according to the particular assumption employed, the velocities in all cases being those corresponding to the same potential gradient. As the graphical process above gave the dilution of each electrolyte in the mixture, the values of u and u' were readily determined by the aid of Kohlrausch's table of ionic velocities*. I found that according to assumption (1) the conductivity would be greater than Bender's reduced value by 1·6 per cent., and that according to assumption (2) it would be greater by 1·3 per cent. Similar calculations could not be carried out in the case of solutions stronger than No. 18 owing to lack of data. Such calculations are of course of little value; but they strengthen the suspicion that the excess of the observed values of the conductivity of mixtures over the calculated values is due to the impossibility of taking into account the effect of mixing on the velocities of the ions.

* Wiedemann's *Annalen*, l. p. 385 (1893).

XXXIII. *Thermodynamic Properties of Air.* *By* A. W. Witkowski*.

[Plates I. & II.]

Part I.

A. *Thermal Expansion of Compressed Air.*

§ 1. *Aim of the Work.*—It was an important advance in the theory of gaseous matter, when the experimental investigations of Despretz, Pouillet, Rudberg, Regnault, and several others demonstrated the approximate character of the fundamental laws of Boyle and Charles. Instead of a common and single law of compressibility and thermal expansion for different gases, there arose the necessity for determining specific corrections of these laws for every one of them.

But as soon as the range of temperatures and pressures had been enlarged, new analogies between the physical properties of these bodies became once more evident. First of all, the theory of the critical state may be mentioned here, supported by the discoveries of Andrews, Cailletet, Wróblewski, Olszewski, and many others. On the other hand, these analogies found their expression in the laws of compressibility, thanks to the investigations of Natterer, Cailletet, Amagat, and Wróblewski.

At the present time the theory of gases seems to be entering a third phase of its development, initiated by Van der Waals, and supported by Wróblewski, L. Natanson, Ramsay, Young, and many other investigators. There are many facts which seem to show that possibly there may be found, not merely an analogy, but even an identity of properties of different gases, provided that for every one of them special specific units of measure (of temperature, pressure, and density) be employed: all matter seems to be built on the same plan; but the scale is different for various bodies.

It is difficult to judge nowadays whether this grand law is an exact truth, or only an approximation: whether it is really universal, or confined only to certain classes of bodies. To confirm or disprove it, numerous and very exact experimental data are greatly needed.

Investigations of the compressibility and thermal expansion in very extended limits prove to be the best means of comparing the thermodynamic properties of gases: in every case

* Translated from the xxiii. Vol. (1891) of the '*Rozprawy*' of the Cracow Academy of Science (Math. Class), and communicated by the Author.

they give fuller and more exact information than observations of critical points, or other singular states. Amongst recent work in this direction the very important investigations of Amagat, extending to very high pressures, must be placed in the first rank. They contain exceedingly valuable data, relating to the compressibility and expansion of gases at ordinary temperatures and at higher ones. Amagat's results give a very clear and extensive idea of the behaviour of several gases, chiefly of those the critical point of which is not far from ordinary temperatures.

Until quite recently the so-called permanent gases had not been investigated at very low temperatures. So far as I know the important paper by Wróblewski, "On the Compressibility of Hydrogen," published after the author's death in the *Sitzungsberichte* of the Vienna Academy, is the only one relating to this range of temperatures. This want of information as regards the compressibility and expansion of permanent gases in the vicinity of the critical state has induced me to undertake the experimental investigation of which the present forms an account. The properties of atmospheric air are described here, at temperatures ranging from $+100°$ to $-145°$ Cent., and for pressures from 1 to 130 atmospheres.

§ 2. *Outline of Method.*—In order to determine the compressibility at different temperatures, and through it also the expansion of gases, two experimental arrangements were chiefly used: in one of them the quantity of gas remains constant, in the other its volume. We might call them the manometric and the volumetric method. In the first of these methods (Andrews, Amagat, &c.) a long calibrated capillary glass tube, enlarged at the open end into a bulb of known volume, is employed. A certain quantity of gas is shut up in the tube by mercury. By means of a compressing arrangement the volume of the gas may be varied at will. The experiment consists in determining the volume and temperature of the gas and the amount of the applied pressure.

The method of constant volume was invented, so far as I know, by Natterer. A vessel of known volume was filled with gas under a known pressure. The experiment consisted in measuring the quantity of gas contained in the vessel at a certain temperature. This was done in the so-called pneumatic trough, under atmospheric pressure. This method has been used also by Wróblewski in his experiments on the compressibility of hydrogen at low temperatures (the first method being useless here on account of the freezing of the mercury).

The determination of the compressibility of a gas at a set of temperatures, together with its thermal expansion under a single pressure (say the atmospheric), gives at once the expansion under any of the pressures employed. And *vice versa*, if the compressibility at one chosen temperature be known, it is sufficient to determine the thermal expansion of the gas under different pressures, in order to obtain directly the compressibility at any of the temperatures employed.

The first of these two ways was followed by Wróblewski in his above-mentioned work on hydrogen. In the present investigation I have used experimental appliances modelled on those of Wróblewski, and my theoretical aim was a similar one; but I preferred to apply the second way of experimenting, viz., the thermal expansion, instead of the compressibility, has been considered as the principal subject of the experiments.

The following considerations have induced me to make this change. First, the difficulty of measuring pressures exactly. Since absolute manometers, suitable for laboratory use and sufficiently trustworthy, are still to be invented, we are compelled to use gas-manometers, founded on the compressibility of gases (air or nitrogen). The pressure calculated according to the indications of an instrument of this kind, as well as the law of compressibility of the gas under investigation, depend, in this case, directly on the assumed law of compressibility of the gas used in the manometer. Suppose we take this law as known, say through the experiments of Amagat, then our results will be inextricably mingled with these results. The method to be described presently, depending on determinations of expansion under constant pressure, on the other hand, furnishes values of expansion entirely independent of any accepted law of compressibility: a dependence of this kind remains only in the values of the pressures applied.

Another reason which induced me to depart from Wróblewski's combination of the gas-manometer with the constant-volume method of Natterer, was the wish to invent a method of constant sensibility as regards pressure-meaurements, and the determinations of expansion and compressibility as well. The gas-manometer is an instrument of variable sensibility; the higher the pressure measured, the less is the exactness of measurement. On the other hand, the volumetric method is one of constant sensibility; *i. e.* like increments of pressure yield approximately equal increments of the quantity of gas.

Instead of combining two methods of such opposite cha-

racter, I preferred to measure both pressures and expansions by the volumetric method. By this device, as I hope, the exactness and homogeneity of the results were materially increased.

§ 3. *The Coefficient.*—In order to determine the thermal expansion of air at constant pressure (1 to 130 atmospheres), I have applied the following arrangement:—Two vessels of known capacity are filled simultaneously with gas, under any desired pressure, by connecting them with a reservoir containing a sufficient quantity of condensed gas. One of these vessels is cooled or heated to any temperature θ; the other is kept at the constant temperature of melting ice. Let p atmospheres be the common pressure in both vessels; s_1 and s_2 their capacities, at the respective temperatures $\theta°$ and $0°$, under the common pressure p.

The quantities M_1 and M_2 of gas contained in the vessels are then brought under atmospheric pressure and temperature: having measured their volumes, we calculate M_1 and M_2 in the usual way. As unit quantity of gas I take here, and in the following pages, the mass contained in unit volume (cub. mm.) at 0° Centigr. under the pressure of one atmosphere.

The densities of the two masses, when compressed in the vessels s_1 and s_2, are unequal; the colder one is also the denser. Let their densities be ρ_1 and ρ_2, then we have

$$\rho_1=\frac{M_1}{s_1}; \quad \rho_2=\frac{M_2}{s_2}.$$

But

$$\rho_1=\frac{\rho_2}{1+\alpha_{p,\theta}.\theta};$$

therefore

$$\alpha_{p,\theta}=\frac{1}{\theta}\frac{M_2}{M_1}\frac{s_1}{s_2}-\frac{1}{\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This formula may be employed to calculate $\alpha_{p,\theta}$; *i. e.*, the mean coefficient of expansion of the gas between 0° and $\theta°$, when under the constant pressure of p atmospheres. It will be remarked that the value of $\alpha_{p,\theta}$ is made here to depend on two *ratios*, $\frac{M_2}{M_1}$ and $\frac{s_1}{s_2}$, and on the temperature θ.

In most of the experiments to be described the temperature of the vessel s_2 was not 0°, but $t°$ (usually $+16°$). In this case we have

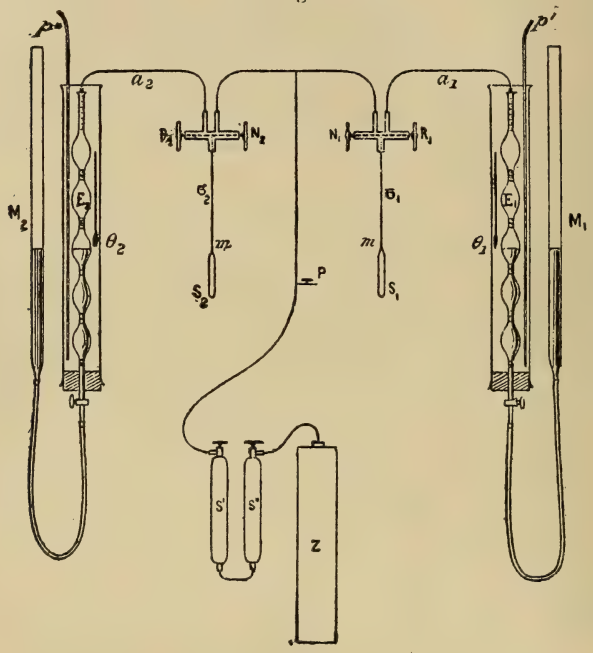
$$\rho_1=\frac{M_1}{s_1}=\frac{\rho_0}{1+\alpha_{p,\theta}.\theta}; \qquad \rho_2=\frac{M_2}{s_2}=\frac{\rho_0}{1+\alpha_{p,\ }.t};$$

therefore

$$\alpha_{p,\theta}=(1+\alpha_{p,t}\,.\,t)\frac{M_2 s_1}{M_1 s_2}-\frac{1}{\theta}\quad . \quad . \quad . \quad . \quad . \quad (2)$$

§ 4. *Description of Apparatus.*—A general representation of the apparatus I have used to obtain the values of the coefficient α will be found in fig. 1.

Fig. 1.



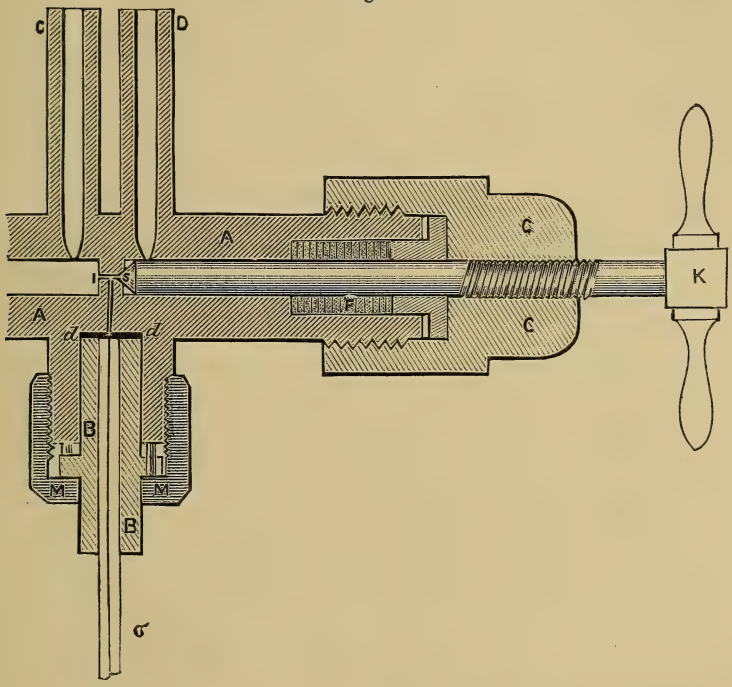
The vessels denoted above by s_1 and s_2 are two thick-walled glass bulbs, melted on to capillary stems σ_1 and σ_2, the diameter of bore of which is less than $\frac{1}{5}$ mm. On each of the capillary tubes a mark m is drawn at a distance of 3–4 cm. from the bulb, to limit the capacity s. The capacities of bulbs and stems were measured repeatedly, before and during the experiments, by mercury weighings. The capacities s_1 and s_2 were nearly 2000 c.mm. (in some experiments at the lowest temperatures, only 1000 c.mm.) ; the capacities of the capillary stems (σ_1 and σ_2) were not more than 8 to 10 c.mm.

The upper end of each stem (σ_1 and σ_2) is connected with a sort of distributing apparatus, provided with two screw-valves, of which one is used to fill the bulb with gas, the other

to transfer the gas to the eudiometer, where its quantity is to be measured. This arrangement is quite similar in plan to that used by Wróblewski in his already-mentioned investigations on hydrogen. As regards details, I have adopted, after several unsuccessful trials, the following construction, which proved quite trustworthy and convenient :—

A brass cylinder, A (fig. 2), is bored along its axis from

Fig. 2.



both ends ; in its centre the borings are connected by a very short and narrow channel *r s*, communicating with another vertical channel, *t*, of very small capacity. The channel *t* is connected with the stem σ in the following manner:—On the upper end of σ there is cemented, by means of fine sealing-wax, a thick-walled brass tube B, provided near its lower end with a collar. The end of the glass stem, cut perpendicularly to its length, protrudes some $\frac{2}{10}$ of a millimetre from the flat end of B. The glass bulb *s*, with its stem and brass tube B, form a separate piece, which is to be connected tightly with the cylinder A. This is done by means of the external screw M, in a way sufficiently indicated by the figure. To ensure

perfect tightness a circular perforated plate of lead, *d d*, carefully cleaned, is placed between the flat end of B and the bottom of the corresponding boring in A. Through the pressure of the screw the hole in the lead plate gets reduced to a very small size, and at the same time a perfect pressure-tight connexion is ensured. It is necessary, however, that the metallic surfaces in contact be quite clean and bright.

To shut up the compressed gas in the bulb two steel screw-valves, K, are employed (one is shown in the figure). They consist of a cylindrical spindle, carefully turned and polished, provided with a conical end; near the outer end a screw-thread is cut, working in the screw-head G. By means of the same screw-heads G a number of leather rings, cleaned with ether and impregnated with beeswax and paraffin-oil, are compressed in the stuffing-boxes around the cylindrical polished parts of the spindles. This mode of tightening proved perfectly capable of withstanding the highest pressures employed, although the tightening of the screw-valves is far less important than that of the channel *r s t*, which is effected solely by contact with metallic surfaces.

On both sides of the channel *r s* there are two metallic outlets, C and D, soldered to A. Through C (fig. 2) the bulb s_1, and similarly s_2, is charged with compressed gas; to do this simultaneously in both bulbs, stout copper tubes of small bore, soldered in C and C, are connected by means of a metallic T-tube with the reservoir Z (fig. 1) containing compressed air. In the connecting-piece another screw-valve P is inserted, through which the interior of the bulbs may be made to communicate with the atmosphere. The valves N_1 and N_2, facing C and C, will be called here for brevity the "charging valves."

Through the second pair of outlets D and D (fig. 2) the charges of air, compressed in the bulbs *s*, are delivered for the purpose of measurement; this is done by unscrewing the "discharging valves" R_1 and R_2 (fig. 1).

The measurement of gas-quantity has been effected in special glass flasks E_1 and E_2 (fig. 1), which may be called eudiometers. For the sake of distinctness of drawing, E_1 and E_2 are represented in the figure on opposite sides of the apparatus; in reality they were placed near one another.

The discharging-tubes D and D were connected with the eudiometers by means of capillary copper tubes a_1, a_2 (fig. 1), ending on the eudiometer side in perforated steel stoppers; these were cemented in the upper ends of the eudiometers by means of sealing-wax.

Each eudiometer consists of a vertical glass tube, some

80 cm. in length, on which five bulbs of nearly equal capacity have been blown; the connecting-pieces are provided with divisions in millimetres. The eudiometers are surrounded by wide cylindrical jackets of clear glass filled with water. Two thermometers, θ_1 and θ_2, immersed in the water give the temperature of the gas; air-streams blown in through p' and p'' serve to equalize the temperature of the water.

The eudiometers are connected by means of thick-walled indiarubber tubing with open mercury-manometers M_1 and M_2, fixed to two sliding pieces on a vertical wooden support (not shown in the figure).

The calibration of the eudiometers was effected after they were put in their place and surrounded with water. To do this by mercury-weighing, the glass stop-cocks near the lower ends of the eudiometers were provided. Positions of the mercury meniscus were read by means of a cathetometer, placed in a position where it was to remain during the rest of the experiments. In this manner both eudiometers were calibrated twice. From the results of calibration I prepared a table giving capacities in cubic millimetres (at the standard temperature of $+17°$ C.) corresponding to every scale-division. Each of the five glass bulbs had a capacity of nearly 40 c.cm., the whole eudiometer about 210 c.cm. One millimetre on the glass necks corresponded to 60 c.mm. capacity. Therefore the smallest volume which could be appreciated was about 3 c.mm.

After calibration of the eudiometers, I made another set of measurements in order to determine the capillary depressions &c. These corrections (of small importance) have been determined for every scale-division in an obvious manner.

§ 5. *Preparation of Gas.*—To complete the description of the apparatus, a few words remain to be said about the reservoir of compressed air and the mode of compressing and purifying the gas. One of the well-known carbonic-acid bottles of 11 litres capacity, resisting 250 atmospheres, is used as air-store. The usual screw-valve is replaced by a simple stopper of brass with a copper tube soldered in it, connecting the reservoir Z with two drying-tubes S′, S″. These are made of steel, and are filled with finely grained chloride of calcium and potassium hydroxide, stoppered by thick layers of cotton-wool. A considerable quantity of these substances is also contained in the reservoir Z in a vertical tube of wire-gauze. The air submitted to experiment is compelled to pass the drying-tubes twice—during the charging of the reservoir, and again on its way to the apparatus. Besides that, it remains a long time in contact with the drying-substance in the reservoir itself.

To condense the gas, a Natterer condensing-pump has been

employed. The usual oiled leather piston is replaced by one packed with fibrous asbestos, and lubricated by a very small quantity of water (a device used also by Wróblewski). Although this arrangement renders the working of the pump exceedingly tedious, nevertheless I found it absolutely necessary, as the oiled piston yielded a very impure gas.

The air was taken from the outside of the laboratory; it passed through several washing-bottles filled with solutions of potassium hydroxide and sulphuric acid. After condensation, it was dried in the manner described above. The air treated in this way was submitted repeatedly to chemical tests, and proved to be sufficiently pure and dry.

§ 6. *Determination of* s_1 *and* s_2.—Before commencing the experiments, it was necessary to determine very exactly the capacities of the bulbs s_1 and s_2. This was done by mercury-weighings at the temperatures 0° and +100°. The true capacity during experiment depends on the temperature of the bulb and on the pressure employed.

To calculate the capacity at low temperatures, I used the coefficient of expansion of glass, determined in the interval 0° to +100°, and corrected it in accordance with the experiments of I. Zakrzewski on the expansion of solids at low temperatures *.

The influence of pressure on the capacity of the bulbs has been determined directly by submitting them to known pressures while filled with well-boiled mercury. From the observed variations of position of the meniscus in the calibrated stem, it is easy to calculate the variations of capacity with the aid of Tait's and of Amagat's determinations of the compressibility of mercury. In the limits of pressure used this dependence was a linear one. Denoting the corresponding coefficient by x, we may write

$$s_p = s(1 + xp).$$

For a pair of bulbs of very unequal capacity (1 : 2), I have found $x = 0{\cdot}0000062$ and $x = 0{\cdot}0000064$ (per atmosphere). These values refer to ordinary temperatures. As there is nothing known about the variations of elasticity of glass at very low temperatures, I have always used the above value of x; I consider the error resulting from this proceeding as altogether insignificant, the more so as the correction itself, depending on pressure, is very small.

§ 7. *Mode of Experimenting.*—I will now describe the mode of experimenting and calculation of the results.

* Transac. Cracow Acad. vol. xx. (math. class.).

The apparatus having been filled repeatedly with pure and dry air, I bring the movable manometric tubes M_1 and M_2 to such levels as to fill both eudiometers E_1 and E_2 with mercury; the menisci are then brought as near as possible to the zero marks of the eudiometer-divisions. (The stopcocks P, N_1, N_2, R_1, R_2 are opened for this purpose.) The small volumes in the eudiometer-tubes left between the menisci of mercury and the zero marks will be denoted here by w_1 and w_2.

As soon as equality of pressure in the interior of the apparatus and in the atmosphere ($=b$) has been established, I close the discharging-valves R_1 and R_2, as well as P, and charge the apparatus with compressed air at that particular pressure for which the expansion is to be measured. During the charging both bulbs s_1 and s_2 remain in connexion with the large store of air contained in the reservoirs Z, S′, S″. This connexion ensures great constancy of pressure in case there should be a small leakage of air somewhere.

During the charging of the bulbs I determine their temperatures (t and θ), as well as the temperature ($=\tau$) of the surrounding air. Next, I shut the charging-valves N_1 and N_2 both at the same instant, and immediately after that I open the discharging-valves R_1, R_2. The interval of time during which the charge remains imprisoned in the bulbs is less than one second.

The air flows now from the bulbs into the eudiometers. During this I lower the manometric tubes M_1 and M_2, so as to reduce the gas (which fills now several bulbs in each eudiometer) as nearly as possible to atmospheric pressure. The temperatures of the water-baths surrounding the eudiometers having been equalized by air-currents blown in through p' and p'', I proceed to determine, with the aid of the telescope of the cathetometer,—

(1) The temperatures ϑ_1 and ϑ_2 of the gas in both eudiometers;

(2) The volumes u_1 and u_2 of air in both eudiometers, reckoned from the zero marks;

(3) The differences of level of mercury in the eudiometers E and in the corresponding manometric tubes M_1 and M_2.

Taking into account the corrections mentioned in § 4, as well as the barometric pressure, it is easy to find the pressures of air in the eudiometers E, and in the bulbs s, which are now in communication with E. I shall denote these pressures by B_1 and B_2.

During these determinations the temperatures of the bulbs are again measured ($=t'$ and θ').

In this manner every single determination of the coefficient

α has been conducted. It takes some time, from 10 to 20 minutes, to perform the necessary measurements. At the lowest temperatures this time was considerably longer, because in this case it is necessary to provide a sufficient quantity of the freezing substance (solid carbonic acid, liquid ethylene); to regulate the speed of the pneumatic pumps and of the gas-motor by which they were driven, in order to obtain the required temperature; to compare the electric thermometer for low temperatures with the hydrogen thermometer, &c. Under such conditions it happened that during a day I was not able to make more than one or two determinations—to say nothing of the frequent occurrences, when, after several days of laborious preparations, the intended experiment has been entirely lost, through some mischance or other. I think it necessary to mention these difficulties of experimenting in order to explain the comparatively small number of results obtained at the very lowest temperatures.

§ 8. *Calculation of Results.*—By means of the experimental data gathered in the manner described above, the results have been calculated as follows:—

The bulbs being charged, and the charging-valves closed, we have the following quantities of gas in the apparatus:—

1. In the bulb of capacity s (index 1 or 2 omitted, since the same formula applies to both) there is the quantity M (§ 3) under the pressure of p atmospheres, at θ and t (respectively) degrees.

In the stem σ, and the narrow channels $r\,s\,t$ connected with it, there is confined a quantity m; under the same pressure p, but at the temperature τ of the circumambient air. I shall use the letter σ to denote the small capacity of this space.

Adding together we find in the apparatus the quantity

$$M+m=A$$

under the pressure p.

2. In the copper tubes a_1 and a_2 (fig. 1)—I shall use the same letters to denote their capacities—and also in the eudiometers themselves (in the spaces denoted above [§ 7] by w_1, w_2), we have air under the barometric pressure b.

I shall use for brevity the symbol $w(\vartheta, b)$ to stand for the expression

$$\frac{w}{1+\gamma\vartheta}\cdot\frac{b}{760},$$

γ being the coefficient of expansion of air of ordinary density at ordinary temperatures ($=0{\cdot}00367$, very nearly a constant).

Then the above quantities are

$$a(\tau, b) + w(\vartheta, b)$$

expressed in normal volume-units.

This same quantity of air which occupied after charging the spaces s, σ, a, and w, fills after discharge a space increased by that capacity in the eudiometer (u is now the volume occupied in it) in which the mercury has been replaced by air, plus a small capacity (K=42 c.mm.) laid open by unscrewing the valve R (the valves were always opened by the same number of turns of the spindle). The whole quantity of air supports now the common pressure B. Consequently we have a new expression for the whole quantity of air, namely:

$$s(\theta', \mathrm{B}) + \sigma(\tau, \mathrm{B}) + \mathrm{K}(\tau, \mathrm{B}) + a(\tau, \mathrm{B}) + u(\vartheta, \mathrm{B}).$$

Equating we find the formula :

$$\mathrm{A} = \mathrm{M} + m = u(\vartheta, \mathrm{B}) - w(\vartheta, b) + a(\tau, \mathrm{B} - b)$$
$$+ s(\theta', \mathrm{B}) + [\mathrm{K} + \sigma](\tau, \mathrm{B}) \quad . \quad . \quad . \quad . \quad (3)$$

§ 9. *Corrections.*—We have now to calculate the small amount m which is to be subtracted from A. Consider the bulb No. 2, charged at temperature $t = +16°$ C. Suppose the temperature of the corresponding space σ were also exactly $+16°$, then obviously we could write

$$m_2 = \mathrm{A}_2 \frac{\sigma_2}{s_2 + \sigma_2}.$$

Taking into consideration the small difference of temperature between bulb ($+16°$) and stem (τ_2) we write instead

$$m_2 = \mathrm{A}_2 \frac{1 + \gamma t}{1 + \gamma \tau} \frac{\sigma_2}{s_2 + \sigma_2}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Simultaneously in the space σ_1, corresponding to bulb No. 1, we have the following quantity of air :

$$m_1 = \mathrm{A}_2 \frac{1 + \gamma t}{1 + \gamma \tau} \frac{\sigma_1}{s_2 + \sigma_2},$$

because in every case

$$\frac{m_1}{m_2} = \frac{\sigma_1}{\sigma_2}.$$

In this manner we find, finally,

$$M_1 = A_1 - m_1 \; ; \; M_2 = A_2 - m_2 \; ;$$

these values substituted in (1) or (2) lead directly to the coefficient $\alpha_{p,\theta}$.

§ 10. *Corrections continued.*—The first of the small capacities denoted above by a and σ I determined by a simple volumetric method, with the aid of the calibrated eudiometer E and manometer M. It is superfluous to enter into particulars. I will mention only that the value of a has no influence on the final result, provided we have exactly $B = b$; in general any small error in a influences the result quite insignificantly.

The second of these capacities, viz. σ, is composed of two parts: the capacity of the glass stem, and of the narrow channels $r\,s\,t$, together with the very small hole in the tightening lead plate. The capacity of the glass stem, reckoned from the mark m, is obtained directly by weighing of mercury. In order to find the rest (3 or 4 c. mm.) of the space σ, I stopped the connexion between the glass stem and the rest of the space σ by interposing another (not perforated) lead plate and charged the channels several times with compressed air, at 50 or 60 atmospheres. The quantity of the collected gas having been measured with the aid of the eudiometer, it is easy to calculate the capacity of the channels, by applying Boyle's law, or better Amagat's results, for the compressibility of air at ordinary temperatures.

§ 11. *Determination of Pressures.*—The just mentioned results of Amagat* render also possible the determination of pressures. The pressures p, as already mentioned, I measured by applying the "constant volume" method, with variable quantity of gas. This I will now explain.

If a given quantity of air be compressed more and more, by application of increasing pressure, then, as is well known, the product of volume and corresponding pressure—far from being constant, as it would be according to Boyle's law—diminishes at first, until a certain pressure is reached, and increases afterwards indefinitely (so far as known at present).

Let us denote by v_0 the volume occupied under the pressure of one atmosphere ($=p_0$), then we may express the law of compressibility of air by the formula

$$pv = \epsilon \cdot p_0 v_0,$$

ϵ denoting a coefficient variable with the pressure p. The

* *Comptes Rend.* 1884, p. 1154.

values of ϵ at ordinary temperature ($+16°$) have been determined with great care by Amagat with the aid of a huge mercury-manometer.

Consider now a closed vessel, capacity s, charged with compressed air, under a pressure p, at t degrees. Since a unit volume of air, exerting the pressure of one atmosphere (at $+16°$), gets reduced to $v=\frac{\epsilon}{p}$ when submitted to p atmospheres (by the last formula), then we infer that the air-mass compressed in the closed vessel when liberated and submitted to unit-pressure would occupy the volume $\frac{ps}{\epsilon}$; or, in the normal condition, when cooled to 0°, the volume:

$$M=\frac{ps}{\epsilon(1+\gamma t)}.$$

It will be seen now that the apparatus described in the foregoing paragraphs may be used also as a manometer, recording pressures to be calculated by the formula:

$$p=\epsilon\frac{M(1+\gamma t)}{s},$$

provided that t equals that temperature for which the values of ϵ were determined, or at any rate that the difference be not too great. For this reason one of the bulbs of my apparatus was always immersed in a water-bath at $+16°$, together with a stirrer and a mercury-thermometer divided in $\frac{1}{20}$ of a degree. Since the temperature τ of the space σ differed but slightly from $+16°$, we may write with sufficient accuracy:

$$p=\epsilon\frac{A(1+\gamma t)}{s+\sigma}\quad\ldots\ldots\ldots\quad(5)$$

The method chiefly employed till now for measurements of high pressures consists in the use of manometers charged with a constant quantity of gas. I thought it interesting to compare this method with the constant-volume method used by myself. For this purpose I connected my apparatus with a gas-manometer, charged at first with dry air, afterwards with pure dry nitrogen. From a large number of comparisons at various pressures, there resulted a slight but systematic difference of results: the values of pressures, as determined by my method, were constantly less, by several tenths per

cent., than the values recorded by the gas-manometer; this applies as well to the air- as to the nitrogen-manometer. The sign of the difference led me to suspect a leakage of gas in some part of my apparatus. Therefore it was indispensable to submit the apparatus to a severe test, the more so as any leakage in it would vitiate the results, as regards expansion, to an incalculable extent.

To test the apparatus I charged it (both bulbs at $+16°$) with compressed air, under a sufficiently high pressure, the value of which was determined simultaneously with the aid of the gas-manometer. The bulbs were then immediately discharged into the eudiometers. Next I repeated this same experiment, using the same pressure (as indicated by the gas-manometer), but instead of discharging the bulbs immediately I left them charged for a relatively long time (an hour or two). I was satisfied to find that the quantity of air collected after long imprisonment was not less than in the first case.

As an instance of this sort of testing, the following numbers may be given:—

First experiment.—Nitrogen-manometer = 89·06 atm.
The bulbs discharged immediately after charging.

Bulb.	Temp. of bulb.	s c. mm.	σ c. mm.	A c. mm.	p (calc. by 5).
No. 1...	$\theta=16$	901·90	8·7	77723	88·60
No. 2...	$t=16$	1919·47	15·0	164903	88·49

Second experiment.—Nitrogen-manometer = 89·10.
Bulbs kept charged 1 hour 15 min.

Bulb.	Temp. of bulb.	s.	σ.	A.	p.
No. 1 ...	+16	901·90	8·7	77783	88·67
No. 2 ...	+16	1919·47	15·0	165096	88·60

It will be remarked that the pressure indicated by the gas-manometer exceeds in both cases that calculated by (5) by $\frac{1}{2}$ per cent. nearly. At the same time it is apparent that there was no sensible leakage, since in the long-charge experiment the difference of pressure is even less than in the first. I am not able to give a sufficient explanation of the observed difference, but considering that the constant-volume method employed in the present work deals with larger quantities of gas, and enables us to measure them with greater accuracy, I am inclined to think that the results obtained with my apparatus are at least not less trustworthy than those recorded by an ordinary gas-manometer.

§ 12. *Determination of Temperatures.*—All temperatures referred to in this paper are reduced to the scale of the constant-volume hydrogen-thermometer. In all experiments at temperatures below zero there was placed in contact with the cooled bulb s, in the same freezing-mixture, the bulb of a hydrogen-thermometer. Yet the temperatures of the bulb during the experiments on expansion were not read directly on the hydrogen-thermometer—because of the waste of time unavoidable in such readings, and because of the slowness of indications, which renders the hydrogen-thermometer unsuitable to follow rapid variations of temperature. For this purpose I constructed a small working thermometer based on the variations of electric resistance of a fine platinum wire. A description of this instrument will be found in the appendix to the present paper. Here it will be sufficient to say that its sensibility was about $\frac{1}{20}$ degr. Centigr. and its quickness very considerable.

This electric thermometer has been compared very often with the hydrogen-thermometer, and a table has been drawn interpreting its indications in terms of the hydrogen-scale. Nevertheless I never used the electric thermometer otherwise than under control of the hydrogen-thermometer, because slight secular changes of its resistance manifested themselves. Comparisons of the working and the hydrogen-thermometer were made in the intervals between two consecutive experiments on expansion.

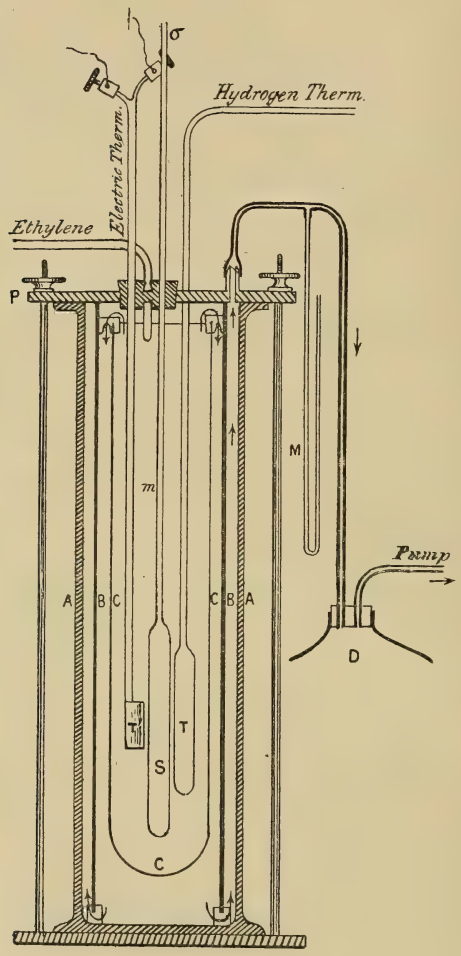
During the experiments themselves an assistant read the electric thermometer a first time simultaneously with the charging of the bulbs, a second time immediately after discharging them (it will be understood that the freezing-mixtures used to obtain very low temperatures do not keep their temperature quite steady); the mean of these two readings has been accepted as the temperature (θ) of the bulb. Finally, the temperature of the bulb was determined a third time (θ') during the measurement of the gas-quantity in the eudiometers; this temperature has been used to calculate the exact value of the gas-quantity remaining in the bulb.

§ 13. *The Low Temperatures.*—In the manner explained above I executed some hundred and twenty determinations of the coefficient of expansion $\alpha_{p,\theta}$, using different pressures up to 130 atmospheres. One of the bulbs of the apparatus being kept at $+16°$, the other was heated or cooled to the following temperatures:—$+100°$ (steam), $0°$ (ice), $-35°$ (a freezing-mixture of pounded ice and crystallized chloride of calcium), $-78°{\cdot}5$ (solid carbonic acid and ether), $-103°{\cdot}5$ (liquid ethylene, boiling under atmospheric pressure), $-130°$, $-135°$,

$-140°$, $-145°$ (liquid ethylene boiling under reduced pressure).

The arrangement of the thermostat in which liquid ethylene has been boiled under reduced pressure is represented in fig. 3. A tall cylindrical glass vessel A, standing on a horizontal

Fig. 3.



brass plate, is covered with another brass plate PP, the diameter of which is somewhat larger than the diameter of the upper flat edge of the glass vessel. By means of three brass pillars, fastened to the lower plate, and screw-heads pressing

on the upper one, the cover can be made air-tight. Near the centre of the upper brass cover there are four circular openings. A fifth opening, with a short brass tube soldered in it, is placed near the circumference; it is connected by means of a lead tube with the pneumatic pumps.

The just mentioned four openings are intended to introduce into the apparatus (1) the bulb *s*, which is to be cooled; (2) the bulb T of the hydrogen-thermometer; (3) the electric thermometer T′. The stems of these three pieces are held by indiarubber stoppers cut into two halves and tightened by a beeswax cement. Through the fourth opening enters in like manner the delivering tube of the well-known apparatus of Wróblewski and Olszewski for liquefying ethylene.

The cold liquid ethylene flows down into the thin-walled tall glass beaker C, surrounded by a wide glass tube B. This tube rests on three elastic pieces of indiarubber which press its upper edge firmly against the brass plate P. This double walling is intended to cool the surroundings of the beaker C by cold ethylene vapour, which is forced to circulate as indicated by the arrows, until finally it is drawn out by the pumps.

Liquid ethylene is very liable to get superheated and to evaporate in an explosive manner, especially when boiling under diminished pressure. This property is a great inconvenience when it is to be used to obtain constant temperatures. After trying different contrivances to avoid it (air-currents, Gernez's air-bubbles, &c.), I contented myself with the use of a large flask D inserted just before the pumps, in order to diminish the fluctuations of pressure accompanying the strokes of pistons.

With the aid of this arrangement, it is possible to vary the boiling temperature of ethylene through a range of some forty degrees. Any desired temperature may be obtained by varying the number of air-pumps (I had three large Bianchi at my disposal), by inserting Babinet's stopcocks, or by varying the speed of the gas-motor driving the pumps. With some practice it is possible to limit the range of fluctuations of temperature in one series of experiments to no more than 2 or 3 degrees.

§ 14. *Construction of Isothermal Curves.*—The temperature θ of the bulb *s* in different experiments belonging to one series was not exactly the same. To obtain strictly isothermal values of the coefficient α, the following graphical method of interpolation was used. All series of experiments having been completed I constructed, on a large scale, a diagram similar to that represented on Pl. I., using uncorrected values

of α. By means of graphical interpolation isothermals were then drawn, differing but little from the final ones given on Pl. I. With the aid of these isothermals I constructed another large diagram on which the values of α were collected on lines of equal pressure; *i. e.* temperatures were drawn on the axis of abscissæ, the values of α belonging to like pressures formed the ordinates.

The inclination of these lines to the axis of abscissæ could now be used to find the correction which was to be added to the coefficient $\alpha_{p,-33}$ say, in order to obtain the value of $\alpha_{p,-35}$, belonging to another isothermal, but to the same pressure p ($-35°$ in this example being the mean value of temperatures in the chloride-of-calcium series). This I did by drawing through the point $a_{p,\theta}$ (experimental value) a short piece of the corresponding isopiestic line, to its intersection with the ordinate $-35°$, or any other desired temperature. In most cases these lines were nearly straight; instead of drawing them I found it convenient to use a glass plate on which a straight line had been drawn with a diamond point. In some cases, however, the corrections were considerably larger, so that it was no longer possible to disregard the curvature of these lines; they were then drawn with reference to the inclination and curvature of the neighbouring isopiestic lines.

§ 15. *Results.*—The whole of the results obtained are represented in a graphical form on Pl. I. The isothermal lines for the nine temperatures experimented on have been drawn by hand along the dots, representing experimental results*. Although some of these dots fall off rather considerably from their respective mean curves, yet, considering the whole of the diagram, I suppose the final results may be considered accurate, at least to four decimals. It is scarcely possible at present to aim at a greater accuracy: this will be admitted on considering the discrepant values of the coefficient of expansion of air given by different experimenters in the relatively simple case of atmospheric pressure and temperature of boiling water.

From the diagram Pl. I., drawn on a large scale, I took the mean values of the coefficient $\alpha_{p,\theta}$, reproduced in the following table:—

* In the original memoir extensive tables are reproduced giving full information on the particular data belonging to every experiment.

TABLE *of the mean Coefficients of Expansion of Atmospheric Air.* (Values of 100000 . $\alpha_{p,\theta}$.)

Pressures atm.	Temperatures θ.								
	+100·0	+16·0	−35·0	−78·5	−103·5	−130·0	−135·0	−140·0	−145·0
10...	375	376	...	..	...	...	...	...	...
15...	379	382	...	...	...	...	...	420	427
20...	383	387	...	401	410	427	...	440	450
25...	388	392	...	411	422	443	...	463	479
30...	392	398	...	420	434	462	477	492	519 *
35...	397	403	...	429	448	483	506	538	
40...	402	408	...	438	461	508	544	632	
45...	406	414	...	448	474	536	594		
50...	410	419	430	457	487	569	619		
55...	414	424	436	467	500	598	623		
60...	418	429	442	476	512	610	622		
65...	421	434	448	485	525	612	621		
70...	425	438	454	494	536	612			
75...	428	442	461	503	547	610			
80...	431	446	467	512	557	607			
85...	434	449	473	520	566				
90...	437	452	479	527	572				
95...	439	455	485	532	577				
100...	441	458	489	537	579				
105...	443	460	493	542	580				
110...	445	462	497	545	580				
115...	447	463	499	548	579				
120...	449	465	501	550	577				
125...	...	466	503	551	574				
130...	...	468	...	551	571				

* Corresponds to 29 atmos.

This table, or, better still, the diagram on Pl. I., shows very clearly the kind of variation of the coefficient of expansion in its dependence on pressure and temperature. For increasing pressures the coefficient increases, at every temperature, to a maximum, and decreases subsequently. The higher the temperature, the less pronounced is this maximum.

The pressure corresponding to the maximum of expansion diminishes with the temperature. From this it follows, that the rate of increase of expansion is the higher, the less the temperature. In the vicinity of the critical state (temp. = −141°, pressure 39 atm. nearly) the rate of increase of expansion is extremely great, the corresponding isothermal runs here nearly vertically. The same holds good for temperatures below the critical, at points where liquefaction occurs, for instance at −145° and 30 atm.

The isothermal lines of the coefficient of expansion form a fan-shaped bundle converging nearly to one point, namely, to the value 0·00367 at 1 atm. of pressure. If this were

strictly true, the expansion of air, subject to a constant pressure of 1 atm., would be independent of temperature. Now this is known to be approximately the case; but on the other hand it is certain that the isothermals cannot converge *strictly* to this point, since the pressure at one atmosphere has nothing in common with the internal constitution of air. From some experiments it would appear that the isothermals do not intersect at all low pressures; they come very near one another, descending to a minimum at certain low pressures, and probably return afterwards in a steep course towards higher values.

B. *Compressibility at Low Temperatures.*

§ 16. *Definitions.*—It is now a simple matter to calculate the compressibility of atmospheric air at any one of the nine temperatures investigated above; the compressibility at $+16°$ being assumed as known through the work of Amagat (§ 11).

Let v_0 denote the normal volume (at 0° and 1 atm.) of a certain quantity of air. When heated or cooled to any temperature θ, and submitted to a pressure of p atmospheres, the air assumes a volume v. Admitting Boyle's and Charles's law, we should write

$$v=\frac{v_0}{p}(1+\gamma\theta), \text{ or } pv=v_0(1+\gamma\theta).$$

Now, in reality, the product pv is not independent of the pressure p, and its dependence on θ is not a linear one. Instead of the above, we must write:

$$pv=\eta \,.\, v_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$\eta(p\theta)$ denoting a certain function of p and θ, the values of which are to be calculated.

This we may do as follows:—Consider a volume v_0 of air in the normal state. Heat it, at constant pressure (=1 atm.) to $+16°$; the volume increases to

$$v_0(1+\gamma \,.\, 16).$$

At this temperature submit it to a pressure of p atmospheres, then according to the notation of § 11 we obtain the volume

$$\epsilon \,.\, \frac{v_0(1+\gamma \,.\, 16)}{p};$$

finally heat it at the constant pressure p to θ degrees. Denoting by $\alpha_{p,16}$ and $\alpha_{p,\theta}$ the coefficients of expansion corresponding to the pressure p and to the ranges 0–16 and 0–θ, we get

$$\epsilon \,.\, \frac{v_0(1+\gamma \,.\, 16)}{p} \; \frac{1+\theta \,.\, \alpha_{p,\theta}}{1+16 \,.\, \alpha_{p,16}}.$$

Now this is identical with

$$\eta \frac{v_0}{p};$$

therefore

$$\eta = \epsilon \cdot \frac{1+16 \cdot \gamma}{1+16 \cdot \alpha_{p,16}} (1+\theta \cdot \alpha_{p,\theta}) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

§ 17. *Results.*—The values of the function $\eta(p, \theta)$ might be also obtained directly from experimental data. This will be evident when we remark, that the quantity of air $=$M (expressed in normal volume-units) contained, at a temperature θ under p atmospheres, in a vessel of capacity s is

$$M = \frac{ps}{\eta}.$$

I preferred to calculate the values of η according to (7) because it is easier to obtain values of the coefficient α free from accidental errors than those of η. The following table contains values of the function $\eta(p, \theta)$ calculated in the manner indicated.

TABLE *of Compressibility of Air.*

Pressures atm.	Temperatures.									
	+100	+16	0	−35	−78·5	−103·5	−130	−135	−140	−145
1...	1·367	1·0587	1·0000	0·8716	0·7119	0·6202	0·5229	0·5046	0·4862	0·4679
10...	1·3678	1·0550	0·9951	...	...	...	...	...	...	...
15...	1·3685	1·0529	9923	...	...	...	...	...	0·4095	0·3786
20...	1·3691	1·0509	9897	...	0·6778	0·5697	0·4410	...	3808	3447
25...	1·3698	1·0488	9869	...	6689	5559	4183	...	3476	3015
30...	1·3704	1·0468	9842	...	6599	5417	3936	0·3502	3063	2444*
35...	1·3713	1·0449	9816	...	6510	5270	3650	3115	2419	
40...	1·3725	1·0433	9793	...	6423	5125	3329	2598	1128	
45 ..	1·3738	1·0419	9772	...	6335	4980	2963	1942		
50...	1·3754	1·0408	9754	0·8288	6252	4839	2544	1605		
55...	1·3770	1·0399	9738	8253	6170	4701	2171	1553		
60...	1·3784	1·0390	9723	8219	6089	4567	2013	1556		
65...	1·3802	1·0384	9710	8187	6011	4439	1985	1576		
70...	1·3821	1·0381	9701	8158	5937	4318	1989			
75...	1·3842	1·0379	9694	8132	5863	4206	2013			
80...	1·3866	1·0379	9688	8105	5796	4103	2043			
85...	1·3887	1·0380	9684	8081	5734	4014				
90...	1·3908	1·0382	9681	8058	5680	3948				
95...	1·3929	1·0386	9680	8038	5634	3903				
100...	1·3951	1·0390	9681	8023	5600	3881				
105...	1·3977	1·0397	9685	8013	5568	3874				
110...	1·4004	1·0406	9690	8006	5544	3877				
115...	1·4034	1·0418	9699	8004	5530	3892				
120...	1·4065	1·0432	9710	8006	5520	3914				
125...	...	1·0448	9722	8012	5520	3944				
130...	...	1·0467	9738	...	5528	3981				

* Corresponds to 29 atmos.

These numbers are represented in a graphical form on Pl. II. by means of so-called curves of compressibility (abscissæ = pressures, ordinates = isothermal values of $\eta = pv$; the axis of abscissæ through the point 0·1). It may be well to remark that, on assuming Boyle's law to hold for all pressures and temperatures, these curves would be straight lines parallel to the axis of abscissæ.

Every one of these curves shows a minimum of the product pv for a certain value of p (depending on the temperature of the corresponding isothermal). This expresses the fact, verified for many gases at higher temperatures, that with increasing pressure the compressibility exceeds at first that given by Boyle's law until a maximum is reached, afterwards it diminishes indefinitely. In the vicinity of the critical point the curves of compressibility run downwards very steeply; at points where liquefaction occurs their course is vertical.

§ 18. *Comparison with other Gases.*—The general properties of atmospheric air as regards expansion and compressibility are quite analogous—apart from the large difference of critical temperatures—with those of other gases which have been investigated hitherto in this respect. It is interesting to inquire whether this resemblance of properties is merely a qualitative one, or whether it is more deeply rooted; in other words, is it possible to calculate beforehand the properties of a gas, assuming the properties of another to be known?—this is the thesis of van der Waals. Wróblewski, in his memoir on the Compressibility of Hydrogen, asserts the possibility of such predictions. Take for any one gas the critical pressure for the unit of pressure; for the unit of temperature its critical temperature (absolute): then Wróblewski's theorem asserts that the dependence between the temperature and that value of the pressure for which the product pv is a minimum, is the same for all gases. A more general theorem is due to L. Natanson, namely, that all gases have a common characteristic equation, *i. e.*, a common relation (not necessarily that of van der Waals) between pressure, temperature, and volume, provided that these elements be measured by means of units specially adapted to the nature of every gas; the critical elements form one of the infinitely numerous groups of such units.

The critical elements of air are given by Olszewski as follows:—crit. pressure = 39 atm.; crit. temperature = −140° Cent. These data are also confirmed by my own experiments; I should only consider −141° as a nearer approximation to the true value of the critical temperature. A comparison of atmospheric air with other gases may be best effected on the

basis of Wróblewski's theorem. With the aid of the table of compressibility given in the preceding paragraph, we calculate the following values of the pressures for which the product pv is a minimum :—

Temperature.	Pressure.	Min. pv (approx.).
+100	Less than 10 atm.	1·367
+ 16	79	1·0379
0	95	0·9680
− 35	115	0·8004
− 78·5	123	0·5519
−103·5	106	0·3873
−130	66	0·1985
−135	57	0·1551

The points corresponding to these values of p and pv are connected on Pl. II. by a dotted curve.

Now, changing units, take the values 132° (abs. crit. temperature) and 39 atm. as new units of temperature and pressure (which will be denoted on this convention by τ and π), then the foregoing table turns into the following :—

$\tau = 2·82$	$\pi < 0·25$
„ 2·19	„ = 2·03
„ 2·07	„ 2·44
„ 1·80	„ 2·94
„ 1·48	„ 3·15
„ 1·29	„ 2·72
„ 1·08	„ 1·69
„ 1·05	„ 1·46

Draw a curve taking the τ's and π's for abscissæ and ordinates, then according to Wróblewski this should be one curve for all gases. Now it will be found that the curve plotted in accordance with the above numbers runs really very near that drawn by Wróblewski *, and based chiefly on experiments on carbon dioxide and methane. It is difficult to tell if the remaining differences are real or else depend merely on experimental errors. At all events it may be taken for granted that these coincidences, as they stand now, are most remarkable.

I have limited my experiments to the gaseous states of air, but at the same time I tried to approach to the limits of liquefaction as near as possible. It is a very difficult matter to experiment near those limits ; very constant temperatures are of great importance. It happened frequently that, in consequence of a slight variation of temperature, the glass bulb of my apparatus was found full of liquid air, instead of compressed gas. For the present time I abstained from an ex-

* Pl. iv. of the paper on Hydrogen.

ploration of the region of liquefaction; I intend to go through it on another occasion, when investigating the properties of simple gases.

In concluding I wish to express my thanks to Dr. J. Zakrzewski, to whom I owe many valuable suggestions, and who undertook a great part of the very considerable labour which was necessary to obtain the results given in the present paper.

Cracow, Physical Laboratory of the
Yagellonian University.
May 1891.

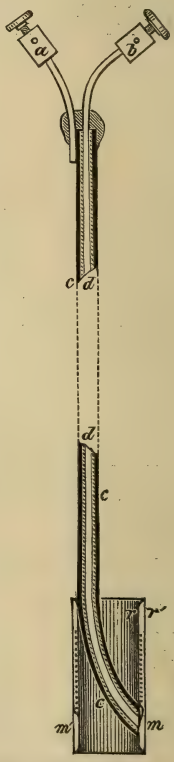
[To be continued.]

Appendix.

*Electric Thermometer for Low Temperatures**.

Variation of the electric resistance of wires depending on variation of temperature has been often employed to construct thermometric apparatus. Fig. 4 shows (nearly true size) a disposition of electric thermometer which the author has found very useful in low-temperature work, on account of its sensibility and quickness. A short cylinder r of thin sheet-copper is soldered at one end of a narrow brass tube c through which passes a rather thick silk-covered copper wire d cemented with a mixture of resin and india-rubber. The outer end of the brass tube ought to be carefully covered with this mastic in order to prevent condensation of moisture on the thermometric wire. The copper wire and the brass tube are furnished with binding-screws a, b to introduce the current. On the outer side of the cylinder r there are wound 2 or 3 metres of a very fine silk-covered platinum wire (diameter about $\frac{6}{100}$ millim.); one of its ends is soldered to the cylinder r, the other to the end of the copper wire d. To protect the coiled wire, another sheet-copper cylinder r' of somewhat greater diameter is pushed over it. Both cylinders r and r' are joined by a small quantity of solder applied round the circumference of their bases.

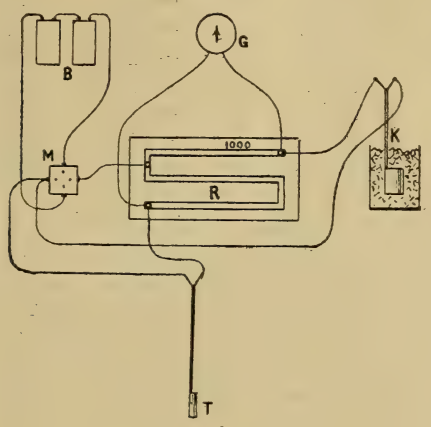
Fig. 4.



* *Bulletin internat. de l'Acad. de Sc. de Cracovie*, May 1891.

In this manner a platinum resistance is obtained of some 220 ohms at 0°. It forms one branch T (fig. 5) of a Wheatstone bridge. A second branch is formed by a resistance of

Fig. 5.



german-silver wire contained in a case K of exactly the same construction as shown on fig. 4, but on a larger scale; it is kept at the constant temperature 0° by melting ice. The third and fourth branches of the Wheatstone bridge are resistance-coils in one and the same box R, namely, a coil of 1000 ohms and a variable resistance R. B is a battery of two elements of Leclanché, M a commutator, G a sensitive galvanoscope.

To every temperature of T there corresponds a determinate resistance R in the box. By comparison with a normal hydrogen-thermometer, it is possible to construct a diagram, or a table, with the aid of which one may find by inspection the temperature T corresponding to any observed resistance R. Such a table does not cease to be true when the apparatus, after having been dismounted, is to be used again several days or months later; the relation of T and R is of course independent of the electromotive force of the battery or the sensitiveness of the galvanoscope.

A table of this kind is given below; at the same time it shows the relation of resistance and temperature (hydrogen-scale) for a certain platinum wire.

T.	R.	T.	R.
+50°	1105·9	− 90°	801·8
0	1000·0	−100	778·9
−10	978·5	−110	755·8
−20	956·9	−120	732·4
−30	935·2	−130	708·9
−40	913·4	−140	685·3
−50	891·4	−150	661·5
−60	869·3	−160	637·3
−70	847·0	−170	612·7
−80	824·5	−180	588·0

The variation of resistance is about 2 ohms per degree; therefore it is easy to obtain a sensibility of $\frac{1}{20}$° Cent.

Experience has shown that the relation between resistance and temperature undergoes slight changes when the thermometer is employed at widely different temperatures. For this reason it is better to avoid heating a thermometer destined for low temperatures; this might cause a variation of resistance which would not disappear until after several months.

Note added by the Author.

It has been found since that fine iron wire answers the purposes of low-temperature thermometry still better than platinum. Both constancy and sensitiveness are greater. The following data apply to an iron thermometer which has been now four years in use. Diameter of wire = 0·035 millim., resistance at 0° about 576 ohms.

Displacement of Zero-point.

January	1892	. . . .	R_0 = 1001·80 ohms.
March	1892	. . . .	1001·50 „
June	1892	. . . .	1001·23 „
February	1893	. . . .	1000·97 „
December	1893	. . . .	1001·10 „
March	1894	. . . .	1001·17 „
February	1896	. . . .	1001·12 „

The sensitiveness is also nearly double that of platinum-thermometers, as will be seen by the following table:—

Temp.	R.	$\frac{dR}{dT}$ (ohms per degree).
0	1001·2	4·65
− 55	761·9	4·20
− 65	720·4	4·12
− 79	663·6	4·02
−100	581·0	3·87
−130	467·3	3·66
−182	291·0	

The author prefers now a slightly different type of electric thermometer, differing from that shown on fig. 4 in this particular, that the brass tube *c* joins the *outer* cylinder r'. Although rather more difficult to construct, it allows the resistance-coil to embrace the bulb of a hydrogen-thermometer or any other piece of apparatus the temperature of which is to be determined.

XXXIV. *An Analytical Study of the Alternating Current Arc.* By J. A. Fleming, *M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London, and* J. E. Petavel*.

ALTHOUGH the physical phenomena of the Electric Arc have received of late years very considerable attention, there are many points in connexion with the study of the alternating current arc which seem to us to have been insufficiently explored. The investigation which is here described is a contribution to the further examination of the mode in which the variation of the luminous and electric effects takes place in the alternating current arc.

The first portion of the research is concerned with an analytical study of the variation of the light thrown out from different radiating regions in the arc, and the delineation of the periodic value of this illuminating power by graphical methods. The object of the first set of experiments was to record and represent by an appropriate graphical method the periodic value of the light thrown out from the carbons and from the true electric arc region when the arc is supplied with electric power of known constant amount and varying magnitude, and at the same time to record the periodic variation of current through the arc, and potential-difference of the carbons. The instrumental appliances involved in the first place the employment of means for keeping constant and

* Communicated by the Physical Society: read February 28th, 1896.

measuring the mean value of the power supplied to the arc. This was accomplished by the use of a suitable non-inductive resistance and a bifilar reflecting wattmeter, the series coil of which was in the circuit of the arc, and the shunt coil of which was connected across the carbons of the arc. The following is a description of this wattmeter:—

The series coil was wound with 10 turns of thick copper wire (No. 10 S.W.G.) and had a resistance of ·01 ohm. The shunt coil was wound with 23 turns of thin wire, and had a resistance of 3·76 ohms, and was placed in series with a non-inductive platinoid resistance of 1200 ohms. The shunt-coil was suspended by two fine silver wires, which also served to conduct the current in and out of the coil. The controlling and deflecting forces so balanced each other at small displacements as to make the angular displacement of the movable coil very nearly proportional to the power passing through the wattmeter. The movable coil was provided with a mirror by means of which the image of a wire illuminated by an auxiliary arc lamp was reflected on to a fixed scale. This scale was carefully graduated, so as to read directly in watts. It was found that the earth's magnetic field caused a small deflexion of the movable coil when using continuous currents and when the shunt current was passing through it, when at the same time no current was flowing in the series coil. This terrestrial field was neutralized by magnets, or else the scale was shifted so that the part of the deflexion of the shunt coil due to the terrestrial field was eliminated. The best way would have been to have turned the wattmeter round through a certain angle, but as it had to be screwed up against the wall in a fixed position for steadiness, it was found that the above method was the simplest plan for obviating this source of error. The vibrations of the movable coil were damped by means of a mica vane dipping into a dash-pot filled with oil. When all the adjustments were made, it was found that this wattmeter produced a deflexion of the spot of light on the scale almost exactly proportional to the power passing through the instrument. This wattmeter was then connected up to the arc lamp, so that the whole current actuating the arc lamp passed through the series coil, the terminals of the shunt coil being connected to the carbons of the arc.

A series of preliminary experiments were then made for the purpose of enabling us to eliminate from the wattmeter-readings, firstly, the power taken up in the wattmeter itself; secondly, the power taken up in the shunt coil of the arc-lamp and other shunt resistances; and, thirdly, the power

taken up in the carbons themselves, right up to the point where the arc is being formed. The observations, when reduced by applying these corrections, gave us at once the true mean power being taken up in the arc. The wattmeter was constructed with all the precautions necessary in making a wattmeter for measuring alternating-current power, so that it served to measure the power taken up either in alternating or continuous current arcs. The current supplied to the arc passed through a series of non-inductive resistances, consisting of carbon plates, in such a manner that the power given to the arc could be regulated with the greatest exactness. In addition to the wattmeter, an ammeter was placed in series with the arc lamp so as to measure the current passing through the arc, and a voltmeter connected to the carbons so as to measure the potential-difference of the carbons, these instruments being suitable both for continuous and alternating currents. In all the experiments the mean power was kept constant in the arc, and this was done by adjusting the current so that the wattmeter took a certain deflexion corresponding to the power desired, and the wattmeter was kept at a constant deflexion by regulating the carbon resistances in series with the arc. In addition to the instruments described above, a lens was fixed so as to enable the length of the arc to be measured in the usual manner.

In the course of the experiments three arc lamps were employed—a hand-regulated arc lamp in which the distance of the carbons was adjustable by a screw with great accuracy; a continuous-current arc lamp (the Waterhouse Arc Lamp); and an alternating-current arc lamp (the Helios Arc Lamp); these last two being selected as excellent arc lamps of their respective types, our object being to select an arc lamp, the mechanism of which enabled it to be worked with electric powers varying over wide limits, and yet to yield a perfectly steady arc. In addition to these instruments there was set up the apparatus for delineating the curves of current of electromotive force, which have been described by one of us in the 'Electrician,' vol. xxxiv. p. 460, and which consists of a synchronizing alternating-current motor, having its fields separately excited and its armature circuit traversed by a shunt current from the circuit operating the arc lamp. This alternating-current motor was set up on the photometer bench, and it was used to drive an aluminium disk pierced by four openings, in such a manner that the disk revolved synchronously with the alternating current operating the arc. In addition to this duty the alternating-current motor carried a

contact-maker on its shaft for the purpose of enabling the curves of current of electromotive force to be delineated. A brief description of the motor is as follows:—

It consists of two sets of field magnets M, M (see fig. 1), having eight poles on each side which are secured to two cast-iron disks. Between these field magnets revolves a small armature A, the iron core of which is formed of a very thin strip of transformer iron wound up into a ring, the armature coils being wound on this ring. The armature coils are joined up in series with one another, so as to give a series of alternating magnetic poles round the ring when a current flows through the armature circuit. The diameter of this armature is about six inches. The field magnets have eight poles, and the armature eight coils. The field magnetic coils are bobbins about 2 inches long and $1\frac{1}{2}$ inch in diameter, and when joined up in series in the proper manner the field magnets take a current of about eight amperes to give them a proper amount of saturation. The armature is carried upon a hard wooden boss fixed to a steel shaft; and the steel shaft is carried through small ball bearings made like bicycle bearings. In order to prevent any side shake of the armature, there are at opposite ends of the base cast-iron pillars with a gun-metal screw through each, against which the rounded end of the shaft bears; the position of the shaft can thus be adjusted with great nicety, and runs with great freedom from friction. The ends of the armature circuit are brought to two small insulated collars fixed on the shaft, against which press two light brass brushes marked B, B kept against the shaft by means of an expanding steel wire W. The armature shaft carries on one side an ebonite disk with a steel slip let into it. Two insulated springs S, S are carried on a rocking arm H; this rocking arm can be traversed through half a circumference, and is centred upon the gun-metal screw which prevents side shake in the shaft, and a pointer and graduated scale enables the exact position of the contact-springs to be determined. One of the stop-screws keeping the shaft from side shake is pierced with a longitudinal hole, and through this hole passes a stiff steel wire; this serves to drive an aluminium disk 27 centims. in diameter and 4 millimetres thick. This disk is carried on a shaft which runs in a cast-iron bearing, and the disk is therefore driven synchronously by the motor. This aluminium disk has four slits in it separated by angular intervals of 90°; the slits are 0·5 centimetre wide and 4·5 centimetres long. If the field-magnets of the motor are excited by a continuous current of

about eight amperes, and if an alternating current of about two amperes is passed through the armature of the motor, then, on turning the motor rapidly round by hand, which can

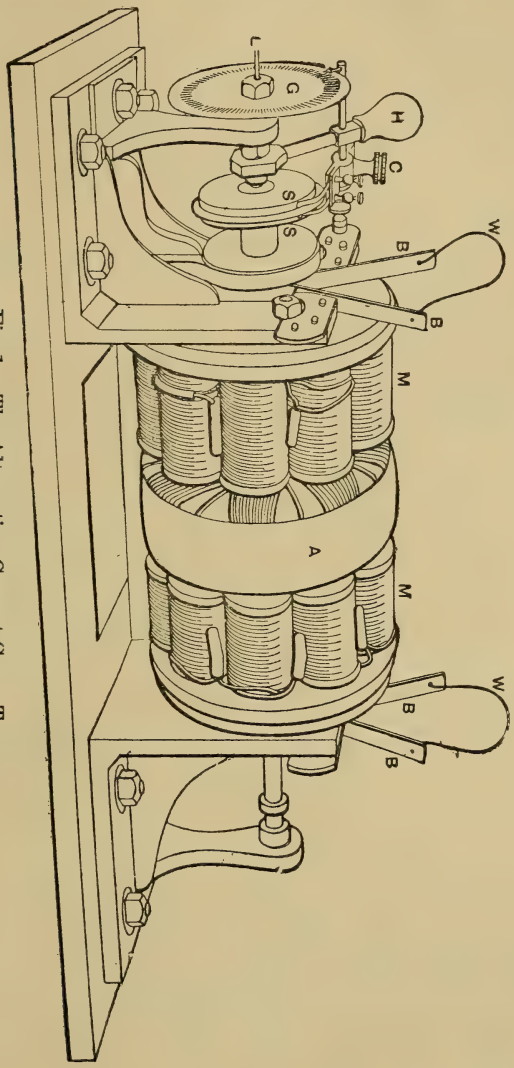


Fig. 1.—The Alternating Current Curve Tracer.
[From the 'Electrician,' vol. xxxiv. p. 460, 1895.]

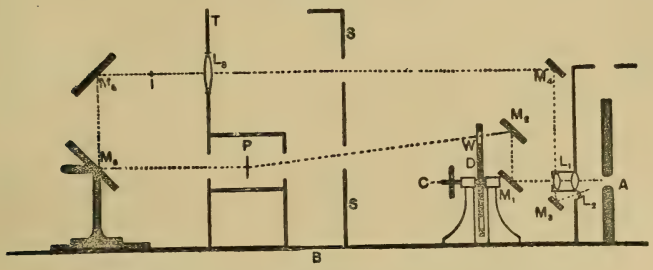
best be done by passing a strap round the shaft and pulling at the strap so as to spin the motor like a top, the motor will,

if sufficient speed be gathered, drop into step with the alternating current driving it. Since the motor has eight magnetic poles, it makes one complete revolution in four complete periodic times, so that if the motor is being driven from an alternating-current circuit having a frequency of 100, then the motor has to run at 1500 revolutions per minute before it will drop into step, but at that speed it will fall into step with the current passing through its armature, and will be driven as a synchronous motor. Under these circumstances, if a ray of light is passed transversely to the disk in such a manner as to pass through the slits of the aluminium disk during the progress of rotation of the disk, the beam of light will be interrupted, but will obtain passage four times during each revolution through the slits in the disk as it goes round.

If the motor is being driven by the same alternating current circuit which supplies the alternating current to an arc lamp, it is evident that, on looking through the slits in the revolving disk at the alternating current arc, it will be seen in one constant condition during its periodic variation, such instant being determined by the position of the slits with reference to the phase of the current. Without entering into a longer description, it will be evident that this synchronizing motor driving a disk and a contact-breaker enabled two things to be done—first, to delineate all the current and electromotive-force curves of the arc taken in the usual way; and, secondly, a ray to be taken from the arc selected at one particular instant during the complete period through which the variation of illumination passes. These arrangements were completed by the construction of a photometer of a particular kind. Owing to the slow variation of position of the electric discharge in the alternating current arc, it would have been useless to photometer the instantaneous value of the light coming from the alternating current arc against any fixed standard of light ; but it was found possible to make a very exact comparison between the intensity of the light coming from any part of the arc, and selected at any one constant instant during the complete phase, with the mean value of the light coming from that same part of the arc during the complete period ; in other words, it was found possible to photometer the arc against itself, and so eliminate to a large extent the difficulties arising from slow periodic variations of the light sent out from the arc in any one direction. It is well known that the light of an alternating current arc, taken in any one direction, undergoes a slow periodic variation quite independently of the variation of

current during the phase, neither is it dependent upon any variation of the mean square value of the current, because it takes place even when that current is perfectly constant. It appears to be due to slow changes of position of the points on the carbons between which the discharge takes place. The discharge is as it were seeking out new points between which to take place, and it continually changes these positions as the arc burns. The photometric arrangements finally adopted were as follows, and are shown in outline in fig. 2 :—

Fig. 2.



Arrangement of the Photometer and Revolving Disk.

A represents an alternating current arc. This arc was enclosed in a metal lantern in which were three openings. The light from this arc passed through a lens L_1 in a horizontal direction and fell upon a mirror M_1 placed at an angle of 45°, capable of being rotated at this constant inclination round a horizontal line co-linear with the axis of the revolving disk, and the motor placed in front of the lens L_1. The ray was then reflected upwards into another mirror M_2, and by this mirror reflected at an angle very nearly equal to 45° in such a manner as to pass through the slits in the revolving disk D, when any one of the slits was in a position to allow the ray to pass. The two mirrors M_1 and M_2 were rigidly connected to a rocking-arm so centered that the line M_1 M_2 could be rotated round into any required position, always moving parallel to a radial line of the disk D. The disk D was the disk carried on the shaft of the synchronizing motor above described ; the motor, together with its associated disk, was placed on the photometer bench in the required position opposite to the arc-light lantern. In fig. 2 the motor itself is not shown, but its position is indicated by the letter C. The angular position which the rocking-arm carrying the two mirrors M_1 and M_2 occupied with respect to the vertical line passing through the centre of the revolving disk could be

observed on a graduated scale. It will be clear, then, that if the disk driven by the motor was in synchronism with the alternating current producing the alternating arc at A, an observer, looking through holes in the rapidly revolving disk, would see by reflexion in the mirrors M_1 and M_2 the alternating current arc at A; but he would see it, not as it is seen when looked at directly, but in some constant condition taken at one definite instant during the phase, which instant would depend upon the position of the line $M_1 M_2$ with regard to the vertical line through the centre of the disk. Thus by rocking over the arm $M_1 M_2$ in various positions the observer could see, through the window W as the disk revolved, the arc at A, either in the condition when the electric discharge is taking place, or when the true arc is extinguished, according to the position in which the arm $M_1 M_2$ was set. If, instead of observing with the eye, a disk of paper with a photometric grease spot upon it was placed at P, the lens L could be so adjusted as to throw an enlarged and well-defined image of the arc upon the disk at P; and by rocking over the arm carrying the mirror into successive positions, the observer would see the image of the arc pass slowly through all those successive phases which in the arc itself actually take place during one periodic time. As the image of the arc is much larger than the photometer disk, it was possible, by slightly shifting one of the mirrors, to bring any desired part of the image of the true arc or of the craters of either of the carbons to cover the grease spot. In addition to this interrupted ray, another ray was gathered from the same part of the arc by a lens L_2, placed on the same level with the lens L_1 but slightly to one side of it. This lens gathered a beam which was reflected by a mirror M_3 placed at an angle of 45°, and which reflected the ray upwards to another mirror M_4. In fig. 2, for the sake of clearness the lens L_2 is shown beneath the lens L_1, but it must be understood that in the real apparatus the lenses L_1 and L_2 were on the same level and placed side by side. The ray reflected from the mirrors M_3 and M_4 was set horizontally, so as to be received on a lens L_3, and by this lens L_3 was gathered to a focus at a point I. The lens L_2 was so adjusted as to form a large image of the arc on the screen which carried the lens L_3, and by slightly moving the mirror M_4 any part of this image could be made to cover the lens L_3. It will be seen, therefore, that the light gathered together at a focus at the point I could be made to be light coming from any assigned area in the arc or from the craters. A movable stand carried two other mirrors M_5 and M_6 fixed at

angles of 45° in such a way as to reflect the ray coming from focus at 1, and reverse its direction so as to bring it round and make it fall on the left-hand side of the photometer-disk P. It will thus be seen that, by moving the mirrors M_5 and M_6, the light falling on the left-hand side of the photometer-disk P could be made to have any desired intensity within certain limits, and could be gathered from any desired part of the arc or craters ; and, moreover, this illumination was the mean illumination, or proportional to the mean illuminative power of any part of the arc selected for examination. It will thus be clear that the arrangement enabled us to project on to the right-hand side of the photometer disk P the rapidly intermittent ray taken from any part of the arc, and always gathered at one constant phase condition during the complete period ; whilst on the left-hand side of the photometer disk we could project a ray gathered from the same part of the arc, but not interrupted. We could therefore compare the mean value of the light proceeding from any part of the arc with the instantaneous value of the light taken from the same part of the arc and selected at any assigned instant during the period. Thus the arc itself became its own standard, and difficulties due to slow fluctuation of the mean light of the arc disappeared. At the same time the contact-maker on the motor enabled us to delineate in the usual manner the curves of current and potential-difference of the arc, and thus to record the variation in the arc of the arc current, the carbon potential-difference, the power expended in the arc, the resistance of the arc, and the luminous intensity of any part of the arc. A long series of experiments was then made with alternating current arcs of different lengths and powers, the periodic electric quantities being delineated and the light being taken, either from the centre of the true arc halfway between the carbons or from one of the craters of the carbon terminals,—generally the bottom carbon. The process of taking measurements was as follows :—

After setting the lenses and the mirrors so that the lens L_1 and the mirrors M_1 and M_2 gave a sharp image of the arc on the right-hand side of the photometer disk with the selected area of the image covering the grease spot, the mirrors M_5 and M_6 were moved backwards and forwards until the balance was obtained between the illumination falling on the right- and on the left-hand side of the photometer disk. The right-hand side of the photometer disk being illuminated by an intermittent stream of light always selected in the same phase, when considered as belonging to a periodically varying illu-

minating beam, whilst the light falling on the left-hand side of the photometer disk is a uniformly illuminating beam of the same quality and colour coming from the same part of the arc but not interrupted, and representing therefore the mean value of the light emitted from that selected area of the arc. Observers were deputed to measure all the various quantities by the different instruments, and a power of constant definite amount was supplied to the electric arc from an alternating-current machine driven by a continuous-current motor. By means of the carbon resistance and the reflecting wattmeter this power was kept constant at a selected value for a sufficient time to enable all the various periodic quantities to be observed at sufficiently frequent intervals during the phase. In all cases the arc was allowed to burn quietly for half an hour to get the carbons into a constant position before any observations were taken. It is hardly necessary to go into the details of delineating the current and electromotive-force curves, as the process of doing this is now well understood. The Kelvin vertical multicellular voltmeter, having a half microfarad condenser placed across its terminals, was employed for the measurement of the potential difference of the carbons in the following manner:—

The voltmeter, with its associated condenser, was connected, through the contact-maker driven by the shaft of the alternating motor, to the carbons of the arc; the contact-maker thus closed the circuit at a certain instant during the phase, and the voltmeter gave the instantaneous value of the potential-difference of the carbons. In series with the arc was placed a non-inductive resistance of suitable magnitude. A switch was arranged so that the voltmeter with the condenser in parallel with it, both being in series with the revolving contact-maker, could be put across either the terminals of this non-inductive resistance, or else between the carbons of the arc. By rocking over the arm carrying the spring-brushes of the contact-breaker, the voltmeter circuit was closed at a particular instant during the phase, and the voltmeter reading gave therefore, when corrected, the instantaneous value either of the potential-difference of the carbons or the instantaneous value of the current through the arc. As the Kelvin multicellular voltmeter used by us only begins to read at 60 volts, in order to get readings for lower values than 60 volts, it is necessary to add a known electromotive force to the voltmeter circuit in order to block up the needle of the voltmeter to a false zero. This was done by connecting a known number of small Lithanode secondary batteries, the

potential of which was determined by the same voltmeter, in series with the voltmeter. In this way the series of observations were successively taken of the following quantities:—First, the instantaneous value of the potential difference of the carbons taken at equidistant intervals throughout a complete period; secondly, the instantaneous values of the current through the arc taken throughout the complete period; and, thirdly, the instantaneous values of the luminous intensity of a certain selected portion of the arc taken at intervals throughout the complete period and expressed in terms of the true mean luminous intensity of the same portion of the arc at that time. These quantities having been obtained, it was then possible to plot them down in a series of curves, and to deduce therefrom curves representing the periodic variation of power through the arc, and the periodic variation of the resistance of the arc. Five sets of experiments were made, taken for different frequencies and different lengths of arc, each set comprising an observation taken with the light proceeding from the centre of the true arc, and also an observation taken with the light proceeding from the centre of the crater of the lower carbon. The frequencies employed were 83, 50, and 26. In all cases the current was kept at 14 amperes (mean square value); the results of these observations are embodied in the following 10 tables arranged in 5 pairs. Table I. A, for instance, gives the results of the observations taken with an arc having a frequency of 83·3, the light being taken from the centre of the true arc. Table I. B gives similar results of observations for the same arc, the light being taken from the centre of the lower crater. By holding a magnet at the back of the arc, noticing which way the instantaneous image of the true arc was projected by the magnet, and noting the pole of the magnet presented to the arc, it was possible to determine when the lower carbon was positive and when it was negative; and the diagrams corresponding to the above 10 tables are marked so as to show the half of the wave when the lower carbon is positive and when it is negative, in all those diagrams which refer to the light coming from the crater. The results of the observations given in the 10 tables are delineated graphically in diagram in figs. 3 to 12; and on referring to these diagrams a periodic line will be seen in each, delineating the variation of the potential difference of the carbons, and another periodic line indicating the variation of the current through the arc, whilst a third line indicates the periodic variation of the luminous intensity of the selected

portion of the arc. In the diagrams figs. 13 to 17 are given curves representing the periodic variation of the light from the crater in the various cases, and extending over several periods. It will be noticed that those diagrams representing the periodic variation of the light from the arc show that this light undergoes a regular fluctuation between a maximum and a minimum, the maxima having equal values. The light in the centre of the true arc never falls quite to zero. This seems to be due to a little luminosity which hangs in the interspace between the carbons, but at the present moment it is difficult to say whether this persistence is due to a very small admixture of stray light (although every effort was made to keep this out), or to a persistence of the illuminating-power of the incandescent vapour in which the arc has been formed. On examining the true arc during its complete periodic variation, it is found that the blue or purple strip of light forming the true arc undergoes a periodic variation in intensity. As far as the eye can judge, the blue or purple light completely vanishes at a certain instant during the phase; but there is, outside the true arc, a dim halo of golden light which is persistent; and it is therefore probably on account of this persistent aureole of faint light round the true arc that the ordinates of the curve representing the periodic variation of the luminous intensity of the arc never become zero, but always indicate the outstanding constant amount of light. On the other hand, in the diagrams which represent the periodic variation of light coming from the centre of the crater of the lower carbon, we find the luminous intensity of the crater varies between a minimum value and two maximum values of different magnitude. During the time when the crater is positive it reaches a higher maximum intensity of illuminating power than during the time when it is negative, and, moreover, the curve representing the periodic variation of light rises more steeply than it comes down, which indicates a slow cooling of the carbons after they have been heated; in other words, they heat more quickly than they cool. This is particularly noticeable in that part of curve corresponding to the crater being negative; and it is only what would be expected, because after the carbon has reached its negative maximum and is beginning to cool, the opposite carbon is cooling from a condition in which it has been positive, and as it has been heated to a higher temperature than the negative carbon, it must assist by its radiation in keeping up the temperature and retard the cooling of the negative carbon. These diagrams will also show many other

interesting facts : they show, for instance, that in the case of the long arc the self-induction of the arc is more marked than when the short arc is employed, and in that case there is a very distinct lag of current behind the potential-difference.

It may also be noted in comparing the diagrams V. B and IV. B in figs. 12 and 10, which represent the curves for arcs of the same frequency but of lengths 1·2 centim. and ·32 centim. respectively, that in the case of the long arc there is no lag of light behind power as far as regards the points of maximum when the carbon is positive, but in the case of the short arc there is a sensible lag. This might be expected to be the case, because for a long arc the influence of the opposite carbon in keeping up the temperature of its neighbour as this last is cooling is less felt than in the case of a short arc. Broadly speaking, the facts may be summed up as follows :—The purple light of the true arc undergoes a periodic variation, and, as far as the eye can judge, is completely extinguished during a certain interval during the phase ; it has equal maxima values during the period, at instants slightly lagging behind the instants of maximum power-expenditure in the arc. On the other hand, the illuminating-power of the carbon crater varies between a minimum value and two maxima of unequal values ; the greatest maximum occurring when the carbon is positive and at an instant slightly lagging behind the instant of maximum power-expenditure in the arc. A series of curves are also given (see figs. 18, 19, and 20) in which the periodic variation of the current, potential-difference, power, and apparent resistance of the arc for various powers and frequencies are represented ; and it will be seen from these curves that the resistance of the arc, including in this any counter electromotive force which may exist, varies periodically, the resistance being a minimum when the current is a maximum, and *vice versa.*

Table I. A.

Observations on the Periodic Variation of the Intensity of the Light of an Alternating Current Arc.

Light from centre of Arc. Short Arc.

Frequency . . . . . = 83·3 ~
Length of arc . . . . . = 0·55 cm.
Potential-difference (P.D.) of carbons (mean square value) . . . . . } = 39 volts.
Current (mean square value) . . . . = 14 amperes.
Power expended in arc . . . . . = 546 watts.
Pressure at alternator terminals . . . = 104·5 volts.

Angle of Phase.	Current through Arc (instantaneous value).	P.D. of Carbons (instantaneous value).	Angle of Phase.	Intensity of Light from Arc.
0			26	38
20			38	23
40	+ 5·6	+20·2	58	16
60	+ 0·7	+ 0·2	76	6
80	− 3·7	−24·2	80	16
100	− 9·7	−34·5	102	24
120	−16·2	−38·0	132	47
140	−20·3	−43·0	146	60
160	−21·0	−44·8	156	56
180	−18·4	−43·5	168	68
200	−13·0	−32·5	176	57
220	− 6·2	−19·2	196	48
240	− 0·4	+ 1·9	202	31
260	+ 4·2	+25·5	242	11
280	+10·2	+38·1	252	19
300	+16·3	+40·8	274	12
320	+20·0	+46·5	304	50
340	+20·4	+48 3	320	58
360	+18·4	+50·5	326	55
			340	65
			346	63
			356	68

TABLE I. B.

Observations on the Periodic Variation of the Intensity of the Light of an Alternating Current Arc.

Light from Crater. Short Arc.

Frequency . . . . . = 83·3∼.
Length of arc . . . . . = 0·42 cm.
Potential-difference (P.D.) of carbons (mean square value) . . . . . } = 36 volts.
Current (mean square value) . . . . = 14 amperes.
Power expended in arc . . . . . = 504 watts.
Pressure at alternator terminals . . . = 98 volts.

Angle of Phase.	Current through Arc (instantaneous value).	P.D. of Carbons (instantaneous value).	Angle of Phase.	Intensity of Light from Crater of lower Carbon.
0	+16·5	+42	12	60
20	+10·2	+25·3	33	34·5
40	+ 3·1	+10·2	53	28
60	− 1·1	−14·2	74	31
80	− 5·4	−27·7	95	24
100	−13·4	−32·4	116	44
120	−18·7	−38·4	137	71
140	−20·4	−48·9	157	101
160	−19·8	−52·3	178	148
180	−15	−48·4	200	91
200	− 8·8	−31	205	93
220	− 3·3	−11	226	61
240	+ 1·5	+11·4	247	42
260	+ 6·3	+26·4	268	31·5
280	+13·0	+34·9	289	41
300	+18·7	+39·4	309	56
320	+21·5	+44·4	330	65
340	+21·2	+45·8	351	55
360	+16·5		119	163
160	−19·8	−49		
140	−20·3	−48		

Table II. A.

Observations on the Periodic Variation of the Intensity of the Light of an Alternating Current Arc.

Light from centre of Arc. Medium Arc.

Frequency . . . . . = 50 ∼.
Length of arc . . . . . = 0·63 cm.
Potential-difference (P.D.) of carbons (mean square value) . . . . . = 39 volts.
Current (mean square value) . . . . = 14 amperes.
Power expended in arc . . . . . = 546 watts.
Pressure at alternator terminals . . . = 99 volts.

Angle of Phase.	Current through Arc (instantaneous value.)	P.D. of Carbons (instantaneous value).	Angle of Phase.	Intensity of Light from centre of Arc.	
0	+19·8	+44	10	44	
20	+14·1	+38·5	32	13	Arc out.
40	+ 6·3	+27	55	11	" "
60	+ 2·7	+10·5	77	18	
80	− 2·2	−16	100	17	
100	− 6·3	−36	122	45	
120	−13·5	−37·5	144	74	
140	−18·3	−47	165	68	
160	−19·8	−54	187	52·5	
180	−19·5	−54	208	32	
200	−14·8	−50	230	8·5	
220	− 7·9	−34	252	9	
240	− 1·9	−11·2	273	11	Arc out.
260	+ 1·9	+13	294	26	" "
280	+ 7·2	+31	315	(71)?	
300	+12·6	+36	336	68	
320	+19·5	+40	356	42·5	
340	+21·7	+47·3	348	68	
360	+19·8	+46·3	327	45	

N.B.—Owing to an accidental shift of the brushes, in this table the angle corresponding to the instantaneous light is not the same as that corresponding to the current. The position of the curve of instantaneous light with regard to the E.M.F., current, and power curves cannot therefore in this case be determined.

TABLE II. B.

Observations on the Periodic Variation of the Intensity of the Light of an Alternating Current Arc.

Light from Crater. Medium Arc.

Frequency . . . . . = 50 ∼.
Length of arc . . . . . = 0·63 cm.
Potential-difference of carbons (mean square value) . . . . . = 38 volts.
Current (mean square value) . . . . = 14 amperes.
Power expended in arc . . . . . = 532 watts.
Pressure at alternator terminals . . . = 96 volts.

Angle of Phase.	Current through Arc (instantaneous value).	P.D. of Carbons (instantaneous value).	Angle of Phase.	Intensity of Light from Crater of lower Carbon.
			0	56
			20	43
30	+ 6·3	+17	40	45
50	− 0·5	− 2·5	60	32
70	− 3·9	−22	80	39
			100	52
			120	137
			140	277
			160	277
			180	227
190	−11·3		200	165
210	− 5·5	−22	220	63
230	+ 0·1	− 1	240	64
250	+ 4·2	+16	260	61
270	+11	+28·4	280	78
			300	85
			320	72
			340	87
			360	(37)?
			340	69
			320	75

Table III. A.

Observations on the Periodic Variation of the Intensity of the Light of an Alternating Current Arc.

Light from centre of Arc. Medium Arc.

Frequency	= 26 ∼.
Length of arc	= 0·63 cm.
Potential-difference (P.D.) of carbons (mean square value)	= 37 volts.
Current (mean square value)	= 14 amperes.
Power expended in arc	= 518 watts.
Pressure at alternator terminals . . .	= 62 volts.

Angle of Phase.	Current through Arc (instantaneous value).	P.D. of Carbons (instantaneous value).	Angle of Phase.	Intensity of Light from Centre of Arc.	
0	+19	+48	7	45	
20	+12·5	+39·5	28	26	
40	+ 5·8	+21·5	49	6	Arc out.
60	+ 0·5	+ 2·5	71	12	" "
80	− 2·2	−22·5	92	7	" "
100	− 9·7	−24	114	14	Arc just starting.
120	−16·8	−34·5	136	67	
140	−19·5	−45·5	158	71	
160	−21·8	−47·5	180	66	
180	−19·3	−50	202	24	
200	−11·5	−42	224	19·5	Arc out.
220	− 4·8	−22	246	4	" "
240	− 0·5	− 2	268	8	" "
260	+ 2·0	+17	290	22	
280	+10·8	+26	312	40	
300	+15·8	+36	333	57	
320	+20·7	+47·5	354	75	
340	+21·3	+50	15	53	
360	+19	+48			
340	+21	+51	345	62	
320	+20	+48	323	70	

Table III. B.

Observations on the Periodic Variation of the Intensity of the Light of an Alternating Current Arc.

Light from Crater. Medium Arc.

Frequency . . . . . = 26 ∼.
Length of arc . . . . . = 0·63 cm.
Potential-difference (P.D.) of carbons (mean square value) . . . . . = 39 volts.
Current (mean square value) . . . = 14 amperes.
Power expended in arc . . . . . = 546 watts.
Pressure at alternator terminals . . = 62 volts.

Angle of Phase.	Current through Arc (instantaneous value).	P.D. of Carbons (instantaneous value.)	Intensity of Light from Crater of lower Carbon.
0	+18	+52	64
20	+14·3	+55	77
40	+ 6·7	+24·5	46
60	+ 1	+ 4·5	42
80	− 1·2	−21	43
100	− 5	−27	52
120	−17·5	−37	64
140	−18·2	−47	145
160	−20·6	−52	285
180	−20	−54	310
200	−11	−47	260
220	− 5	−25·5	200
240	− 1	− 5·5	119
260	+ 2	+15	87
280	+ 8·5	+30	50
300	+14	+40	104
320	+19	+45	110
340	+22	+53	130
360	+21		120

Table IV. A.

Observations on the Periodic Variation of the Intensity of the Light of an Alternating Current Arc.

Light from centre of Arc. Short Arc.

Frequency	= 50 ~
Length of arc	= 0·32 cm.
Potential-difference (P.D.) of carbons (mean square value)	= 36 volts.
Current (mean square value) . . .	= 14 amperes.
Power expended in arc	= 504 watts.
Pressure at alternator terminals . .	= 85 volts.

Angle of Phase.	Current through Arc (instantaneous value).	P.D. of Carbons (instantaneous value).	Angle of Phase.	Intensity of Light from Centre of Arc.	
0	+18·4	+41·4	12	28	
20	+13·8	+38·4	34	15	
40	+ 5·7	+21	56	24	Arc going out.
60	+ 0·6	− 8·6	78	13	Arc out.
80	− 6	−22	100	14	Arc out.
100	−13·1	−25	122	18	
120	−19	−43	144	28	
140	−19	−46·5	166	20	
160	−18·3	−49	188	26	
180	−14·4	−46	210	17	
200	−10·2	−44·6	232	15	Arc going out.
220	− 5·0	−20	253	6	Arc out.
240	+ 1·1	+ 6·8	274	4	Arc out.
260	+ 8·6	+22·6	296	16	
280	+14·5	+29	317	26	
300	+18·9	+37·6	338	23	
320	+22·1	+40·6	360	30	
340	+20·3	+44	350	29	
360	+18·1	+42·6	328	23	
340	+21·5	+45·4			
320	+21·1	+40·4			

Table IV. B.

Observations on the Periodic Variation of the Intensity of the Light of an Alternating Current Arc.

Light from Crater. Short Arc.

Frequency . . . . . = 50 ~
Length of arc . . . . . = 0·32 cm.
Potential-difference of carbons (mean square value) . . . . . } = 38 volts.
Current (mean square value) . . . = 14 amperes.
Power expended in arc . . . . . = 532 watts.
Pressure at alternator terminals . . = 93 volts.

Angle of Phase.	Current through Arc (instantaneous value).	Angle of Phase.	Intensity of Light from Crater of lower Carbon.
0			
20	+13·1	20	61
40	+ 7·2	41	56
60	+ 0·8	63	45
80	− 5·0	84	58
100	.....	105	51
120	.....	126	101
140	.....	148	220
160	.....	170	227
180	.....	191	216
200	.....	212	111
220	− 6·3	232	64
240	− 0·4	253	73
260	+ 4·6	273	66
280	+12·6	294	63
300	+17·6	315	77
320	+19·0	335	60
340	.....	359	63
360	.....	338	78
		317	88
		36	79

Table V. A.

Observations on the Periodic Variation of the Intensity of the Light of an Alternating Current Arc.

Light from centre of Arc. Long Arc.

Frequency . . . . . = 50 ~.
Length of arc . . . . . = 1·2 cm.
Potential-difference (P.D.) of carbons (mean square value) . . . . . } = 53 volts.
Current (mean square value) . . . = 14 amperes.
Power expended in arc . . . . . = 742 watts.
Pressure at alternator terminals . . . = 88 volts.

Angle of Phase.	Current through Arc (instantaneous value.)	P.D. of Carbons (instantaneous value.)	Angle of Phase.	Intensity of Light from Centre of Arc.	
0	+20·4	+58·6	10	40	
20	+15·9	+48·6	31	44	
40	+ 8·4	+26·4	52	9	
60	+ 0·7	− 7·6	76	12	Arc out.
80	− 3·6	−46	98	19	„ „
100	− 8·1	−53·6	120	24	
120	−15·1	−63·6	142	72	
140	−18·6	−67·6	164	50	
160	−19·6	−68	187	63	
180	−17·6	−67	209	44	
200	−12·6	−53·6	232	12	Arc out.
220	− 5·6	−21	253	7	„ „
240	− 0·6	+24	275	7	„ „
260	+ 4·4	+47	297	34	
280	+ 8·4	+53	320	62	
300	+17·4	+55	343	87	
320	+21·4	+59·6	348	63	
340	+23·4	+59·6	326	78	
360	+20·4	+55·6			
340	+22·4	+59·6			
320	+21·4	+58·6			

Table V. B.

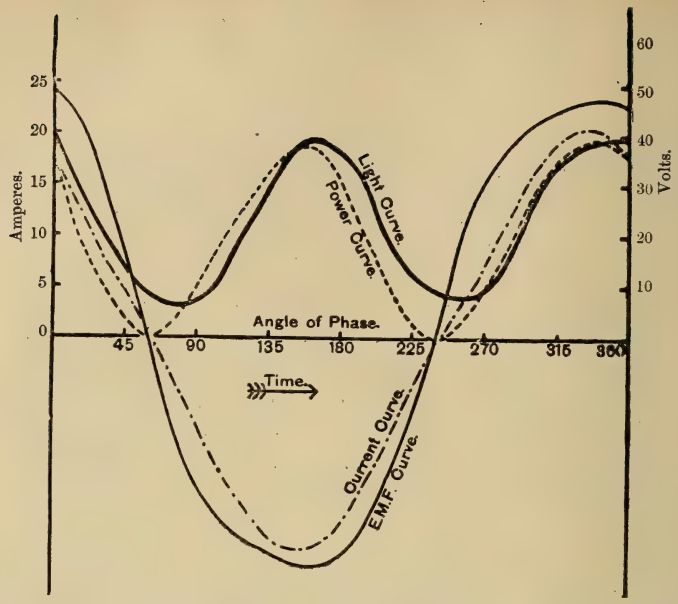
Observations on the Periodic Variation of the Intensity of the Light of an Alternating Current Arc.

Light from Crater. Long Arc.

Frequency . . . . . = 50 ∼.
Length of arc . . . . . = 1·2 cm.
Potential-difference (P.E.) of carbons (mean square value) . . . . . = 53 volts.
Current (mean square value) . . . = 14 amperes.
Power expended in arc . . . . . 742 watts.
Pressure at alternator terminals . . = 93 volts.

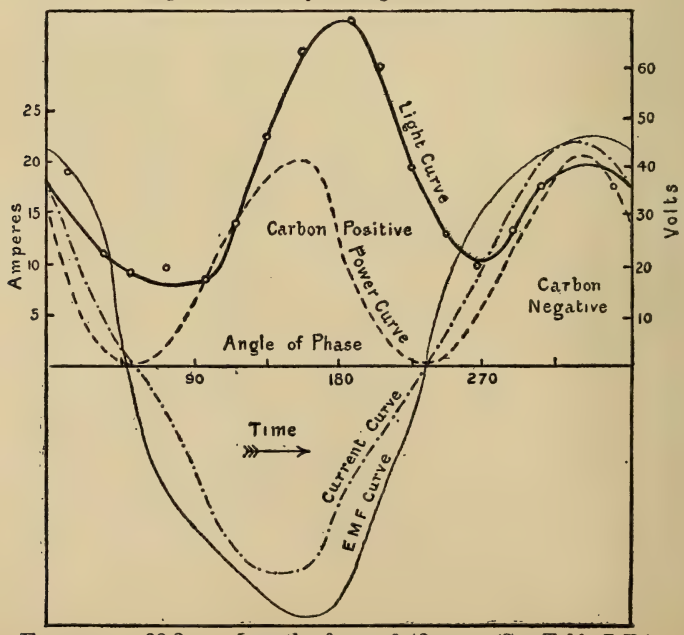
Angle of Phase.	Current through Arc (instantaneous value).	Angle of Phase.	Intensity of Light from Crater of lower Carbon.
0		359	92
20	+15·9	18	92
40	+11·7	36	98
60	+ 1·6	53	84
80	− 4·4	70	70
100		87	76
120		105	74
140		123	207
160		142	175
180		160	280
200		179	265
220	− 6·5	199	193
240	0·0	218	76
260	+ 4·4	238	84
280	+10·4	257	71
300		277	66
320		296	101
340		315	98
360		334	107
		340	120

Fig. 3.



The ordinates of the firm line marked "Light Curve" represent the periodic variation of illuminating-power of the centre of the *true arc* of an alternating-current arc. The other curves marked E.M.F., Current, and Power represent the variation of potential difference of the carbons, current, and power in the arc. Carbons, 15 mm. cored; $\sim = 83{\cdot}3$; amperes=14; volts=39; length of arc, 0·55 cm.; volts at alternator terminals=104. (See Table I. A.)

Fig. 4.—Intensity of Light from Crater.



Frequency=83·3 ~. Length of arc=0·42 cm. (See Table I. B.)

Fig. 5.—Intensity of Light from Arc.

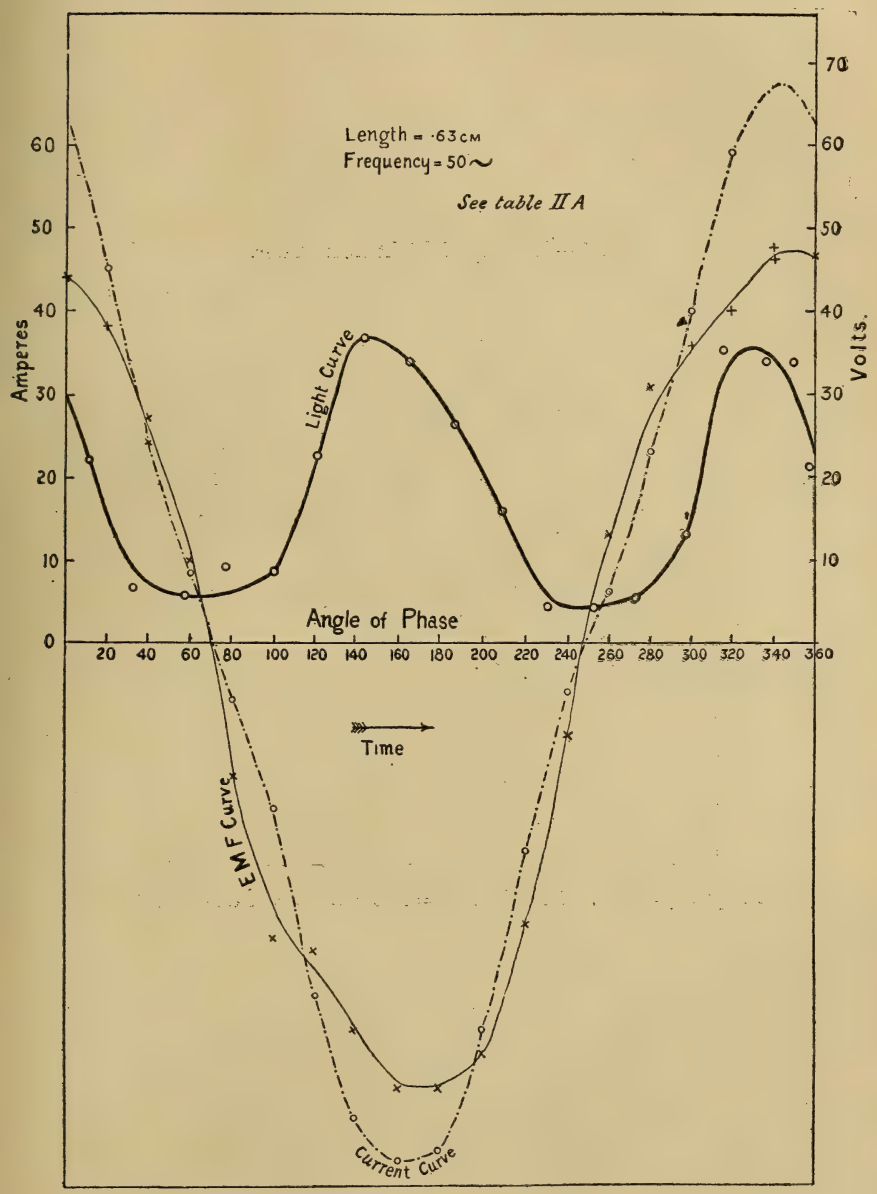
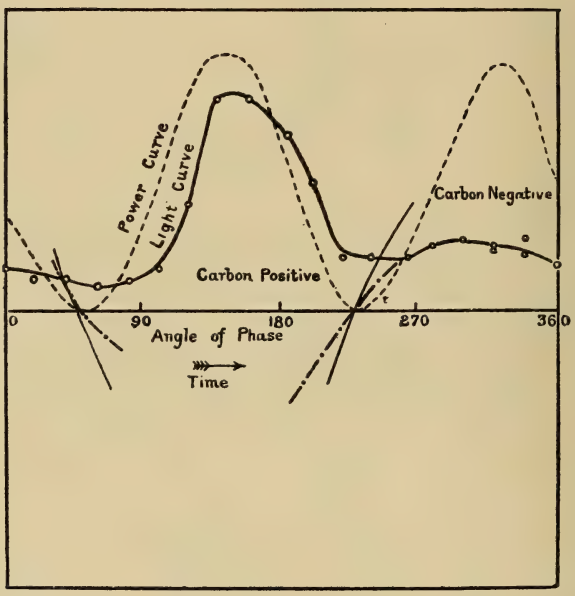


Fig. 6.—Intensity of Light from Crater.



Frequency = 50 ~. Length of arc = 0·63 cm. (See Table II. B.)

Fig. 7.—Intensity of Light from Centre of Arc.

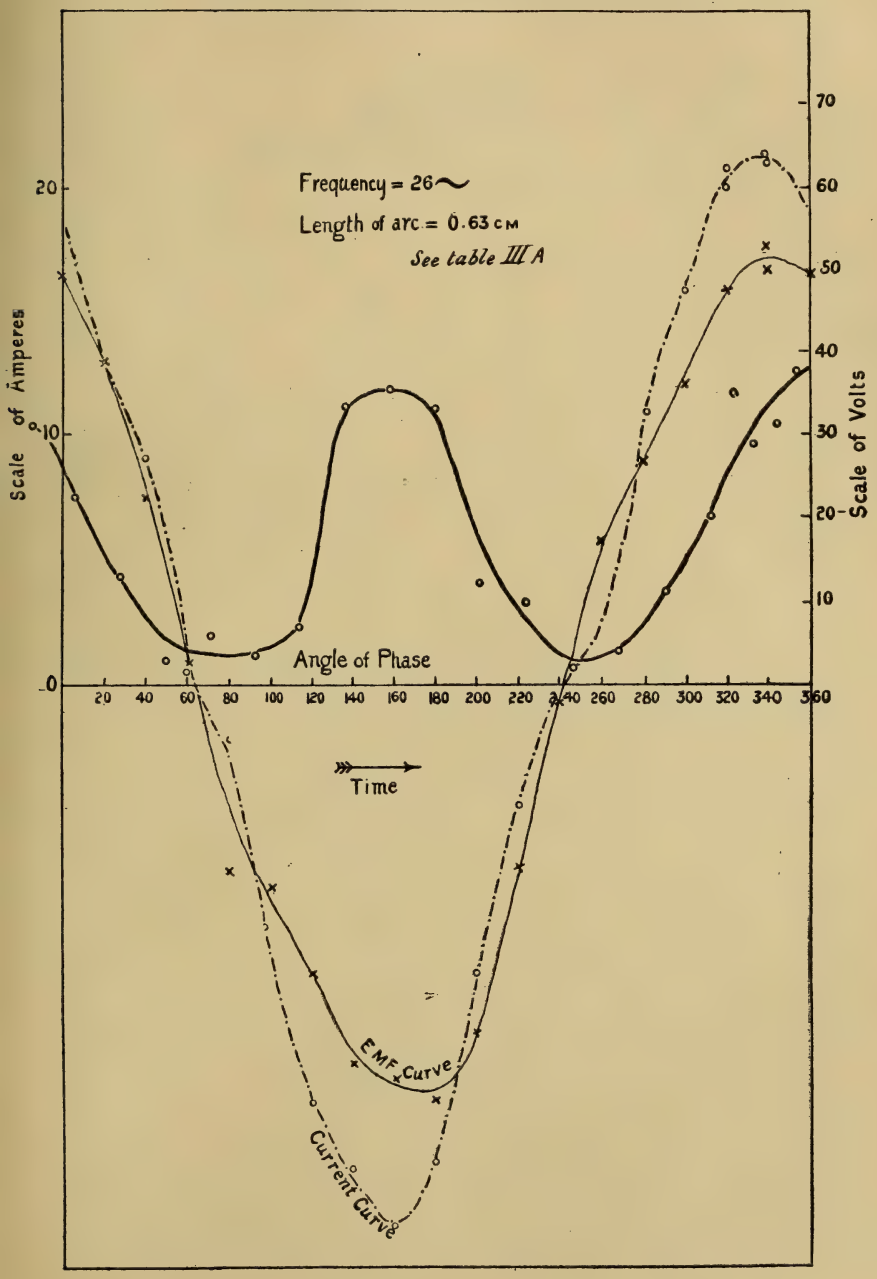


Fig. 8.—Intensity of Light from Lower Carbon.

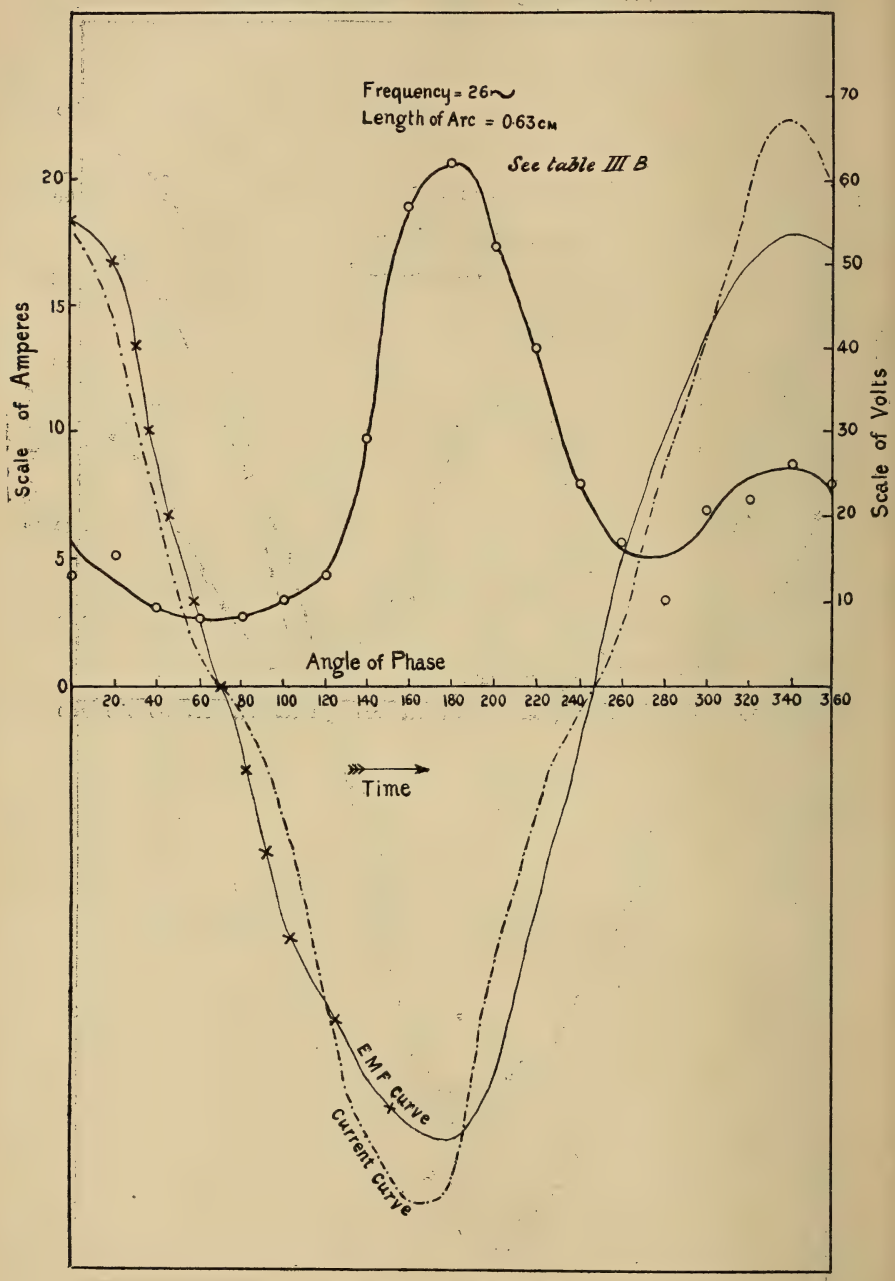


Fig. 9.—Intensity of Light from Centre of Arc.

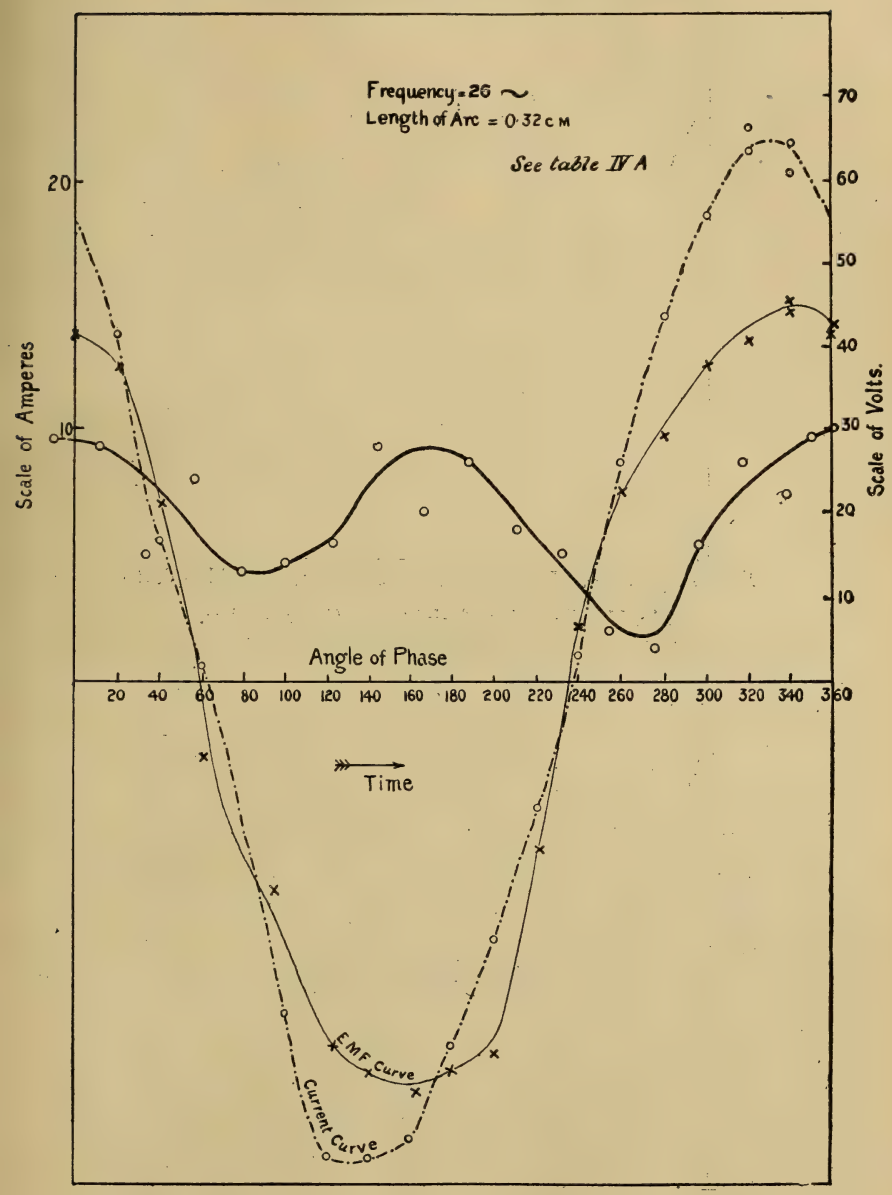
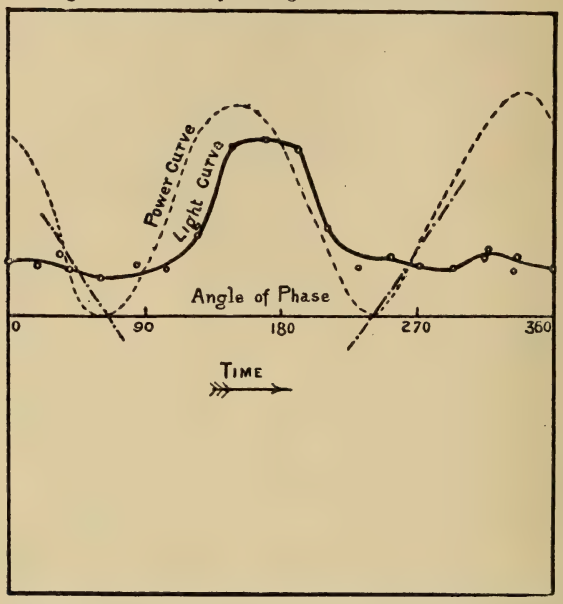
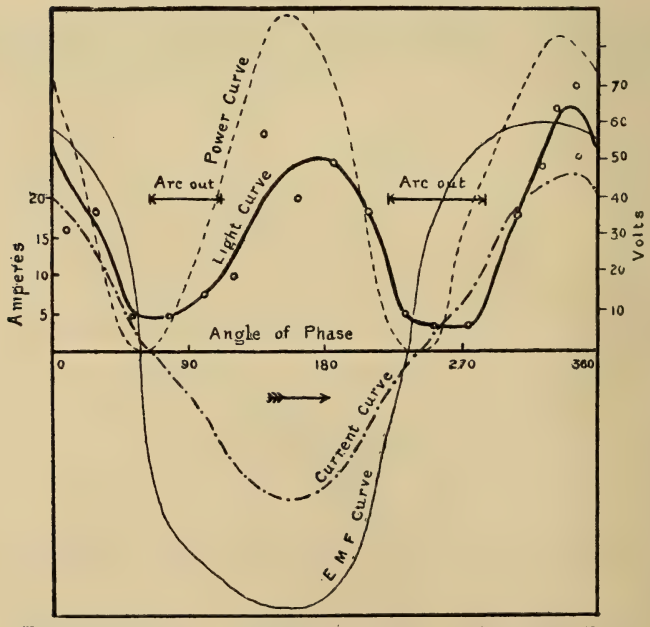


Fig. 10.—Intensity of Light from Centre of Arc.



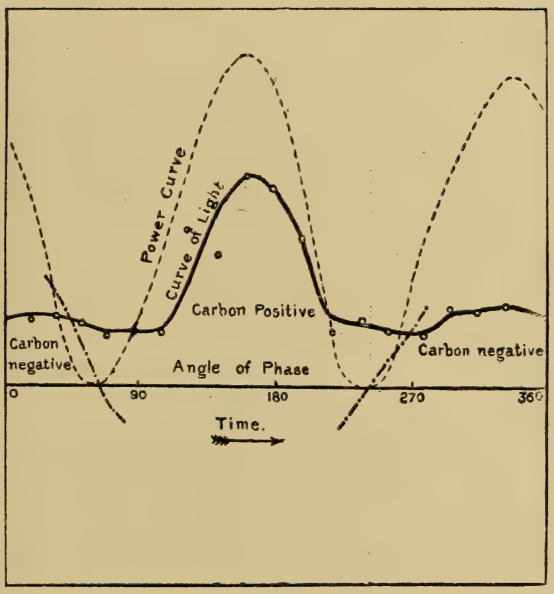
Frequency=50 ~. Length of arc=0·32 cm. (See Table IV. B.)

Fig. 11.—Intensity of Light from Centre of Arc.



Frequency=50 ~. Length of arc=1·2 cm. (See Table V. A.)

Fig. 12.—Intensity of Light from Crater.



Frequency = 50 ~. Length of arc = 1·2 cm. (See Table V. B.)

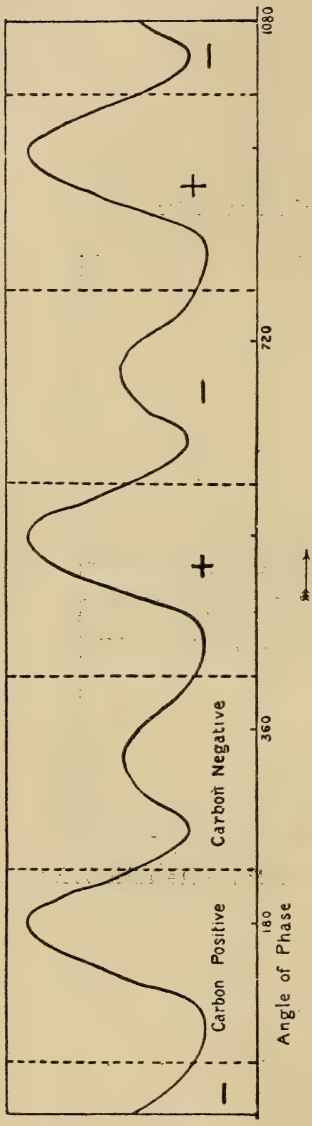


Fig. 13.—Variation of Light from Crater of Alternating Current Arc.

Frequency = 83·3. Length of Arc = 0·42 centim. (Table I. B.)

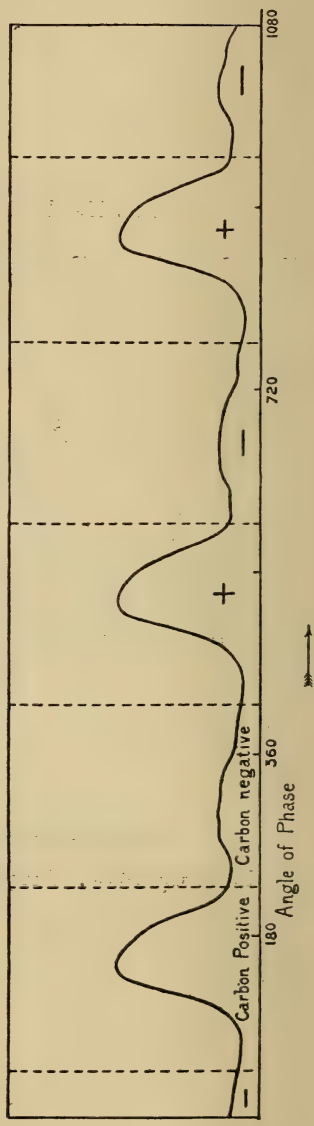


Fig. 14.—Variation of Light from Crater of Alternating Current Arc.

Frquency = 50. Length of Arc 0·63 centim. (Table II. B.)

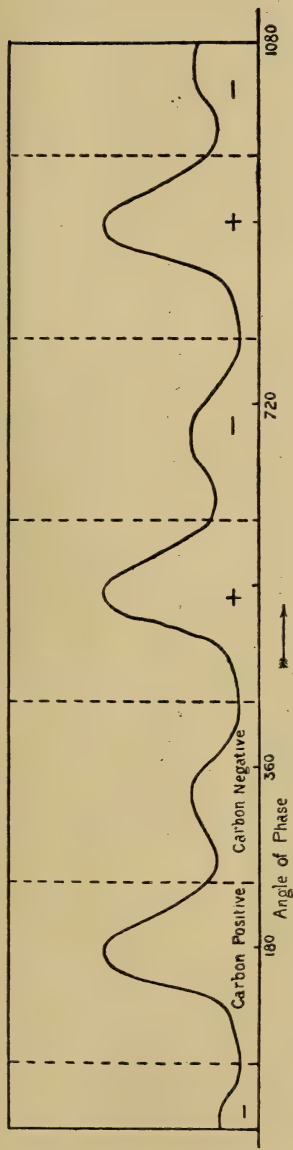


Fig. 15.—Variation of Light from Crater of Alternating Current Arc.

Frequency = 26. Length of Arc = 0·63. (Table III. B.)

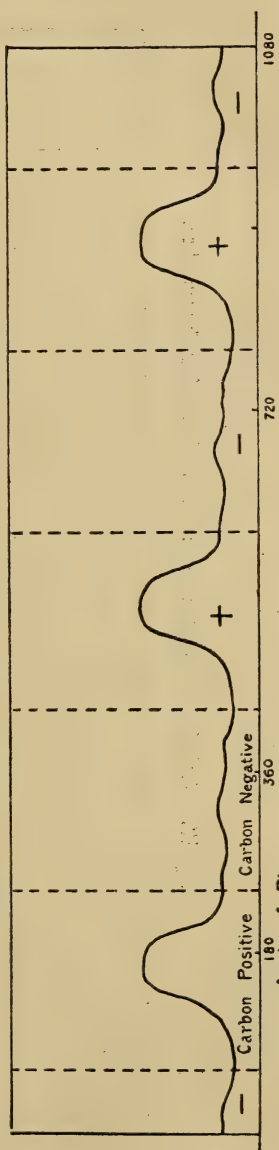


Fig. 16.—Variation of Light from Crater of Alternating Current Arc.

Frequency = 50. Length of Arc = 0·32 centim. (Table IV. B.)

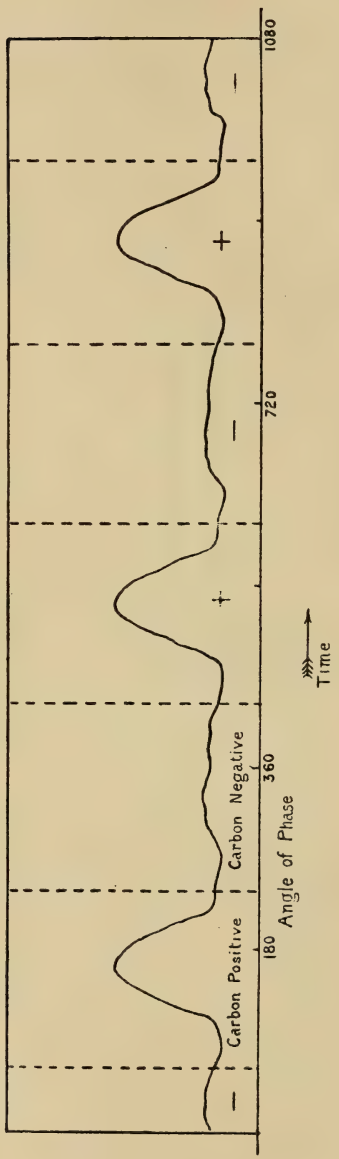


Fig. 17.—Variation of Light from Crater of Alternating Current Arc.

Frequency = 50. Length of Arc 1·2 centim. (Table V. B.)

The second portion of the experimental work here recorded consisted in an investigation of the efficiency of the alternating-current arc as a light-giving agent, when compared with the continuous-current arc taking up the same mean power. The question in all its generality, whether alternating-current arcs are less efficient as light-giving agents for a given amount of power expended in them than continuous-current arcs absorbing the same power, is not one which we have attempted to settle, for the simple reason that a far more extended series of experiments would be necessary before it would be possible to say whether an alternating-current arc can or cannot be made to give the same mean spherical candle-power as a continuous-current arc absorbing the same mean power. Limitations of time compelled us to reduce our investigation to one definite problem. Taking alternating-current arcs, formed with the length, voltage, and the current and the carbons as described below, and taking continuous-current arcs also with the length, current, and voltage and carbons most usually employed in practice, we have investigated the relative magnitude of the mean spherical candle-power produced by these arcs for equal expenditure of powers in the arcs varying from 200 to 600 watts. This, it will be observed, is an investigation which has nothing whatever to do with the relative mechanical or electrical efficiency or excellence of the arc-lamp mechanisms, but it is purely a physical measurement made of the two arcs themselves under the conditions which are found to obtain in practice. In order to settle this question, an alternating-current arc lamp was taken with good regulating qualities; it was furnished with cored carbons 15 mm. in diameter, and being placed under the conditions above described, in which the power supplied to the arc could be regulated and measured, a series of observations was made of the mean spherical candle-power of the arc for different powers expended in the arc and for two frequencies of 83·3 and 50 $\sim$. A similar set of experiments was made with the continuous-current arc, using in one case the same carbons—15 mm. carbons, both cored—and in the other case the positive carbon 15 mm. cored and a negative carbon 9 mm. solid. These sizes were chosen because the 15 mm. cored and 9 mm. solid sizes are those that are frequently employed in continuous-current arc lighting. The arrangements for obtaining the mean spherical candle-power consisted of a mirror—the coefficient of reflexion of which had been determined—which was employed to reflect the light from the arc proceeding in different directions to the horizon into the photometer, the ray always falling upon the mirror at an incidence of 45 . The standard of comparison

was an incandescent lamp, which was worked at rather a high temperature in order to diminish the difficulty of making the photometric comparison by diminishing as much as possible the colour-difference in the lights compared. This incandescent lamp was kept constantly standardized against another standard incandescent lamp worked at a rather lower temperature. The means of comparison was a Sugg's Star Disk Photometer. It was found that by focussing the eye to a point rather nearer the eye than the images of the Star Disk as seen in the two mirrors, the difficulty of discriminating between a small difference in the brightness of the two images in spite of a small colour-difference was to a great extent diminished. We abandoned as perfectly useless any comparison of the two lights in terms of red and green candle-power. By the employment of the reflecting wattmeter and carbon resistance as above described, it was perfectly possible to keep the power expended in the arc constant to a certain number of watts throughout long periods. In the case of the alternating-current arc lamp experiments, the alternating-current arc lamp was turned round its vertical axis so as to take a series of observations quickly in different directions, but at the same angle to the horizon, and the mean of these observations was taken as the effective illuminating-power in any angular direction; at the same time, the current through the arc and the potential-difference of the carbons were observed. These observations having been taken, involving many hundreds of photometric measurements, the results were set out in a series of photometric diagrams, as shown in figs. 21 to 24, which delineate the respective form of the photometric curves for the two arcs and for different wattages expended in the arc. These photometric diagrams were then integrated, and the mean spherical candle-power calculated in the usual way by means of a Rousseau's diagram, and finally the results embodied in one complete table and diagram, as given below (pp. 356, 357).

The results in the Table are graphically embodied in the diagram in fig. 25, from which it will be seen that, taking the alternating-current arc as employed, the total mean spherical candle-power is always considerably less than that of a continuous-current arc, taking the same mean power. Lowering the frequency seems to increase the efficiency of the alternating-current arc, as one might naturally assume it would do, and it is obvious that increasing the diameter of the lower carbon of the continuous-current arc would diminish its total mean spherical candle-power at any given wattage. This table and diagram therefore, we think, settle the question that for a given expenditure of power in the arc a greater

Fig. 18.—Curves of Current, Potential Difference, Power, and Resistance of 360 Watt Alternating Current Arc.

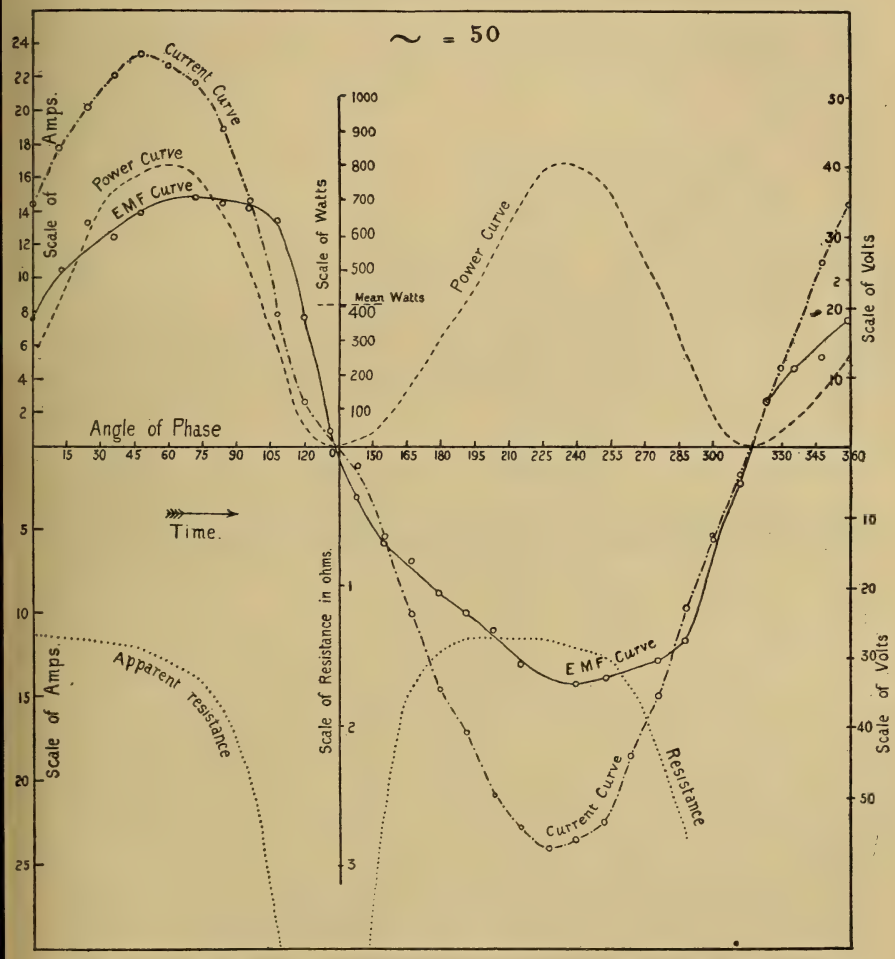


Fig. 19.—Curves of Current, Potential Difference, and Power of 500 Watt Alternating Current Arc.

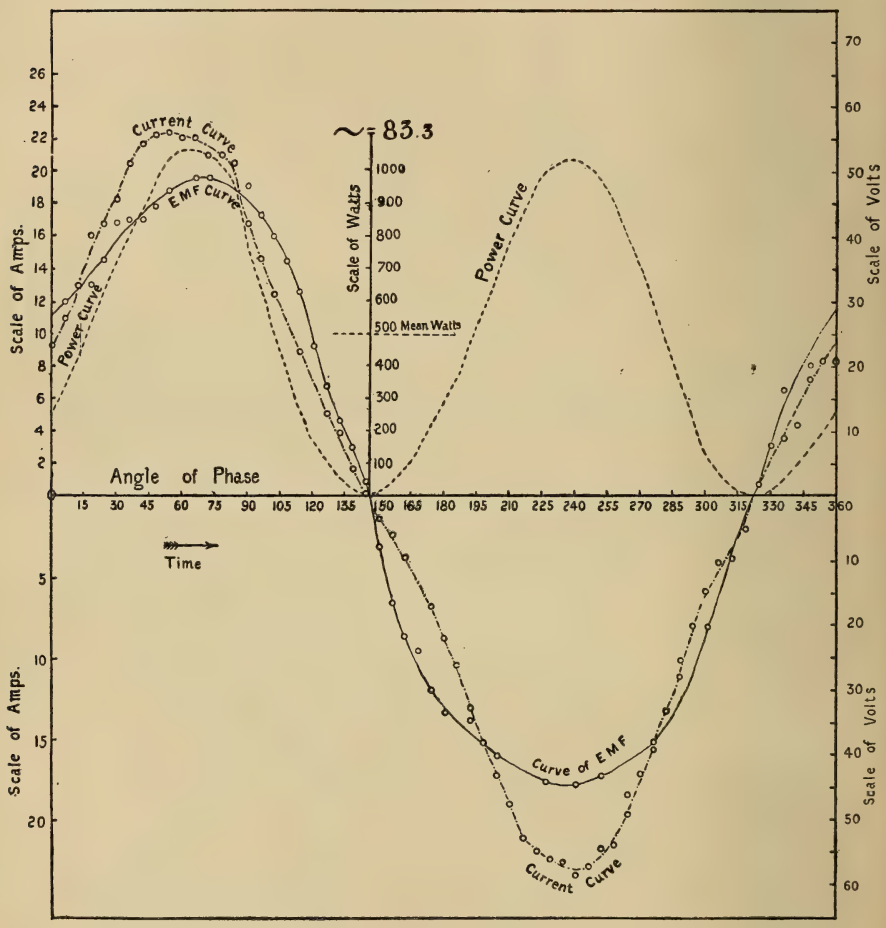


Fig. 20.—Curves of Current, Potential Difference, Power, and Resistance of 600 Watt Alternating Current Arc.

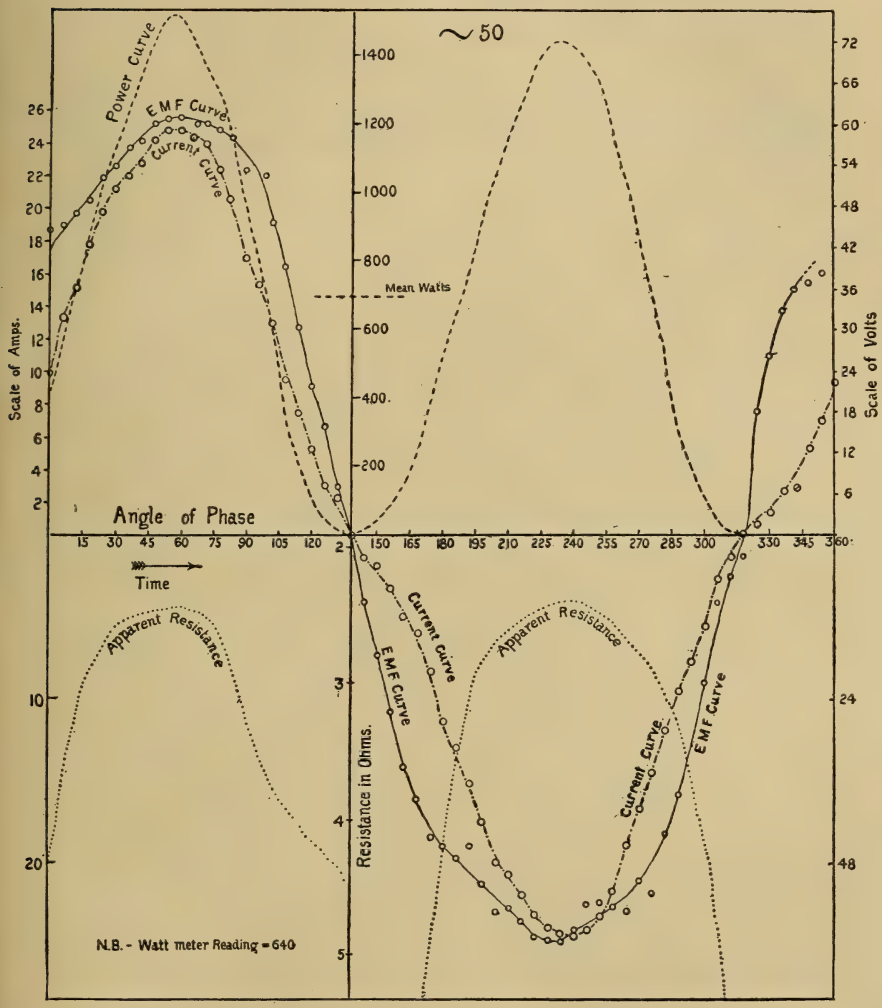


Fig. 21.—Photometric Curves of Continuous Current Arcs. Both Carbons 15 mm. cored.

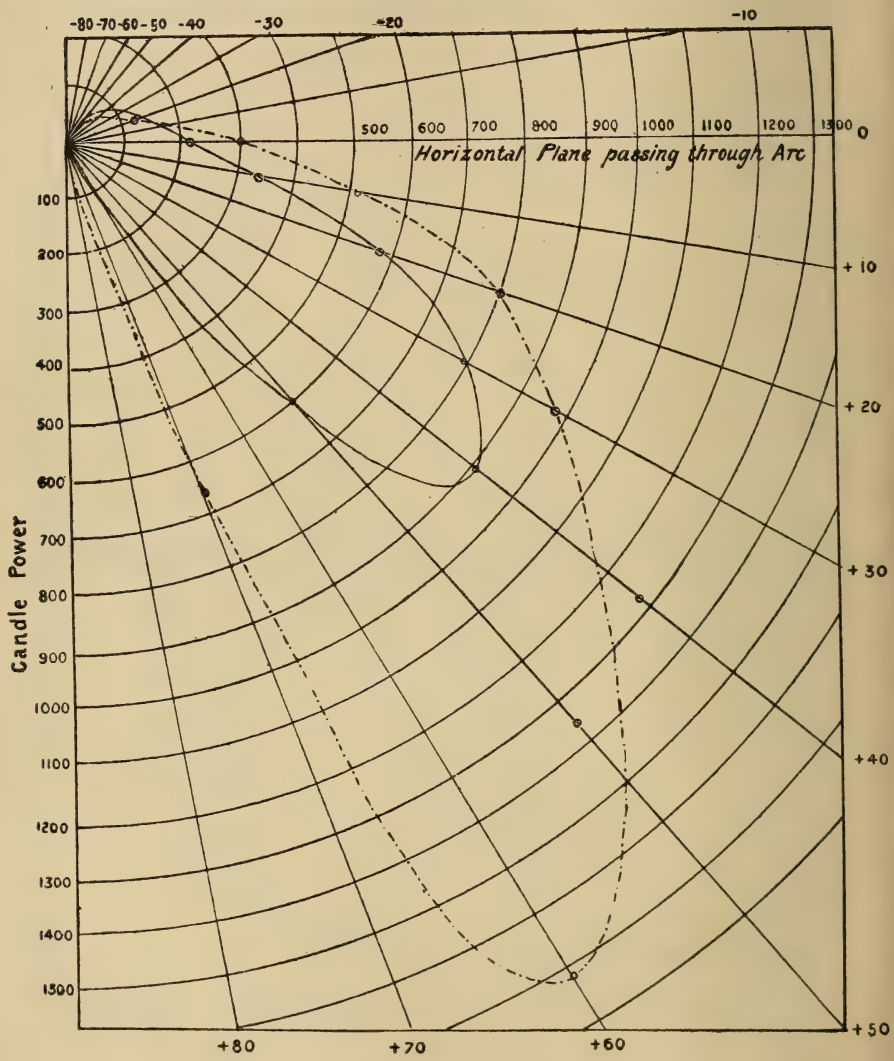


Fig. 22.—Photometric Curves of Continuous Current Arc.
Negative carbon 9 mm. uncored.
Positive carbon 15 mm. cored.

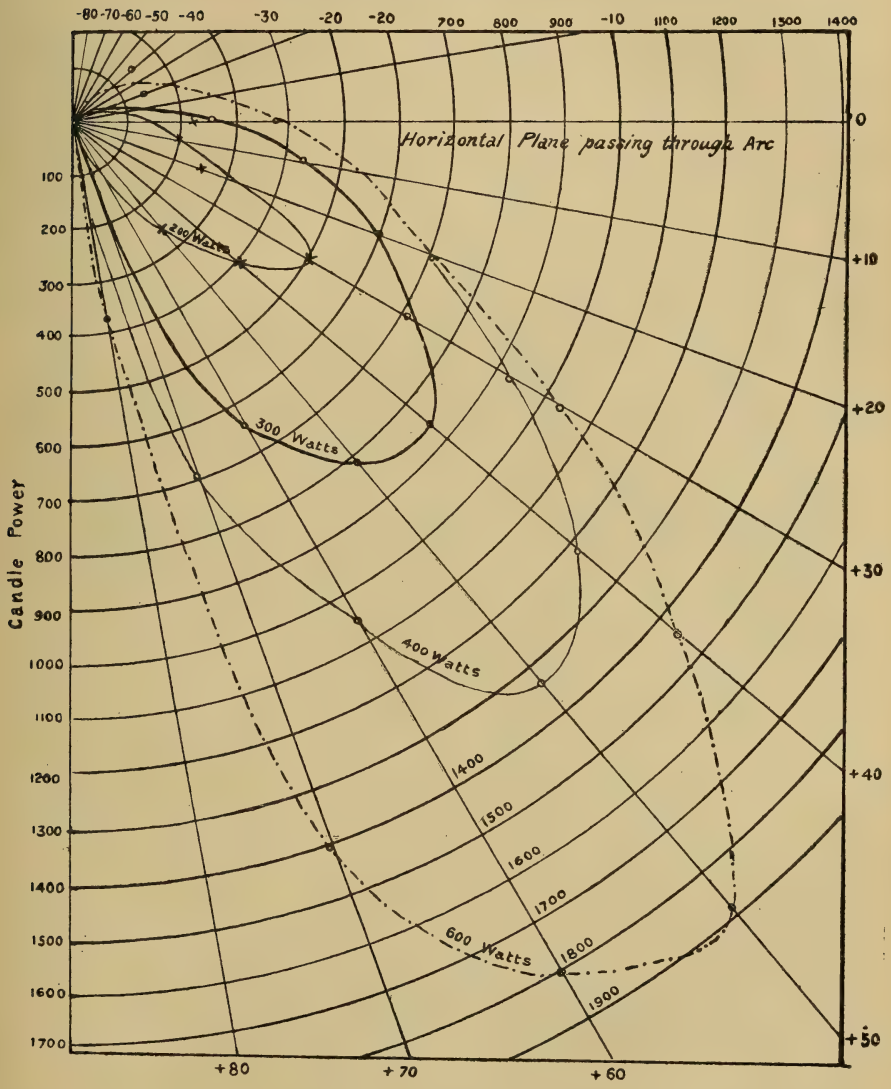


Fig. 23.—Photometric Curves of Alternating Arc Lamp ~ 83·3.

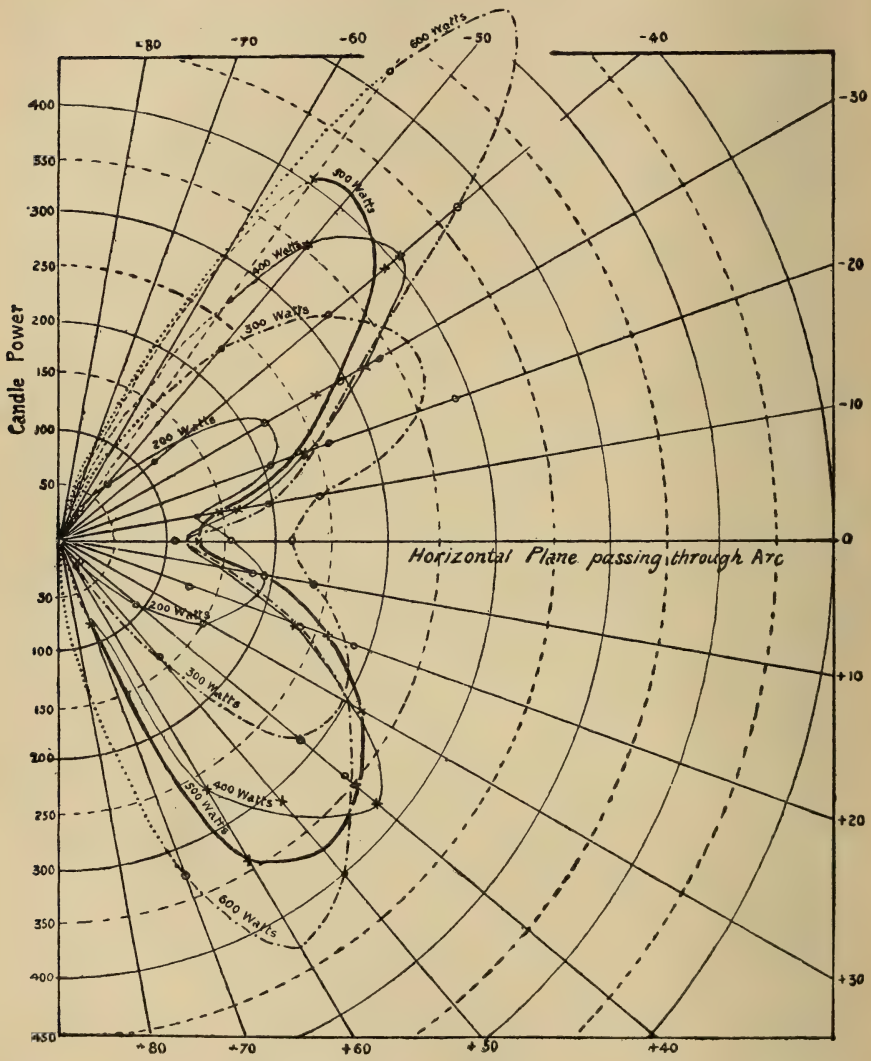
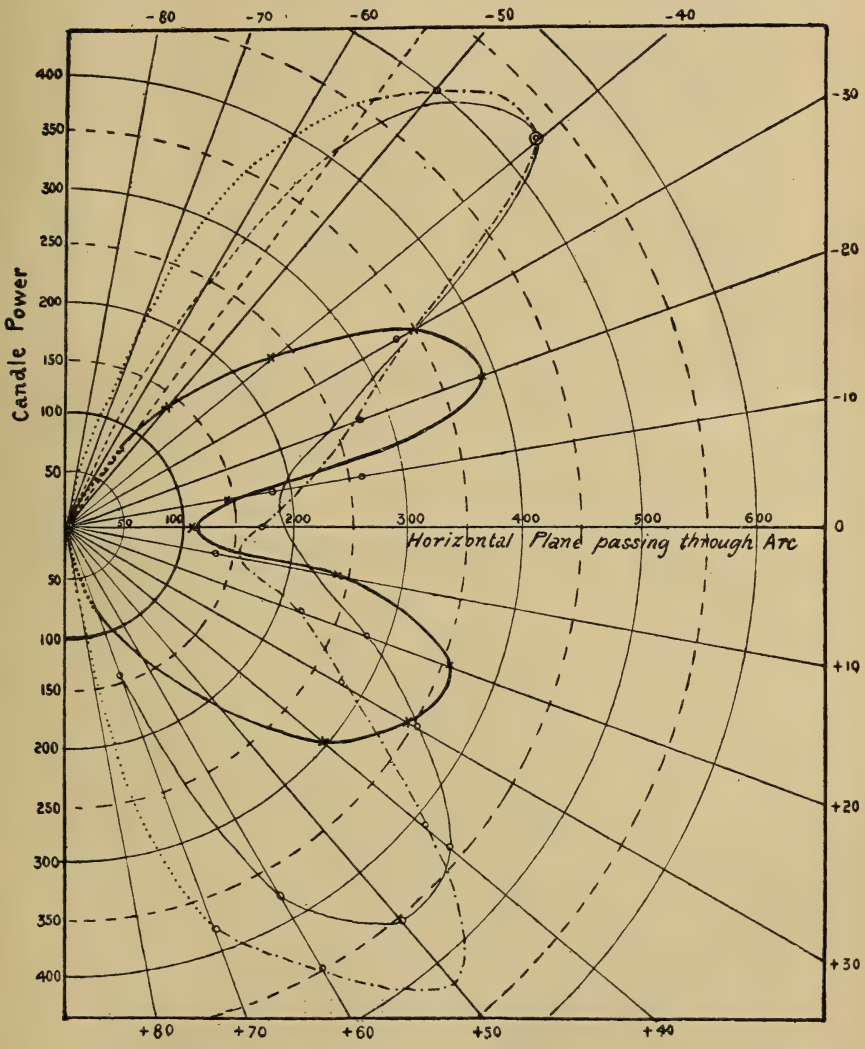


Fig. 24.—Photometric Curves of Alternating Arc Lamp ~ 50.

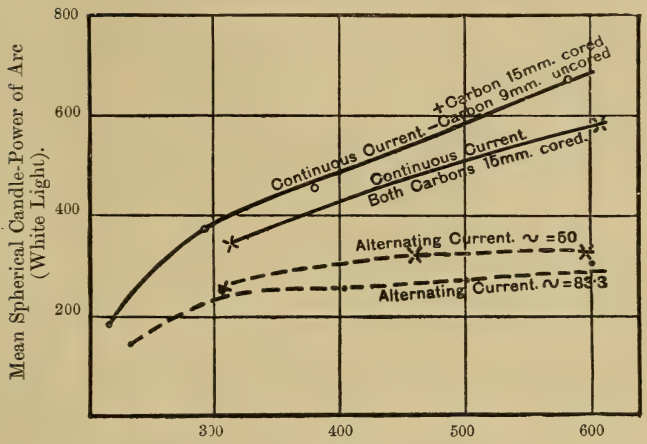


	Continuous-current arc.						Alternating-current arc. Both carbons 15 mm. cored.							
	+Carbon 15 mm. cored. −Carbon 9 mm. solid.				Both carbons 15 mm. cored.		Frequency 83·3 ∼.					Frequency 50∼.		
Power expended in the arc in watts	582	380	292	*215	607	313	601	501	404	305	*233	596	459	308
Mean spherical candle-power	675	455	372	181	582	344	307	274	256	250	144	326	322	254
Potential difference of carbons in volts	56	44	36	26	60	40	37·5	34	28	21	15·3	39	31	22
Current through arc in amperes	10·6	8·7	8·1	8·2	10·2	8	16·2	15·1	15·1	15·5	15·4	16	15·1	14·6
Length of arc in centimetres	0·715	1·0	0·07	0	0·807	0·061	0·628	..	0·125	0·016	0·01	0·7	..	..
Power factor.....	1·0	0·25	1·0	1·0	1·0	1·0	..	..	0·96	0·96	0·99	0·97	0·99	0·97

* In these observations the arc was hissing during the whole of the time.

In reading the above table it must be noted that it was the power, measured in watts, taken up in the arc which was kept constant. The current and difference of potential of the carbons were controlled so as to secure this result. The current and volts were approximately constant, and had the mean values given in the table, but the product of these mean values does not quite agree with the true mean power measured in watts taken up in the arc.

Fig. 25.
(From 'Electrician,' Dec. 20, 1895, p. 247.)



amount of mean spherical illumination can be obtained, if that power is supplied in a continuous-current form than if it is supplied in an alternating form. It has been suggested that, with proper carbons and under proper conditions, the alternating and continuous-current arc should give the same mean spherical candle-power for the same expenditure of power, as has been shown to hold true in the case of the incandescent lamp by the investigations of Profs. Ayrton and Perry. We think, however, that no *à priori* reasoning can apply in the case of the arc. It is perfectly clear that, owing to the interval of cooling that elapses, the mean temperature of the two carbons in the case of the alternating-current arc must be less than the temperature of the positive crater of the continuous-current arc, and that therefore the result we have obtained is only what might be expected. If the question is asked, how do we account for the difference in efficiency? the answer must be that the energy absorbed in the case of the alternating-current arc is radiated at a lower temperature, and the two arcs were therefore exactly in the same condition as regards comparison that two incandescent lamps would be, both taking the same total power but worked at different temperatures, and therefore different watts per candle, and therefore giving different mean spherical candle-powers with the same total power expended in them.

It may be observed in all cases the alternating-current arcs we employed in our experiments took 16 to 17 amperes, and

the power was varied by varying the potential-difference of the carbons, and the alternating-current arc lamp used was one which effected this variation automatically, even although the power expended in the arc was varied from 200 to 600 watts. In order to complete the comparison of the continuous and alternating-current arcs, it will be necessary to compare the behaviour as regards illuminating-power of alternating-current arcs, taking the same mean power but formed with larger currents and less carbon potential-differences; that is to say, comparing alternating arcs of equal power-absorption, but taking very different currents and therefore having different lengths. We hope to extend this investigation to cover these additional questions at some future time.

The above observations have necessitated an enormous number of photometric and electrical measurements, and we have in the above work been very efficiently aided by Messrs. L. Birks, W. H. Grimsdale, A. M. Hanbury, E. N. Griffiths, and others, to whom our thanks are due.

XXXV. *On the Determination of High Temperatures with the Meldometer.* *By* William Ramsay, *Ph.D., F.R.S., Professor of Chemistry, University College, London, and* N. Eumorfopoulos, *B.Sc., Demonstrator of Physics, University College, London**.

THE meldometer, an instrument devised by Dr. Joly, has been sufficiently described by him (Proc. Roy. Irish Acad. 3rd series, ii. p. 38, or Chem. News, vol. lxv.), and we need therefore only give a very brief account of it here. The essential part of the instrument is a length (about 10 centim.) of thin uniform platinum ribbon, about 1 millim. wide. This is heated by a current of adjustable strength, and the increase in length of the ribbon is measured by a delicate micrometer-screw, the ribbon being kept gently stretched by a small spring. The temperature of the ribbon is, of course, lower where the two forceps hold it; but if it is suitably cut at each end nearly to a point, a length of, say, 6 centim. in the middle may be made of a very uniform temperature, as can be proved experimentally by taking the reading of the melting-point of the same substance at different points along the ribbon.

An infinitesimal quantity (scarcely visible with the naked eye) of the substance to be melted is placed on the ribbon and viewed with a low-power microscope. The small quantity of

* Communicated by the Physical Society: read February 14, 1896.

the substance required enables it to be purified very completely. The current is then put on and increased rapidly until the substance melts, and thus an approximate reading is obtained. This is repeated more cautiously, to obtain an exact reading. Several readings can be taken by remelting the same, and also by using fresh substance, the latter method being usually adopted.

To translate the readings into temperatures, it is necessary to standardize the instrument by taking the readings with substances of known melting-points. Of these, unfortunately, there are none known with certainty beyond about 350° C. One reading is obtained by taking the temperature of the air, another is the melting-point of potassium nitrate (339°), and for a third one the melting-point of potassium sulphate (1052°) was adopted, for reasons that will appear. The melting-point of silver is irregular, apparently because of absorption of oxygen, and consequent spitting. Gold can be used, and also palladium. As, however, the expansion of the ribbon is almost a linear function of its temperature, and as the observations hitherto taken did not extend beyond about 1050°, it was considered unnecessary to take readings with palladium, the general character of the expansion of the ribbon being already known from Dr. Joly's observations. The question next arises, what is the melting-point of gold? There have been two or three determinations of value, which unfortunately differ from one another.

M. Violle (*C. R.* 1879) determined it by a calorimetric method, and obtains as a result 1045° (on the air-thermometer).

Messrs. Holborn and Wien (Wied. *Ann.* xlvii. and lvi., 1892 and 1895), who give 1072°, compared a thermo-element with an air-thermometer, and then used the former for determining the melting-points of silver, gold, and copper. This was done by inserting in a porcelain crucible the thermo-element, and also two platinum wires connected by a wire of the substance whose melting-point was to be taken. The platinum wires formed part of a circuit containing a battery and a galvanometer. When the wire melted, the circuit was broken, and the temperature read at the same moment with the thermo-element. The melting of the substance must in general lag a little behind the thermo-element; and as no mention is made of the rate at which the temperature was raised, it is difficult to know how far the results can be trusted. They obtain, however, very concordant readings.

Besides these, there are two determinations with the platinum pyrometer: one by Professor Callendar (Phil. Mag. Feb.

1892), and the other by Messrs. Heycock and Neville (J. C. S. Trans. 1895, p. 160). By a somewhat violent extrapolation * from a formula which, as far as we know, has been compared with the air-thermometer only to about 625°, they obtain :—

	Callendar.	Heycock and Neville.
Freezing- or melting-point of silver ...	982°	961°
" " gold	1091°	1062° †

Callendar then, taking the melting-point of silver as 945°, makes the melting-point of gold 1037°. Now Violle's determination for silver is 954°, which, using Callendar's formula, would give for gold 1049°. Holborn and Wien's value for silver is 968°. In view of the variance between the numbers, it was determined to take Violle's value, though the correct value may be a few degrees higher (compare Le Chatelier, *C. R.* cxxi. p. 323, 1895). In any case, allowance can easily be made, when further researches have determined the true melting-point.

The gold used by us was a very pure specimen obtained from Messrs. Johnson and Matthey. The salts used, with the exception of some of the iodides, were pure specimens, prepared specially by ourselves. In a few cases the salts so prepared were recrystallized, and melting-points were taken both of the recrystallized salt and also that obtained from the mother liquor. No difference could be detected, and hence no further mention is made of these determinations.

Gold on melting alloys with the platinum, and hence must destroy to a certain extent the uniformity of the ribbon. A

* In Holborn and Wien's last paper (*loc. cit.*) the resistance of pure platinum is determined at different temperatures with their thermo-element, and their results cannot be expressed quite satisfactorily (*"nur ungenügend"*) by means of Callendar's formula. The resistance of two of their pure platinum wires began to differ beyond about 900°, while agreeing below this temperature. They therefore consider this property unsuited to extrapolation.

† In a recent paper (J. C. S. Trans. 1895, p. 1025) Heycock and Neville state that "Callendar did not rigidly follow the method of calibration, which was afterwards developed by Griffiths and himself, and which we have always adhered to. That method requires that the resistance of the pyrometer should be determined at three standard temperatures," viz., at 0°, 100°, and the boiling-point of sulphur, 444°·5. And lower down they say, "If Callendar had standardized his thermometer on the boiling-point of sulphur. . . .;" hence they infer that Callendar did not use the boiling-point of sulphur for this determination. But the determination of this boiling-point is given in Phil. Trans. 1891, *i. e.*, it was published before the paper referred to, and the latter is also later than another paper (Phil. Mag. July 1891, p. 109), in which the boiling-point of sulphur is directly referred to. We do not quite see how to reconcile the various statements.

remedy for this is to dust the ribbon lightly with finely powdered talc. This, however, is not very satisfactory, and interferes somewhat with the observations; but with care, fairly good results can be obtained, as the following numbers show:—

	With talc.	Without talc.
NaCl.	799°	792°
K^2CO^3	883	880
KCl	752	762
$Ba(NO^3)^2$ . . .	583	575

It was then found that the melting-point of potassium sulphate is very little different from that of gold, viz., 7 degrees higher or 1052°; and this salt was afterwards used instead of the gold; thus there is no need to use talc.

Some melting-points are sharply marked, others are not. In these cases the lowest point was taken at which spreading over the ribbon could be detected.

For purposes of comparison determinations by other observers are given; some determined the melting-points (*e.g.*, Carnelley, and Meyer, Riddle and Lamb), others the freezing-points (Carnelley, Le Chatelier, Heycock and Neville, and McCrae). The data are taken from the following references:—

Carnelley (calorimetric method): J. C. S. Trans. 1876, p. 489; 1877, p. 365; 1878, p. 273.

Le Chatelier (thermo-electric method, assuming melting-point of gold as 1045°): *Bull. Soc. Chim.* t. xlvii. p. 301; *C. R.* t. cxviii. pp. 350, 711, and 802.

V. Meyer, Riddle and Lamb: *Ber.* xxvii. (1894) p. 3129. —In this method the salt has been previously fused in a platinum tube with a wire down its centre, and to this wire is attached a weight passing over a pulley. When the salt melts, it is pulled out by the weight, and the temperature is determined at the same moment by an air-thermometer.

J. McCrae (Wied. *Ann.* lv. p. 95), relying on Holborn and Wien's results, standardized his thermo-element with boiling diphenylamine (304°) and boiling sulphur (444°·5). For his thermo-element he used platinum against an alloy of platinum rhodium, and also against an alloy of platinum and iridium. The numbers in brackets given below refer to the latter, and the others to the former. It will be noticed that they do not agree perfectly. This may, of course, be due to his metals not being of the same purity as those of Holborn and Wien. The latter also from their observations find that the iridium alloy is not as well suited as the rhodium alloy.

Salts of Lithium.

The lithium carbonate bought as pure gave a conspicuous sodium coloration to the Bunsen flame. It was purified by successive precipitation and washing, until it gave a brilliant carmine coloration to a Bunsen flame, free from yellow fringe, showing an absence of sodium. The melting-points are not very well marked as a rule.

	Ramsay & Eumorfopoulos.	Carnelley.	Le Chatelier.
Li^2SO^4	853	818	830
Li^2CO^3	618	695	710
LiCl.....	491	598	
LiBr	442	547	
LiI	below 330	446	

It will be noticed that there is here no agreement between the results of different observers. Our results are as a rule about 100 degrees lower than Carnelley's, except that of the sulphate. Our resistances did not allow us to take the melting-point of lithium iodide, as we had not arranged for temperatures below 330°, but its melting-point is below that of potassium nitrate (339°). Lithium iodide is very hygroscopic, so that on placing it on the ribbon it quickly liquefies; then on putting on the current, it becomes solid and then liquid again at the temperature given above, *i. e.*, it has melted, and remains so indefinitely.

Salts of Sodium.

The salts of sodium and potassium were prepared from the bicarbonate, precipitated by carbonic acid from a solution of the pure carbonate. The melting-points are well marked, though all are not equally so, *e. g.*, the iodides.

	Ramsay & Eumorfopoulos.	Carnelley.	Le Chatelier.	Meyer, Riddle & Lamb.	Heycock & Neville.	McCrae.
Na^2SO^4...	884	861	860	863	883	883
Na^2CO^3...	851	814	820	849	852	861 (854)
NaCl.....	792	772	778	815	...	813
NaBr.....	733	708	...	758	...	761
NaI	603	628	...	661	...	695 (668)

The agreement between Messrs. Heycock and Neville's results and our own is only apparent, as they find 1062° as the melting-point of gold, while we assume 1045°. The fact that they are determining freezing- and not melting-points may introduce some difference.

Salts of Potassium.

	Ramsay & Eumorfopoulos.	Carnelley.	Le Chatelier.	Meyer, Riddle & Lamb.	Heycock & Neville.	McCrae.
K^2SO^4...	1052	...	1045	1078	1066·5	1059 (1166)
K^2CO^3...	880	834	860	879	...	893 (885)
KCl.....	762	734	740	800	...	800
KBr ...	733	699	...	722	...	746 (709)
KI	614	634	640	685	...	723 (677)

M. Le Chatelier gives an earlier value 1015° for potassium sulphate, and also 885° for potassium carbonate. Our value for potassium sulphate is 7 degrees above that of gold. Messrs. Heycock and Neville's is 4·5 above their value for gold, but they mention that there may be an error of 2 degrees, due to the alkalinity of their potassium sulphate.

Salts of Calcium, Strontium, and Barium.

The salts of calcium, strontium, and barium were prepared from their carbonates precipitated from the purified nitrates. Their melting-points are not well marked, especially those of calcium; and here again the iodides are less well marked than the chlorides.

	Ramsay & Eumorfopoulos.	Carnelley.	Le Chatelier.	Meyer, Riddle & Lamb.	McCrae.
$Ca(NO^3)^2$.....	499	561			
$CaCl^2$	710	719	755	806	802
$CaBr^2$	485	676			
CaI^2	575 (?)	631			
$Sr(NO^3)^2$.....	570	645			
$SrCl^2$	796	825	840	832	854
$SrBr^2$	498	630			
SrI^2.....	402	507			
$Ba(NO^3)^2$ ...	575	593	592		
$BaCl^2$	844	860	847	922	916 (941)
$BaBr^2$	728	812			
BaI^2.....	539				

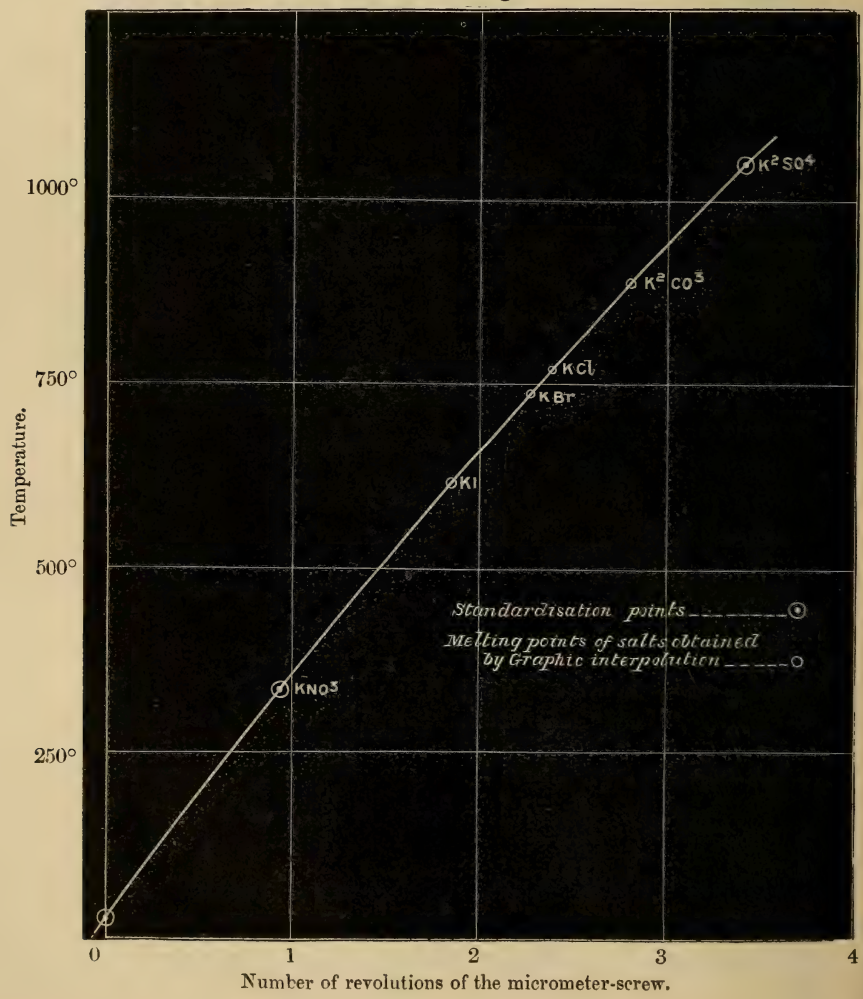
Calcium chloride is very difficult to observe, as it slowly softens. We found it practically impossible to take the melting-point of calcium iodide, as it is exceedingly hygroscopic, and, on heating, it is almost immediately oxidized. We do not think it can be above the value given, though it may be below it.

Salts of Silver and Lead.

These were prepared by precipitation. Their melting-points are well marked.

	Ramsay & Eumorfopoulos.	Carnelley.
Ag^2SO^4	676	654
AgCl	460	451
AgBr	426	427
AgI	556	527
$PbSO^4$	937	
$PbCl^2$	447	498
$PbBr^2$	363	499
PbI^2	373	383

A curve is appended with the melting-points of the salts of potassium marked on it. In drawing this curve there must



of necessity be a small uncertainty, as there is no datum-point between 339° and 1052°; but this cannot amount to more than a very few degrees.

In conclusion, we may point out that plotting our meldometer-readings against other observers' melting-points does not give a smooth curve.

University College, London.

XXXVI. *The Magnetic Field of any Cylindrical Coil.* *By* W. H. Everett, *B.A.**

APPLYING Ampère's formula for the magnetic force at any point due to an element of current, the force perpendicular to any plane circuit, carrying a current i, is found to be, at any point P,

$$Z' = i\int\frac{rd^2\theta}{(r^2+h^2)^{\frac{3}{2}}};$$

h being the distance of P from the plane of the circuit, and r, θ the polar co-ordinates of any point of the circuit referred to the projection of P as origin.

The longitudinal force at any point due to a current in a cylindrical coil, or solenoid, is given by a second integration. It is the sum or difference of two terms, each of the form

$$Z = inh\int\frac{d\theta}{\sqrt{r^2+h^2}};$$

where h denotes the distance of the point from an end plane of the solenoid, and n the number of turns per unit length of h. The depth of the coil, normal to the cylindrical surface, is assumed to be inconsiderable. The limits of integration are 0 and 2π for any point whose projection, taken parallel to the axis, falls within the solenoid.

Similarly, the transverse force at any point, due to a solenoid, is found to be

$$R = in\Sigma\left(\frac{1}{\sqrt{r^2+h_1^2}} - \frac{1}{\sqrt{r^2+h_2^2}}\right)ds,$$

the summation being vectorial.

The latter two formulæ can be readily applied, for approximate calculation, to a cylindrical coil of any cross-section,

* Communicated by the Physical Society, being abstract of paper read November 8, 1895.

including coils of circular and rectangular sections. But in the case of rectangular coils the formulæ become integrable.

Let p be the perpendicular distance of any point P from one of the faces of the rectangular coil, and a one of the two parts into which the corresponding side of a cross-section is divided by the perpendicular from P. Then the longitudinal force at P is given by the algebraic sum of sixteen terms, each of the form

$$Z = inhp\int_0^a \frac{dx}{(x^2+p^2)\sqrt{x^2+p^2+h^2}},$$

$$= in\sin^{-1}\frac{ah}{\sqrt{(a^2+p^2)(h^2+p^2)}}.$$

And the transverse force at P is the resultant of eight terms, each of the form

$$R = in\int_0^a\left(\frac{1}{\sqrt{x^2+p^2+h_1^2}} - \frac{1}{\sqrt{x^2+p^2+h_2^2}}\right)dx,$$

$$= \frac{in}{2}\log\left(\frac{p^2+h_2^2}{p^2+h_1^2}\cdot\frac{a+\sqrt{a^2+p^2+h_1^2}}{a+\sqrt{a^2+p^2+h_2^2}}\right).$$

The first formula in the paper can be used to find an expression for the force due to a circular current, at any point P in the plane bounded by the circle. Draw any chord through P, and call its segments r, r'. Write c for the distance of P from the centre, b for half the minimum chord through P, and a for the radius. Then for the force at P the formula reduces to

$$F = i\int_0^{2\pi}\frac{d\theta}{r},$$

$$= i\int_0^{\pi}\left(\frac{1}{r}+\frac{1}{r'}\right)d\theta,$$

$$= \frac{2i}{b^2}\int_0^{\pi}\sqrt{a^2-c^2\sin^2\theta}\,.\,d\theta.$$

This can be written

$$F = i\frac{s}{b^2},$$

s being the perimeter of the ellipse with a and b as semi-axes, and having some value between $2\pi a$ and $4a$, according to the position of the point considered.

XXXVII. *A Method of Determining the Angle of Lag.* By Arthur L. Clark, *S.B., Prof. of Mathematics and Physics, Bridgton Academy, North Bridgton, Maine, U.S.A.**

THE power or rate of expenditure of energy at any given instant of time, on an electrical circuit, may always be found from the equation

$$W = EI;$$

where W is the power in watts, E the E.M.F. in volts, and I the current in amperes. But if the average power is desired this formula is not general. It suffices only where E and I are constant or nearly so.

It is a problem at the present time to measure the power expended on circuits through which flow alternating currents whose E and I vary harmonically with the time. In this case the formula becomes

$$W = \frac{EI}{2} \cos \phi;$$

where ϕ is the difference in phase or is the angular magnitude of the delay of the rise of I behind E. E and I are the maximum values of the E.M.F. and current respectively.

It is obvious that $\cos \phi$ is a very much desired value, and different methods for determining it have been conceived. There have been several phase-indicators brought before the scientific world during the past year or two, but of these there are very few, if any, which furnish a convenient means of accurately measuring a difference of phase.

The instrument herein described is the outcome of extended investigation carried on by the author during the past year at the Worcester Polytechnic Institute. Many of the different forms of apparatus which depend upon the interference of sound waves, vibrating wires, &c., were constructed and experimented upon, with unsatisfactory results.

It was found that indicators which are influenced to a marked degree by small variations of the vibration-period are of little value. As this variation interferes seriously with results, and as no dynamo furnishes a current absolutely periodic, such an indicator is worthless commercially.

The well-known Lissajous's figures have been used at different times as a means of determining the angle of lag, and are the basis of the herein described apparatus. The current from the dynamo passes through a single loop of wire

* Communicated by the Author.

clamped at one end, and carrying a small mirror on the other end, which is free to vibrate. This loop is suspended between the poles of a magnet (electro or permanent), so that with every change of direction of the current through this loop, it will tend to rotate one way or the other according to Maxwell's rule, *i. e.*, "Every portion of the circuit is acted upon by a force urging it in such a direction as to make it enclose within its embrace the greatest possible number of lines of force."

Now if a beam of light falls on the mirror, the reflexion will be drawn out into a line by the vibration of the mirror. This beam of light coming from this mirror falls on a second mirror, arranged as the first but actuated by another current and with its plane of vibration perpendicular to that of the first. In the resultant reflexion we find our means of measuring the amount by which one mirror leads the other, or, in other words, by how much the phase of the current in the first leads that in the second.

We will call the direction of vibration of the beam of light as given by each mirror alone the axes of X and Y respectively. That is, the axis of X is the figure from the first mirror while the second is stationary, and the axis of Y that from the second while the first is stationary.

The equation of a simple harmonic motion of amplitude a along the axis of X may be expressed

$$x = a \sin \theta,$$

where θ is a linear function of the time.

Also the equation of another harmonic motion of amplitude b, along the axis of Y, whose time differs from θ by an amount ϕ, is

$$y = b \sin (\theta - \phi).$$

Combining by eliminating θ since

$$\sin \theta = \frac{x}{a}, \quad \cos \theta = \frac{1}{a} \sqrt{a^2 - x^2},$$

the resulting equation is

$$y = \frac{b}{a} (x \cos \phi - \sqrt{a^2 - x^2} \sin \phi),$$

an equation in x and y, independent of the periodic time. This equation is the equation of an ellipse. The resultant reflexion, then, is an ellipse whose shape depends upon a, b, and ϕ.

The equation of the long diameter of the ellipse is

$$y=\frac{b}{a}x.$$

Then since the short diameter is perpendicular to this, its equation is

$$y=-\frac{a}{b}x.$$

Treating either of these equations as simultaneous with the equation of the ellipse, the coordinates of the intersection of these diameters with the curve may be found, from which may be deduced the lengths of the diameters in terms of ϕ. This result is general but too cumbersome to be of service.

Suppose $a=b$, a condition which may be easily attained by increasing or decreasing the current in the vibrating loops, or by varying the strength of the actuating magnets. The equation of the ellipse would then be

$$y=x\cos\phi-\sqrt{1-x^2}\sin\phi,$$

the equation of a family of ellipses whose parameter is ϕ.

The equation of the diameters is

$$y=\pm x.$$

Combining this with the equation of the curve, there results as the squares of the coordinates of intersection

$$y^2=\frac{\sin^2\phi}{2(1\mp\cos\phi)};\quad x^2=\frac{\sin^2\phi}{2(1\mp\cos\phi)}.$$

The upper sign in the denominator belonging with the positively sloped and the lower with the negatively sloped diameter. The squared length of the semidiameters is the sum of these squares, or

$$d^2=\frac{\sin^2\phi}{1\mp\cos\phi}.$$

The whole diameter is double this semidiameter, so calling D_1 the positively and D_2 the negatively sloped diameter,

$$D_1^2=\frac{4\sin^2\phi}{1-\cos\phi},$$

$$D_2^2=\frac{4\sin^2\phi}{1+\cos\phi}.$$

Dividing the first by the second we have

$$\frac{D_1^2}{D_2^2}=\frac{1+\cos\phi}{1-\cos\phi},$$

from which

$$\cos\phi=\frac{D_1^2-D_2^2}{D_1^2+D_2^2}.$$

The reflexion is examined in a telescope with micrometer eyepiece, having two separate scales so that the lines on these scales may be made perpendicular to the long and short diameters of the ellipse. It makes no difference what the scale-divisions are, if they are alike on both scales.

When the ellipse becomes a straight line, $D_2^2=0$ and $\cos\phi=1$, from which $\phi=0$, or the currents are in phase. If the ellipse becomes a circle, $D_1^2=D_2^2$ and the numerator becomes 0, consequently $\cos\phi=0$, and $\phi=90°$, or the currents are in quadrature.

The amount of self-induction in the apparatus itself is inappreciable, and the loops keep well in time with the current. Even a considerable variation in period does not hinder the vibrations, which in this case are forced. This is possible since the mirrors, while moving very slightly, still vibrate sufficiently for telescopic observation.

Thus we have overcome the two great difficulties of many forms of this apparatus, namely, self-induction and the inability of the vibrating wires to follow a change of period. And with these objectionable features eliminated and a method of finding the exact value of $\cos\phi$, we have a practical means of determining the difference of phase.

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XXXVIII. *A Note on* Mr. Burch's *Method of Drawing Hyperbolas. By* F. L. O. Wadsworth, *E.M., M.E., Assistant Professor of Physics, University of Chicago**.

In the January number of the Philosophical Magazine Mr. Burch describes a very simple and convenient method of drawing an hyperbola by the use of two similar triangles. This method is very similar to one which I have been using for some time and which I have described in my lectures for the past two years, although I have never published it. Mr. Burch's invention of the method antedates

* Communicated by the Author.

mine, however, by several years, as he states that he first used it in 1885, while it first occurred to me in 1893. In the present note I only wish to call attention to the fact that the particular construction described is only one example of a general class of solutions of this character, and to describe two or three others which are, I think, equally simple and convenient.

In general, if in any two similar triangles two dissimilar sides are kept constant and the other sides varied, it is evident that these two varying sides, which are proportional to the two fixed sides, will be asymptotic coordinates of an hyperbola of which the modulus is the product of the two constant sides. The simplicity of the corresponding graphical or mechanical tracing of the curve depends simply upon our choice of triangles and choice of sides. In the method of construction which I most frequently employ, the two similar triangles have a common vertex at the origin *o*, fig. 1, and the two

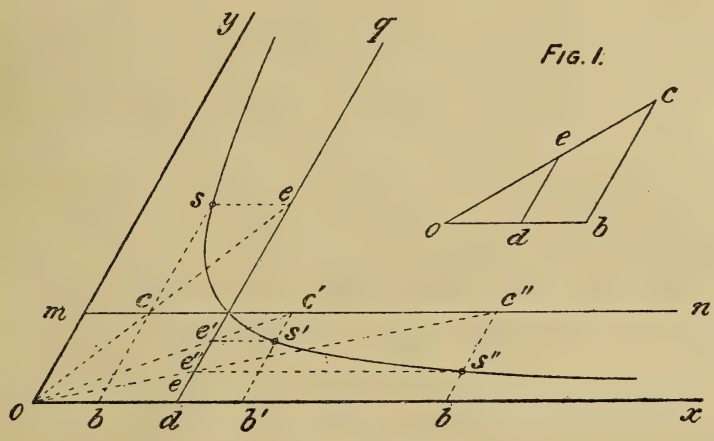


Fig. 2.

sides *ob*, *bc* and *od*, *de* parallel to the asymptotes of the required hyperbola. Then if we put

$$ob = x,$$

$$de = y,$$

$$bc = 1,$$

we have at once

$$xy = od.$$

Hence if we draw a series of triangles in each of which *bc*

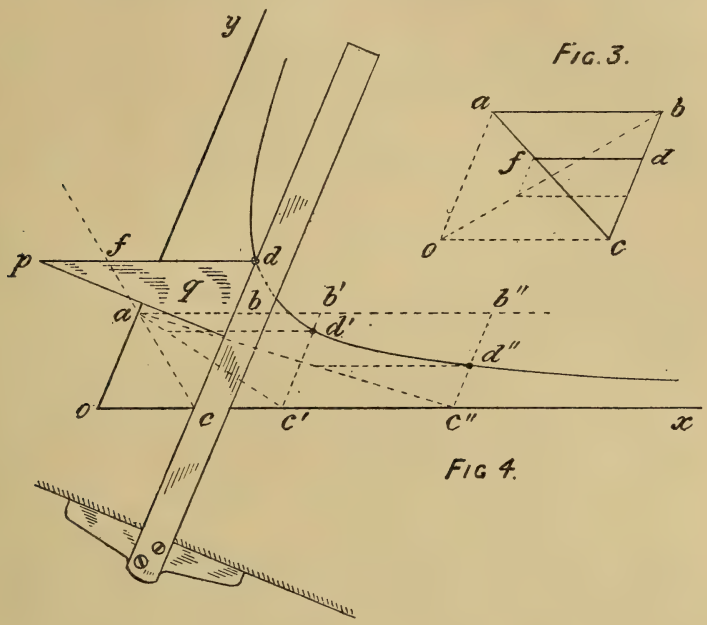
is unity and od is constant and equal to $\frac{a^2+b^2}{4}$, the sides ob, de, of any pair represent coincident values of x and y in the corresponding hyperbola. These coincident values laid off along their respective axes give a series of points on the curve from which it is easy to trace the curve itself. In practice, the whole operation may be rapidly and easily performed by means of a T-square and a single triangle of which one angle is equal to the angle yox between the axes *. First draw the line mn (fig. 2) parallel to and at unit distance from the axis of x, and the line dq parallel to the axis of y and at a distance from it equal to $\frac{a^2+b^2}{4}$. Then to the points c c' c'' on the first line draw the lines oc oc' oc'' &c. by the aid of the edge of the T-square or an auxiliary ruler, and the lines cb, $c'b'$, $c''b''$ by means of the triangle or bevel-gauge. Project the points of intersection e, e', e'' of the first set of lines with the line dq upon the second set of lines, giving us the points s, s', s'' on the required hyperbola. This method is particularly rapid and convenient in plotting rectangular hyperbolas on cross-section paper, the only instrument then necessary being a rule to draw the radial lines oc, oc', &c.

If desired, an instrument can easily be constructed on these lines to trace the curve mechanically, but generally the graphical process is more rapid. A whole series of contours to the thermodynamic surface (pv=const.) can be drawn by this process in a very few minutes, the same set of lines oc, oc', oc'', and cb, $c'b'$, $c''b''$, answering for all the curves.

2nd method.—Make the vertices of the two similar triangles coincide at c instead of o as before, and make $ab=oc=x$, and $cd=y$ (fig. 3). Take a point a on the axis of y at unit distance from the origin and draw from it the lines ac, ac', ac'' to points on the axis of x, and the lines bc, $b'c'$, $b''c''$ parallel to the y axis as before. Mark off a distance equal to $fd=\frac{a^2+b^2}{4}$ on the edge of a triangle (or bevel-gauge), Q, of which the angle at d is equal to the angle yox, and slide this triangle along each of the lines bc, $b'c'$, $b''c''$, &c., until the point f intersects the corresponding line ac, ac', or ac''. The points d, d', d'' will then evidently be points on the hyperbola. In practice it is not necessary to draw the lines bc, $b'c'$, &c. at all, it suffices to place a T-square whose blade is parallel to the axis of y, so that its edge passes through any of the

* It is convenient to use for this purpose an ordinary steel bevel-gauge the two blades of which may be adjusted to the required angle.

points on the axis of x; slide the triangle along this edge until the point f falls on the corresponding line from a, and



mark the position of the vertex d of the triangle (see fig. 4). This method is perhaps even more rapid and convenient than the first, as it involves the drawing of only one set of lines ac, ac', &c. Like the first, it involves the use of only a triangle and a T-square.

3rd method.—Let the vertices of the two similar triangles coincide at b (fig. 5). Then if bc = unity we have as before

$$xy = fb.$$

To determine the points on the curve graphically in this case, we need a triangle or bevel-gauge of an angle equal to yox and a parallel ruler. The side bc of the triangle or bevel-gauge should be of unit length and a distance $fb = \frac{a^2+b^2}{4}$ laid off on the side bq. The triangle is placed with its side bq coinciding with the axis of x, and one edge of the parallel ruler is brought against the point c of the triangle and a pin placed at the origin. The other blade of the parallel ruler is then moved out until it passes through the point f on the

horizontal edge of the triangle, as in fig. 6. The point e at which the blade of the ruler intersects the side bc of the

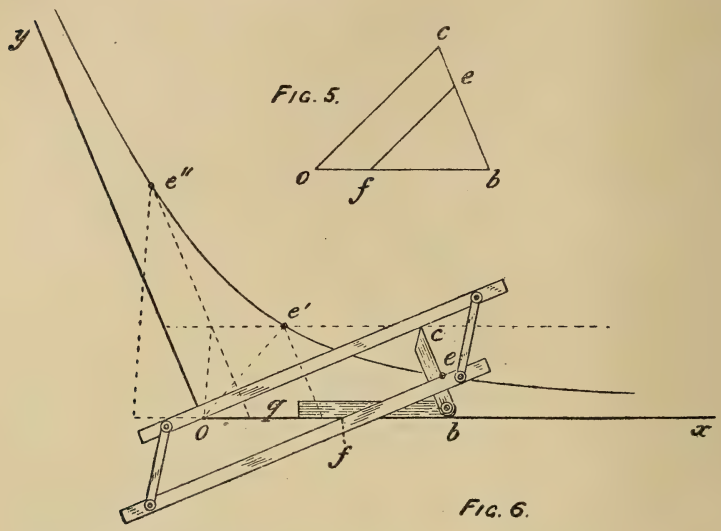


FIG. 6.

triangle will be one of the points on the required curve, the others of which may be found by sliding the triangle along the axis of x, always keeping the two edges of the parallel ruler in contact with the three points o, c, and f. This method is somewhat simpler mechanically, but not quite so rapid and convenient as either of the preceding.

It is evident that there are six other possible solutions to be obtained by combining the sides ac, ae (fig. 1) with each pair of adjacent sides. But these solutions are unsatisfactory graphically, because simultaneous values of x and y will be represented by lines inclined to each other at an angle different from the angle between the axes. If the angle at a is made equal to the angle between the axes, we have one of the three solutions already considered.

The use of two similar triangles in the graphical, and more particularly the mechanical tracing of curves, is of wide application. By their aid we may always express the product or quotient of two variable quantities geometrically as the length (tensor) of one of the sides. Other applications of this principle will be found in a recent paper of the author's on the mountings of concave grating spectroscopes*.

* "Fixed Arm Concave Grating Spectroscopes," F. L. O. Wadsworth. Astro-Physical Journal, vol. ii. p. 370 (Dec. 1895).

The Use of the Quadruplane as a Hyperbolagraph.—The two asymptotic coordinates of any hyperbola evidently form two sides of a parallelogram of constant area. Hence any hyperbola can be readily traced by the use of the Sylvester-Kempe quadruplane linkage, the four vertices of which lie at the four angular points of a parallelogram of constant area and constant obliquity*.

The product of the adjacent sides of this parallelogram, or as Sylvester calls it the "modulus" of the cell, is equal to

$$B^2 - A^2 \frac{\sin \alpha_1 \sin \alpha_2}{\sin^2 \theta} = M,$$

where B and A are the distances between the pivotal points of the long and short links of the cell, α_1 and α_2 the angles adjacent to the line joining the pivotal points, and θ the angle subtended by this line at the intervening vertex. If we make each link of the cell symmetrical we have

$$\alpha_1 = \alpha_2 = 90 - \theta/2,$$

and

$$\therefore\ M = \frac{B^2 - A^2}{4 \sin^2 \theta/2}.$$

The obliquity of the parallelogram is equal to θ.

Hence to describe an hyperbola whose axes are a and b we must make the modulus of the cell equal to the modulus of the hyperbola, or

$$M = \frac{a^2 + b^2}{4},$$

and the angle θ equal to the angle between the asymptotes, or

$$\theta = \sin^{-1} \frac{2ab}{a^2 + b^2}.$$

Then if one vertex of the cell is fixed and an adjoining vertex is moved along a straight line (the edge of a T-square or straight edge for example), passing through the fixed vertex, the vertex diagonally opposite the latter will describe the required hyperbola, having the fixed point as origin and the straight line as one of the asymptotes. To describe different hyperbolas it is necessary to be able to vary both θ and M. The first may be easily done by making each link in two halves, pivoted together at the vertex of the link with a divided sector and clamp, by means of which the desired

* See "The History of the Plagiograph," Sylvester, 'Nature,' vol. xii. p. 214; also Kempe, 'Lecture on Linkages,' p. 25 *et seq.*

angle θ may be laid off. The modulus of the cell is best changed by varying either A or B. This may be conveniently done by making each of either the long pair or the short pair of links like the legs of a pair of proportional dividers.

A small model of an instrument on the above lines has been constructed and found to work very easily and accurately. Like all hyperbolagraphs, however, the range of motion is limited (although larger in this than in most forms), and a considerable amount of time is necessary for the preliminary adjustment. For these reasons I have generally found one of the preceding graphical solutions more rapid and convenient, especially when a number of curves are to be drawn on the same sheet.

This application of the quadruplane, which occurred to me recently while making an application of the Peaucellier linkage to a concave grating mounting *, seems so simple and obvious that I feel sure it must have occurred to others as well as myself; but as I have not been able to find any suggestion to this effect in any of the papers on the subject that I have examined, I have ventured to present the foregoing description as another illustration of the practical application of the beautiful geometrical discovery of Prof. Sylvester and Mr. Kempe.

Ryerson Physical Laboratory,
University of Chicago, U.S.A.,
January 1896.

XXXIX. *A Duplex Mercurial Air-Pump.* *By* R. W. Wood †.

In working with highly exhausted tubes, such as are used for the production of the Röntgen rays, one of the difficulties met with is the speedy deterioration of the vacuum due to the liberation of gas from the electrodes and the glass. If the tube be thoroughly heated, while on the air-pump, this trouble is partially remedied, but even with this precaution the tubes are not very durable and have to be pumped out frequently. In order to overcome this difficulty I have constructed a new form of mercurial air-pump, which can be made on a very small scale and attached permanently to the Röntgen tube. By this arrangement, any traces of gas that

* "On the Use and Mounting of the Concave Grating as an Analyzing or Direct Comparison Spectroscope," The Astro-Physical Journal,' vol. iii. p. 47 (Jan. 1896).

† Communicated by the Author.

make their appearance can be easily pumped out. The apparatus is so compact that it can be held in the hands while in operation, not requiring mounting on a board. The pump is very simple, and a glance at the accompanying diagrams will make its construction clear. It will be seen to consist of

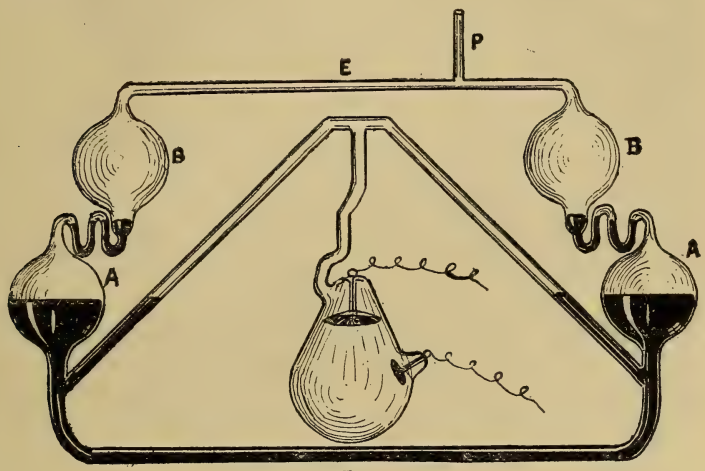


Fig I.

two bulb-pumps joined at the base by a U-tube of glass (fig. 1). The pumping is done by rocking the apparatus, the mercury filling the exhaust-bulbs A A alternately. This duplex action makes the pumping very rapid, for one bulb exhausts while the other fills, there being no lost time. The traces of gas liberated in the discharge-tube are pumped over into the exhausted bulbs B B, where they are stored, being prevented from returning by the mercury which remains in the W traps between A and B. The upper bulbs are joined by a tube E, which has a small lateral tube P blown into it ; this arrangement being necessary for the preliminary exhaustion.

Mercury is first introduced through P until the bulbs A A are half full. A gentle rocking of the apparatus is necessary, as the fluid is held up in the bulbs B by the compression of the air in A.

When enough mercury has been introduced, the apparatus is placed in the position shown in fig. 3, when the fluid should stand at the level indicated in the diagram. The side tube P is now drawn out into a thick-walled capillary in a blast-lamp, in order to facilitate the subsequent closing of the apparatus. This tube being connected with a good mercury-pump by means of a well-greased, thick-walled rubber hose,

the apparatus is exhausted as completely as possible. During the exhaustion it must be supported in the position shown in fig. 3; otherwise the air escaping from the discharge-tube will throw the mercury violently against the top of the bulbs. It is a good plan to carefully heat the bulbs and the discharge-tube by means of a Bunsen burner while the pump is in action, in order to drive off moisture. The current of a fair-sized induction-coil should also be passed through the discharge-tube for a few minutes to rid the electrodes of air as completely as possible. It will be found that the vacuum cannot be made perfect enough to give a Crookes dark space of more than an inch, owing to the leakage through the rubber hose. The capillary part of P is now heated to fusion in a small flame which hermetically seals the entire apparatus. The comparatively poor vacuum in the discharge-tube can now be made as perfect as is possible with any mercury-pump by slowly rocking the apparatus, holding it by the bulbs A A. If the pump is properly made, the traps hold and require no attention: if not, a little dexterity is required, to prevent the mercury from running out into the bulb, and they have to be constantly watched. Care must of course be taken that the

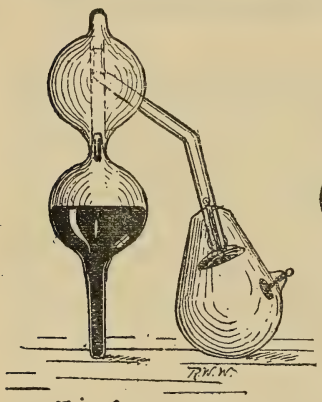


Fig 2.

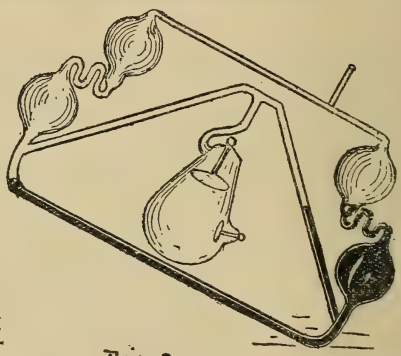


Fig 3.

air-bubble, compressed into the trap by each stroke, is driven entirely around the bend and into the reservoir B. If for any reason the pump requires to be subsequently opened, it must be placed in the inclined position and a file scratch made on the tube P. A bit of red-hot glass pressed against the scratch will cause a crack through which the air will slowly enter. If the tube be broken open suddenly, the mercury will be forced over into the discharge-tube. I con-

structed my apparatus with the Röntgen tube projecting to one side, as shown in the side view, fig. 2. This makes a support for the pump so that it will stand alone. A pump of this description in connexion with an ordinary "Dark-space" tube makes a very convenient piece of lecture-room apparatus for showing the character of the discharge at different pressures. By tipping the pump far enough the upper trap can be emptied, and the air stored in B returned to the discharge-tube again, showing the phenomena at higher pressure.

Owing to the absence of rubber connexions and stopcocks, the mercury remains always clean and there is no leakage.

I am now constructing a pump on this principle on a large scale for general laboratory use in which the rocking motion is to be effected by water-pressure, which, if found serviceable, will be described in a subsequent paper. The chief objection, of course, is that the entire pump is in motion, which makes its connexion with a stationary receiver somewhat difficult. This can perhaps be done by bringing the exhaust-tube into coincidence with the axis of rotation, and using a rubber tube surrounded with mercury as a joint.

The small pump can be ordered with or without the Röntgen tube from Herr Glasbläser R. Burger, Chausseestr. 2 E, Berlin, Germany.

Berlin: Physikalische Institut.

XL. *Intelligence and Miscellaneous Articles.*

NOTES OF OBSERVATIONS ON THE RÖNTGEN RAYS.

BY HENRY A. ROWLAND, N. R. CARMICHAEL, AND L. J. BRIGGS.

THE discovery of Hertz some years since that the cathode rays penetrated some opaque bodies like aluminium, has opened up a wonderful field of research, which has now culminated in the discovery by Röntgen of still other rays having even more remarkable properties. We have confirmed, in many respects, the researches of the latter on these rays and have repeated his experiments in photographing through wood, aluminium, cardboard, hard rubber, and even the larger part of a millimetre of sheet copper.

Some of these photographs have been indistinct, indicating a source of these rays of considerable extent, while others have been so sharp and clear cut that the shadow of a coin at the distance of 2^{cm} from the photographic plate has no penumbra whatever, but appears perfectly sharp even with a low-power microscope.

So far as yet observed the rays proceed in straight lines, and all efforts to deflect them by a strong magnet either within or without the tube have failed. Likewise prisms of wood and vulcanite have

no action whatever so far as seen and, contrary to Röntgen, no trace of reflection from a steel mirror at a large angle of incidence could be observed. In this latter experiment the mirror was on the side of the photographic plate next to the source of the rays, and not behind it as in Röntgen's method.

We have, in the short time we have been at work, principally devoted ourselves to finding the source of the rays. For this purpose one of our tubes, made for showing that electricity will not pass through a vacuum, was found to give remarkable results. This tube had the aluminium poles within 1^{mm} of each other, and had such a perfect vacuum that sparks generally preferred 10^{cm} in air to passage through the tube. By using potential enough, however, the discharge from an ordinary Ruhmkorff coil could be forced through. The resistance being so high, the discharge was not oscillatory as in ordinary tubes but only went in one direction.

In this tube we demonstrated conclusively that the main source of the rays was a minute point on the *anode* nearest to the cathode. At times a minute point of light appeared at this point but not always.

Added to this source the whole of the *anode* gave out a few rays. From the cathode no rays whatever came, neither were there any from the glass of the tube where the cathode rays struck it as Röntgen thought. This tube as a source of rays far exceeded all our other collection of Crookes' tubes, and gave the plate a full exposure at 5 or 10^{cm} in about 5 or 10 minutes with a slow-acting coil giving only about 4 sparks per second.

The next most satisfactory tube had aluminium poles with ends about 3^{cm} apart. It was not straight but had three bulbs, the poles being in the end bulbs and the passage between them being rather wide. In this case, the discharge was slightly oscillatory but more electricity went one way than the other. Here the source of rays was two points in the tube, a little on the cathode side of the narrow parts.

In the other tubes there seemed to be diffuse sources, probably due in part to the oscillatory discharge, but in no case did the cathode rays seem to have anything to do with the Röntgen rays. Judging from the first two most definite tubes, the source of the rays seems to be more connected with the anode than the cathode, and in both of the tubes the rays came from where the discharge from the anode expanded itself towards the cathode, if we may roughly use such language.

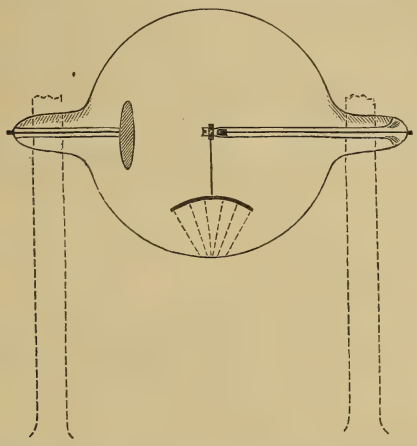
As to what these rays are it is too early to even guess. That they and the cathode rays are destined to give us a far deeper insight into nature nobody can doubt.—*American Journal of Science*, March, 1896.

NOTE ON "FOCUS TUBES" FOR PRODUCING x-RAYS.

BY R. W. WOOD.

The tubes for producing the x-rays which are furnished with a concave kathode for focussing the kathode rays on the glass, in order

to diminish the size of the source and increase its intensity, have the fault that, owing to the great heat developed, the glass is very apt to crack. I have had some success with a tube which I made in which the kathode hangs as a pendulum from the centre of a spherical bulb, by the slow rotation of which one brings a fresh and cold surface into the focus continually, thereby avoiding over-



heating. The concave kathode hangs by an aluminium wire from a short cylinder of aluminium fastened into a glass tube, through which a platinum wire passes which lies in the axis of rotation. The anode is also in the axis of rotation, so that the connexions with the coil can be easily made. My tube has the fault that many kathode rays emanate from the upper surface of the concave plate and are lost. This might be overcome by covering the kathode with a cap of glass. As a suggestion for further experimenting, this note may be of interest to persons working with the new rays.

Berlin, March 8, 1896.

NOTE ON ELEMENTARY TEACHING CONCERNING FOCAL LENGTHS.

To the Editors of the Philosophical Magazine.

GENTLEMEN,

With respect to Prof. Lodge's question on page 152 as to the simplest convention of signs in dealing with focal lengths for junior students, it appears to me to be a matter which can be best settled by one who has had experience in teaching two or more methods. Hence, as this is a qualification to which I can lay no claim, I forbear to dogmatize.

It would seem to me, however, undesirable, apart from weighty reason, to teach an elementary student to let the algebraical sign of a given line depend, *not* upon its *direction*, but upon the *physical*

nature of the optical image at which the line terminates. Still there is a manifest advantage in the convention proposed, since it allows the relation of focal length and conjugate focal distances to be represented by a single equation for both mirrors and lenses.

I give below the adaptation of the graphical method to the convention of signs proposed by Prof. Lodge (and used in Ganot's 'Physics'), viz. :—Distances to real images to be considered positive and distances to virtual images negative.

The fixed points through which the rotating lines pass are now in the right-hand upper quadrant both for convex lenses and concave mirrors, and in the left-hand lower quadrant for concave lenses and convex mirrors.

That is, referring to Fig. 1, page 61, L_x is now shifted so as to coincide with M_a which remains unmoved, and L_a similarly is made to coincide with M_x.

Yours faithfully,
EDWIN H. BARTON.

Univ. Coll., Nottingham,
Feb. 21, 1896.

SOLUTION AND DIFFUSION OF CERTAIN METALS IN MERCURY.

BY W. J. HUMPHREYS.

The investigation, of which this is a summary, was begun with the object of determining the extent to which these phenomena differ, if at all, in this case from the solution and diffusion of non-metallic solids and liquids.

The method of investigation was to fill a vessel of constant cross section with pure mercury, put on its surface a freshly amalgamated piece of the metal to be examined, and after allowing it to stand a definite length of time in a place free from external disturbances and of fairly constant temperature, to remove from known depths below the surface samples of the amalgam and analyse them.

The metals examined were lead, tin, zinc, bismuth, copper, and silver, and the results indicated that there is no essential difference between the solution and diffusion of these metals in mercury and the same phenomena in any other case.

Probably the most interesting results were those given by copper and silver, both of which dissolved to a much less extent than any of the other metals examined, but diffused more rapidly. At 28° C. the silver dissolved to the extent of only about one part in two thousand, and the copper to a still less extent—about three parts in a hundred thousand; while the rate of diffusion of the silver was about twenty millimetres per minute, approximately sixty times that of copper and fully six hundred that of zinc.

This investigation, of which the details will soon be published, was suggested to me by Dr. J. W. Mallet, F.R.S., of the University of Virginia, and carried out there under his supervision during the months of August and September, 1895.—*John Hopkins University Circulars*, February 1896.

THE

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PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[FIFTH SERIES.]

MAY 1896.

XLI. *On the Laws of Irreversible Phenomena.* *By* Dr. LADISLAS NATANSON, *Professor of Natural Philosophy in the University of Cracow* *.

IT can scarcely be doubted that the theory of dissipation of energy is still in its infancy. Reversible phenomena are well understood, but they do not involve dissipation at all; and what is known about irreversible phenomena is merely the qualitative aspect of their general laws. In fact, of the general quantitative laws of irreversible phenomena we are as yet utterly ignorant. Now I venture to think there is a general principle underlying irreversible phenomena which is easily seen to be consistent with fact in various cases well investigated: it is an extension of Hamilton's Principle, and (with much diversity, of course, as to form and generality) has been stated by Lord Rayleigh †, by Kirchhoff ‡, by v. Helmholtz §, and by M. Duhem ‖. It seems that propo-

* From the *Bulletin International de l'Académie des Sciences de Cracovie*, Mars 1896. Communicated by the Author.

† Proceedings of the London Mathematical Society, June 1873. The Theory of Sound,' i. p. 78 (1877).

‡ *Vorlesungen über Math.-Physik. Mechanik*, 1876, Vorlesung xi.

§ *Borchardt-Crelle's Journal f. Mathematik*, Bd. c. (1886); *Wissenschaftliche Abhandlungen*, Bd. iii. p. 203; *ibid.* Bd. ii. p. 958; Bd. iii. p. 119.

‖ *Journal de Mathématiques de Liouville-Jordan* (4) vol. viii. p. 269 (1892); vol. ix. p. 293 (1893); vol. x. p. 207 (1894). See further, Prof. J. J. Thomson's 'Applications of Dynamics to Physics and Chemistry,' London, 1888, where the fundamental standpoint is a very similar one, the chief object of investigation being, however, the theory of reversible phenomena.

sitions equivalent to those indicated by these investigators could be enunciated in the form of a simple and very general formula; we venture to think that the fundamental principle which it embodies is worth attention. Besides, it seems to afford the proper foundation for an attempt to arrive at some deeper insight into the laws of dissipation of energy.

Part I.

§ 1. *Introductory.*—Conceive a system: it may be either finite or infinitely small; it may be an independent system, or it may be only a part of some other system. Let the state of the system, at time t, be determined by the values of certain variable quantities, q_i, and of their first differential coefficients with respect to the time, s_i or dq_i/dt. We shall suppose that the energy of the system consists of two parts, the first of which, T, is a function of the q_i and the s_i, homogeneous of the second degree with respect to the s_i, and the second, say U, is a function of the q_i only. Let ϑ denote the absolute temperature of the system: ϑ may be an independent variable, or otherwise it must be a definite function of the variables. Suppose that the quantities q_i, s_i received certain arbitrarily chosen infinitesimal increments δq_i, δs_i; the energy T will then become $T+\delta T$, and U will become $U+\delta U$. Let then $\Sigma P_i \delta q_i$ be the work done on the system reversibly, during the transformation, by extraneous forces, and let δQ or $\Sigma R_i \delta q_i$ be the quantity of heat simultaneously absorbed by the system from the exterior; P_i will then be the generalized or Lagrangian extraneous "force" in the "direction" of the variable q_i, and R_i will be the "caloric coefficient," as it is called by M. Duhem, or the generalized "thermal capacity" of the system with respect to the variable q_i.

With respect to the quantity δQ we now make the following assumption, which we shall find is in accordance with fact. Let us suppose that every variation δq_i takes the special value dq_i or $s_i dt$; then the values of the variables q_i will become $q_i + dq_i$; the energies T and U will become $T+dT$, $U+dU$; the work done by external forces will be $\Sigma P_i dq_i$; and the quantity of heat absorbed will be dQ or $\Sigma R_i dq_i$. If now the variables be allowed to return to their primitive values q_i, T and U will resume their former values T, U, the external work $-\Sigma P_i dq_i$ will be done, but the quantity of heat absorbed will generally not be dQ but a different quantity, say d_1Q.

Write $$2d'Q = +dQ - d_1Q, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and $$+d^0Q = +dQ - d'Q; \quad . \quad . \quad . \quad . \quad (2)$$

therefore $$-d^0Q = -d_1Q - d'Q. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We have $$+dQ = +d^0Q + d'Q, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and $$-d_1Q = -d^0Q + d'Q; \quad . \quad . \quad . \quad . \quad . \quad (5)$$

thus d^0Q is the reversible, and $d'Q$ the irreversible part of the heat absorbed. Now, if we assume that these quantities are of the form

$$d^0Q = \Sigma R_i^0 dq_i; \quad d'Q = \Sigma R_i' dq_i, \quad . \quad . \quad . \quad . \quad (6)$$

we may consider the new quantities

$$\delta^0Q = \Sigma R_i^0 \delta q_i; \quad \delta'Q = \Sigma R_i' \delta q_i, \quad . \quad . \quad . \quad (7)$$

$\delta^0Q + \delta'Q$ gives again δQ. Let us generally define δQ, δ^0Q, $\delta'Q$ to represent the expressions which result if in the expressions of the quantities dQ, d^0Q, and $d'Q$ (which we suppose to be empirically known *), variations δq_i are substituted in place of the corresponding differentials dq_i.

§ 2. *Statement of the Principle.*—Let us consider a given period of time, from $t=t_0$ to $t=t_1$. Let δq_i, δs_i, δT, δU, $\Sigma P_i \delta q_i$, as usual, represent variations which, between the limits $t=t_0$ and $t=t_1$, are functions of the time susceptible of being differentiated, and which vanish at these limits themselves; finally, let δQ, δ^0Q, $\delta'Q$ be the corresponding infinitesimal expressions calculated as above stated. The following principle seems then to hold in physical phenomena: between $t=t_0$ and $t=t_1$ events which occur in the system must be such that the equation

$$\int_{t_0}^{t_1} dt\{\delta T - \delta U + \Sigma P_i \delta q_i + \delta Q\} = 0 \quad . \quad . \quad . \quad (\text{I.})$$

is satisfied. For brevity, this, when necessary, will be referred to as the *Thermokinetic Principle.*

* To write down the expressions of dQ and $d'Q$, a much greater number of variables would evidently be required in most cases than to write d^0Q; thus in most cases many of the coefficients R_i^0 will be equal to zero. A similar remark applies to the coefficients P_i, $\partial T/\partial q_i$, $\partial T/\partial s_i$, and $\partial U/\partial q_i$.

§ 3. *Lagrangian Equations.*—From (I.), remembering the definitions laid down, we obtain by a well-known calculation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial s_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial U}{\partial q_i}-P_i-R_i=0. \quad . \quad . \quad (II.)$$

These equations, a thermokinetic extension of Lagrange's well-known dynamical equations, have been given implicitly by Helmholtz and explicitly by M. Duhem; the form they take in an important particular case had been previously explained by Lord Rayleigh.

§ 4. *Conservation of Energy.*—Considering a real transformation dq_i, ds_i, multiply each of these equations by $s_i dt$ respectively, and add; we find

$$dT+dU-\Sigma P_i dq_i-dQ=0. \quad . \quad . \quad . \quad . \quad (1)$$

The principle of conservation of energy in its general form is thus seen to follow from the thermokinetic principle. That inversely the thermokinetic principle cannot be deduced from conservation of energy is an obvious proposition which scarcely requires special mention.

§ 5. *Free Energy.*—We shall suppose in the following (except when the contrary is expressly stated) that *one* independent variable is the temperature; and accordingly we shall use q_i to indicate all the *other* variables. That work is not required for merely changing the temperature of a system is an experimental fact; hence, when the variables ϑ, q_i, and s_i receive increments $\delta\vartheta$, δq_i, δs_i, the work done on the system will be still $\Sigma P_i \delta q_i$ (in our present modified notation) and no term including $\delta\vartheta$ will appear. Variables with such properties attributed to them have been employed by Lord Kelvin as long ago as 1855; they have been often adopted in general thermodynamical investigations. M. Duhem calls them "normal" variables.

Let us suppose that ϑ, q_i represent a system of "normal" variables. Write

$$\Sigma\frac{\partial U}{\partial q_i}\delta q_i-\Sigma R_i^0\delta q_i=\Sigma\frac{\partial V}{\partial q_i}\delta q_i. \quad . \quad . \quad . \quad (1)$$

The function V, if it exists, will be called *the free energy* of the system, because, as we shall find hereafter, V defined by equation (1) will agree in the case of Reversible Thermo-

dynamics with what, from Helmholtz, received that designation. Equation (I.) accordingly becomes

$$\int_{t_0}^{t_1} dt \left\{ \delta T - \Sigma \frac{\partial V}{\partial q_i} \delta q_i + \Sigma P_i \delta q_i - \left(\frac{\partial U}{\partial \vartheta} - R_\vartheta^0 \right) \delta\vartheta + \delta' Q \right\} = 0. \quad (2)$$

Now let us further assume that the following equations are true :—

$$\frac{\partial T}{\partial \vartheta} = 0 \; ; \quad \frac{\partial T}{\partial \frac{d\vartheta}{dt}} = 0 \; ; \quad . \quad . \quad . \quad . \quad . \quad (3)$$

they are found to hold good in all cases of which we have precise knowledge ; lastly, let us suppose that there is no term containing $\delta\vartheta$ in the expression for $\delta' Q$. (With respect to this point compare § 12.) Equation (2) may now be divided into

$$\frac{\partial U}{\partial \vartheta} - R_\vartheta^0 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$\int_{t_0}^{t_1} dt \left\{ \delta T - \Sigma \frac{\partial V}{\partial q_i} \delta q_i + \Sigma P_i \delta q_i + \delta' Q \right\} = 0. \quad . \quad (\text{III.})$$

This equation expresses the principle in a form similar to that of equation (I.). It is a useful equation, owing to the readiness with which it admits of application in various cases, but its abstract generality is of course much more restricted than that of the fundamental equation.

§ 6. *Reversible Dynamics.*—In Dynamics properly so-called, *i. e.* in Reversible Dynamics, ideal phenomena of motion are dealt with, and the notion of temperature is not taken into account. Therefore, in Reversible Dynamics a function V can be considered, depending on the remaining variables q_i only, which does not differ, except by a constant, from the "potential energy" U ; this is a remark already made by M. Duhem. Of course it must be restricted to the Dynamics of points and of rigid bodies, since, for instance, in Hydrodynamics and Aerodynamics the difference between the quantities V and U is variable and depends on the compressibility of the fluid.

From (III.) we obtain, leaving out the irreversible term $\delta' Q$, the fundamental principle of Reversible Dynamics.

§ 7. *Electromagnetic irreversible phenomena.*—Energy stored in the æther can be transferred to matter and converted into heat; this phenomenon, when it occurs, is a thoroughly irreversible one.

Here, therefore, we may put $R_i^0=0$ and $\partial U/\partial q_i=\partial V/\partial q_i$; and the equation will be

$$\int_{t_0}^{t_1} dt\{\delta T-\delta U+\Sigma P_i\,\delta q_i+\delta' Q\}=0. \quad . \quad . \quad (1)$$

We shall return to this case in § 13 below.

§ 8. *Reversible Thermodynamics*.—At present the immediate object of the science called Thermodynamics is the study of states of equilibrium. The modifications assumed in Thermodynamics to occur in a system are, for that reason, virtual reversible transformations which lead from one state of equilibrium to another one. Let us admit the following assumptions:—first, that that part of the energy which we call T is a constant quantity; secondly, that the variables are "normal" variables; thirdly, if a function of the variables ϑ and q_i, called the entropy, be denoted by S, that the term $\delta^0 Q$ is of the form

$$\vartheta\frac{\partial S}{\partial\vartheta}\delta\vartheta+\vartheta\Sigma\frac{\partial S}{\partial q_i}\delta q_i; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and, lastly, that the supposed transformation being reversible, the term $\delta' Q$ is equal to zero. Hence the laws of ordinary Thermodynamics must be contained in

$$\frac{\partial U}{\partial\vartheta}-\vartheta\frac{\partial S}{\partial\vartheta}=0, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$\int_{t_0}^{t_1} dt\{-\Sigma\frac{\partial V}{\partial q_i}\delta q_i+\Sigma P_i\delta q_i\}=0, \quad . \quad . \quad . \quad (3)$$

where

$$\frac{\partial V}{\partial q_i}=\frac{\partial U}{\partial q_i}-\vartheta\frac{\partial S}{\partial q_i}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Since the adopted variables ϑ, q_i are "normal" ones, we are at liberty to define the quantity $U-\vartheta S$ as representing what in § 5 has been called the *free energy* of the system; hence

$$-\frac{\partial V}{\partial q_i}+P_i=0;\ \frac{\partial V}{\partial\vartheta}+S=0, \quad . \quad . \quad . \quad (5)$$

and thus we are led to that well-known form of thermodynamical equations which we have learned from MM. Massieu, Gibbs, Duhem, Helmholtz, and others.

§ 9. *Irreversible Dynamics*.—Let us now proceed to consider cases of motion bearing perfect analogy with ordinary

dynamical phenomena, except that, being irreversible, they do not satisfy the condition $d'Q=0$. Lord Rayleigh has shown how in many cases we can put

$$d'Q=-2Fdt, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

employing F to indicate a function of the variables q_i (supposed to be "normal" ones) and s_i, homogeneous of the second degree with respect to the s_i, which he calls the Dissipation Function. The assumption we make is therefore that

$$d'Q=-dt\Sigma s_i\frac{\partial F}{\partial s_i}=-\Sigma\frac{\partial F}{\partial s_i}dq_i\,; \quad . \quad . \quad . \quad (2)$$

and that

$$R'_i=-\frac{\partial F}{\partial s_i}\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

following the rule laid down in § 1 we put

$$\delta'Q=-\Sigma\frac{\partial F}{\partial s_i}\delta q_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and from (III.) we obtain

$$\int_{t_0}^{t_1}dt\{\delta T-\Sigma\frac{\partial V}{\partial q_i}\delta q_i+\Sigma P_i\delta q_i-\Sigma\frac{\partial F}{\partial s_i}\delta q_i\}=0\,; \quad . \quad (5)$$

hence

$$\frac{d}{dt}\left(\frac{\partial T}{\partial s_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}-P_i+\frac{\partial F}{\partial s_i}=0. \quad . \quad . \quad (6)$$

These are Lord Rayleigh's equations, with V written in the place of U.

§ 10. *Irreversible Hydrodynamics.*—Let us now proceed to consider a viscous fluid; we shall call μ its coefficient of viscosity. Owing to the viscosity of the fluid its motion is accompanied by irreversible production of heat; owing to its compressibility, there is reversible production or destruction of heat. We shall suppose that every such loss or gain in every element of the fluid is being immediately and exactly compensated, so that the temperature of the element remains constant. At the interior of a large quantity of fluid we take a portion, of mass $\iiint dx\,dy\,dz\,\rho$, ρ being the density at the point (x, y, z). Let p be the ordinary mean pressure; u, v, w the components of the velocity, X, Y, Z the components of the extraneous acceleration, at the point (x, y, z) and time t. The equations of motion, as given by Navier, Poisson, Stokes, and Maxwell, are as follows:—

$$-\rho\frac{du}{dt}+\rho X-\frac{\partial p}{\partial x}+\mu\nabla^2 u+\tfrac{1}{3}\mu\frac{\partial\theta}{\partial x}=0, \quad . \quad . \quad (1)$$

. . . . . .

. . . . . .

with the usual signification of ∇^2 and θ. Write

$$p_x dS, \quad p_y dS, \quad p_z dS \quad . \quad . \quad . \quad . \quad . \quad (2)$$

for the pressures, parallel to the co-ordinate axes, on the element dS of the boundary of the portion we are considering. If the direction of the inwardly directed normal be denoted by n, we shall have :—

$$p_x=\left[p-2\mu\left(\frac{\partial u}{\partial x}-\tfrac{1}{3}\theta\right)\right]\cos(nx)-\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\cos(ny)$$
$$-\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\cos(nz); \quad . \quad . \quad (3)$$

$$p_y=-\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\cos(nx)+\left[p-2\mu\left(\frac{\partial v}{\partial y}-\tfrac{1}{3}\theta\right)\right]\cos(ny)$$
$$-\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\cos(nz); \quad . \quad (4)$$

$$p_z=-\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\cos(nx)-\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\cos(ny)$$
$$+\left[p-2\mu\left(\frac{\partial w}{\partial z}-\tfrac{1}{3}\theta\right)\right]\cos(nz). \quad . \quad (5)$$

If now a system δx, δy, δz of infinitesimal virtual displacements be imposed upon the fluid, the temperature being kept constant, then the work $\Sigma P_i \delta q_i$ done by extraneous forces will be

$$\iint dS(p_x\delta x+p_y\delta y+p_z\delta z)+\iiint dx\,dy\,dz\,\rho(X\delta x+Y\delta y+Z\delta z); \quad (6)$$

the variation of the energy T will be

$$\delta T=\iiint dx\,dy\,dz\,\rho(u\delta u+v\delta v+w\delta w); \quad . \quad . \quad . \quad (7)$$

the variation of the energy V, which in Hydrodynamics it is usual * to call "intrinsic" energy, will be

* See, for example, that otherwise excellent treatise 'Hydrodynamics' by Prof. Lamb, ed. 1895, pp. 11–12, 469, 507. It is not with the true intrinsic energy U, but with the *free* energy V that we are here concerned; the customary use of the word "intrinsic" seems, therefore, to involve a serious error.

$$\delta V = -\iiint dx\,dy\,dz\,p\left(\frac{\partial \delta x}{\partial x} + \frac{\partial \delta y}{\partial y} + \frac{\partial \delta z}{\partial z}\right), \quad . \quad . \quad (8)$$

and, lastly, the quantity of heat which must be "absorbed" in order to compensate the effects of viscosity will be

$$d'Q = -\iiint dx\,dy\,dz\,2\mu dt \left\{ \begin{array}{c} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 - \frac{1}{3}\theta^2 \\ + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 \end{array} \right\}$$

$$= -\iiint dx\,dy\,dz\,2\mu \left\{ \begin{array}{c} \left(\frac{\partial u}{\partial x} - \frac{1}{3}\theta\right)\frac{\partial dx}{\partial x} + \left(\frac{\partial v}{\partial y} - \frac{1}{3}\theta\right)\frac{\partial dy}{\partial y} + \left(\frac{\partial w}{\partial z} - \frac{1}{3}\theta\right)\frac{\partial dz}{\partial z} \\ + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\left(\frac{\partial dz}{\partial y} + \frac{\partial dy}{\partial z}\right) \\ + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\left(\frac{\partial dx}{\partial z} + \frac{\partial dz}{\partial x}\right) \\ + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\left(\frac{\partial dy}{\partial x} + \frac{\partial dx}{\partial y}\right). \end{array} \right\} (9)$$

In order, therefore, to comply with the rule respecting $\delta'Q$ we have to write

$$\delta'Q = -\iiint dx\,dy\,dz\,2\mu \left\{ \begin{array}{c} \left(\frac{\partial u}{\partial x} - \frac{1}{3}\theta\right)\frac{\partial \delta x}{\partial x} + \left(\frac{\partial v}{\partial y} - \frac{1}{3}\theta\right)\frac{\partial \delta y}{\partial y} \\ + \left(\frac{\partial w}{\partial z} - \frac{1}{3}\theta\right)\frac{\partial \delta z}{\partial z} + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\left(\frac{\partial \delta z}{\partial y} + \frac{\partial \delta y}{\partial z}\right) \\ + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\left(\frac{\partial \delta x}{\partial z} + \frac{\partial \delta z}{\partial x}\right) \\ + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\left(\frac{\partial \delta y}{\partial x} + \frac{\partial \delta x}{\partial y}\right). \end{array} \right\} (10)$$

Let us now verify whether in the present case our general principle applies. From (1) we find

$$\int_{t_0}^{t_1} dt \iiint dx\,dy\,dz \left\{ \begin{array}{c} \left[-\rho\frac{du}{dt} + \rho X - \frac{\partial p}{\partial x} + \mu\nabla^2 u + \frac{1}{3}\mu\frac{\partial \theta}{\partial x}\right]\delta x \\ + [\ldots\ldots]\,\delta y + [\ldots\ldots]\,\delta z \end{array} \right\} = 0; \quad (11)$$

now

$$\nabla^2 u + \frac{1}{3}\frac{\partial \theta}{\partial x} = 2\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} - \frac{1}{3}\theta\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \quad . \quad (12)$$

so that from (8), (10), (12), and again from (3), (4), (5) we obtain

$$\iiint dx\,dy\,dz\left\{\left[-\frac{\partial p}{\partial x}+\mu\nabla^2 u+\tfrac{1}{3}\mu\frac{\partial\theta}{\partial x}\right]\delta x+[\ldots]\delta y+[\ldots]\,\delta z\right\}$$

$$=-\delta V+\delta' Q-\iiint dx\,dy\,dz\left\{\frac{\partial}{\partial x}(p\delta x)+\frac{\partial}{\partial y}(p\delta y)+\frac{\partial}{\partial z}(p\delta z)\right\}$$

$$+\iiint dx\,dy\,dz\,2\mu\left\{\frac{\partial}{\partial x}\left[\left(\frac{\partial u}{\partial x}-\tfrac{1}{3}\theta\right)\delta x\right]+\frac{\partial}{\partial y}\left[\left(\frac{\partial v}{\partial y}-\tfrac{1}{3}\theta\right)\delta y\right]+\frac{\partial}{\partial z}\left[\left(\frac{\partial w}{\partial z}-\tfrac{1}{3}\theta\right)\right]\delta z\right\}$$

$$+\iiint dx\,dy\,dz\,\mu\left\{\begin{array}{l}\frac{\partial}{\partial y}\left[\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\delta z\right]+\frac{\partial}{\partial z}\left[\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\delta y\right]+\\ \frac{\partial}{\partial z}\left[\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\delta x\right]+\frac{\partial}{\partial x}\left[\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\delta z\right]+\\ \frac{\partial}{\partial x}\left[\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\delta y\right]+\frac{\partial}{\partial y}\left[\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\delta x\right]\end{array}\right\}$$

$$=-\delta V+\delta' Q+\iint dS(p_x\,\delta x+p_y\,\delta y+p_z\,\delta z). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Further, we see that

$$-\int_{t_0}^{t_1} dt\iiint dx\,dy\,dz\,\rho\left(\frac{du}{dt}\delta x+\frac{dv}{dt}\delta y+\frac{dw}{dt}\delta z\right)=\int_{t_0}^{t_1} dt\,\delta T, \quad (14)$$

because $dx\,dy\,dz\rho$ does not vary; and collecting our results we see that (11) reduces to

$$\int_{t_0}^{t_1} dt\{\delta T-\delta V+\Sigma P_i\,\delta q_i+\delta' Q\}=0, \quad . \quad . \quad . \quad (14)$$

with equations (7), (8), (6), (10) to define the terms within brackets.

§ 11. *Diffusion.* — We next take two gases which are diffusing into one another. Let the masses of the portions considered be $\iiint dx_1\,dy_1\,dz_1\,\rho_1$ and $\iiint dx_2\,dy_2\,dz_2\,\rho_2$; and S_1 and S_2 their respective boundaries. When the motion of the gases is going on, three irreversible phenomena will occur, viz., internal friction in the first gas, internal friction in the second, and mutual interdiffusion of both; in the following both the first and the second are neglected. Let again u_1, v_1, w_1, u_2, v_2, w_2 denote the velocity components, X_1, Y_1, Z_1, X_2, Y_2, Z_2 those of extraneous acceleration, p_1, p_2 the mean pressure, at the time t and a given point of space, where at that time both the elements $dx_1\,dy_1\,dz_1$ and $dx_2\,dy_2\,dz_2$ of the gases happen to be momentarily situated. The quantities u_1 and u_2, v_1 and v_2,

w_1 and w_2 being, however, quite different, the elements will, of course, separate after a time infinitely short; likewise δx_1 and δx_2, δy_1 and δy_2, δz_1 and δz_2 must be understood to be quite independent variations. Let us write

$$\int_{t_0}^{t_1} dt\{\delta T - \delta V + \Sigma P_i \delta q_i + \delta' Q\} = 0, \quad . \quad . \quad . \quad (1)$$

and let us adopt, as a definition of the terms, the following equations:—

$$\delta T = \delta T_1 + \delta T_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\delta V = \delta V_1 + \delta V_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$T_1 = \tfrac{1}{2}\iiint dx_1\, dy_1\, dz_1\, \rho_1(u_1^2 + v_1^2 + w_1^2), \quad . \quad . \quad (4)$$

$$T_2 = \tfrac{1}{2}\iiint dx_2\, dy_2\, dz_2\, \rho_2(u_2^2 + v_2^2 + w_2^2), \quad . \quad . \quad (5)$$

$$\delta V_1 = -\iiint dx_1\, dy_1\, dz_1\, p_1\left(\frac{\partial \delta x_1}{\partial x_1} + \frac{\partial \delta y_1}{\partial y_1} + \frac{\partial \delta z_1}{\partial z_1}\right), \quad . \quad . \quad (6)$$

$$\delta V_2 = -\iiint dx_2\, dy_2\, dz_2\, p_2\left(\frac{\partial \delta x_2}{\partial x_2} + \frac{\partial \delta y_2}{\partial y_2} + \frac{\partial \delta z_2}{\partial z_2}\right), \quad . \quad . \quad (7)$$

$$\begin{aligned}\Sigma P_i \delta q_i = {} & \iiint dx_1\, dy_1\, dz_1\, \rho_1\, (X_1\, \delta x_1 + Y_1\, \delta y_1 + Z_1\, \delta z_1) \\ & + \iiint dx_2\, dy_2\, dz_2\, \rho_2\, (X_2\, \delta x_2 + Y_2\, \delta y_2 + Z_2\, \delta z_2) \\ & + \iiint dS_1\, p_1\{\cos(n_1 x)\, \delta x_1 + \cos(n_1 y)\, \delta y_1 + \cos(n_1 z)\, \delta z_1\} \\ & + \iiint dS_2\, p_2\{\cos(n_2 x)\, \delta x_2 + \cos(n_2 y)\, \delta y_2 + \cos(n_2 z)\, \delta z_2\}. \quad (8)\end{aligned}$$

The quantity of heat generated in time dt by diffusion may be written

$$\begin{aligned}\iiint dx\, dy\, dz\, A\rho_1\rho_2\{(u_2 - u_1)(dx_2 - dx_1) + (v_2 - v_1)(dy_2 - dy_1) \\ + (w_2 - w_1)(dz_2 - dz_1)\}, \quad (9)\end{aligned}$$

the expression $dx\, dy\, dz$ being understood to mean indifferently $dx_1\, dy_1\, dz_1$ or $dx_2\, dy_2\, dz_2$, and A being a constant coefficient intimately connected with the "coefficient of diffusion" of the gases. If the temperature is to remain constant, the quantity (9) must be taken away; hence

$$\begin{aligned}\delta' Q = -\iiint dx\, dy\, dz\, A\rho_1\rho_2\{(u_2 - u_1)(\delta x_2 - \delta x_1) \\ + (v_2 - v_1)(\delta y_2 - \delta y_1) + (w_2 - w_1)(\delta z_2 - \delta z_1)\}. \quad (10)\end{aligned}$$

Substituting (2), (3), ... (8), and (10) in (1), we find

$$\int_{t_0}^{t_1} dt \iiint dx\,dy\,dz \left\{ \begin{array}{l} \left[-\rho_1 \dfrac{du_1}{dt} - \dfrac{\partial p_1}{\partial x_1} + \rho_1 X_1 + A\rho_1\rho_2(u_2 - u_1)\right]\delta x_1 \\ \qquad\qquad + [\,.\,.\,.\,.\,]\delta y_1 + [\,.\,.\,.\,.\,]\delta z_1 \\ + \left[-\rho_2 \dfrac{du_2}{dt} - \dfrac{\partial p_2}{\partial x_2} + \rho_2 X_2 + A\rho_2\rho_1(u_1 - u_2)\right]\delta x_2 \\ \qquad\qquad + [\,.\,.\,.\,.\,]\delta y_2 + [\,.\,.\,.\,.\,]\delta z_2 \end{array} \right\} = 0, \quad (11)$$

which shows at once that

$$\rho_1 \frac{du_1}{dt} + \frac{\partial p_1}{\partial x_1} = \rho_1 X_1 + A\rho_1\rho_2(u_2 - u_1), \text{ \&c.} \quad . \quad (12)$$

$$\rho_2 \frac{du_2}{dt} + \frac{\partial p_2}{\partial x_2} = \rho_2 X_2 + A\rho_2\rho_1(u_1 - u_2), \text{ \&c.} \quad . \quad (13)$$

These equations have been established long ago by Maxwell and Stefan.

§ 12. *Conduction of Heat.*—Fourier's equation of conduction of heat appears to belong to the class of conservation of energy equations. At first let us avoid employing "normal" variables. Since the motion of the medium and the intervention of extraneous forces are immaterial for conduction of heat, we may put $\delta T = 0$, $\Sigma P_i \delta q_i = 0$, and $d^0 Q = 0$; therefore $\delta U = \delta V$, and

$$\int_{t_0}^{t_1} dt\{-\delta U + \delta' Q\} = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$-\frac{\partial U}{\partial q_i} + R_i' = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence, in any real transformation, we have

$$-\frac{dU}{dt} + \frac{d'Q}{dt} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

that is to say, in "normal" variables :—

$$\frac{\partial U}{\partial \vartheta}\frac{d\vartheta}{dt} = -\Sigma \frac{\partial U}{\partial q_i}\frac{dq_i}{dt} + \frac{d'Q}{dt} \quad . \quad . \quad . \quad . \quad (4)$$

This is the general form of Fourier's equation; usually $\partial U/\partial \vartheta$ is assumed to be of the form $dx\,dy\,dz\rho c_v$ in an element $dx\,dy\,dz$, ρ being the density and c_v the well-known thermal capacity; and the remaining $\partial U/\partial q_i$ are usually neglected. We shall reconsider the present case from a different standpoint in § 19.

§ 13. *Electromagnetic Dissipation.* — In Helmholtz's memoir "Das Princip der kleinsten Wirkung in der Elektrodynamik" *, it is shown in great generality that the thermokinetic principle holds for electromagnetic phenomena; nevertheless we beg leave to consider here the simplest (but for our purpose most important) case, to which in the second part of the paper we shall have again to refer. Consider isotropic conducting substances, at rest. Let us suppose that energy-dissipation of the simplest or Joulean type is the only possible irreversible phenomenon. Let the components of electromotive intensity at the point (x, y, z) be E_x, E_y, E_z; those of extraneous electric forces, F_x, F_y, F_z; and the components of magnetic force H_x, H_y, H_z. Let C be the electric conductivity, K the dielectric inductive capacity, and μ the magnetic permeability. If

$$E_x = -\frac{dA_x}{dt}, \quad E_y = -\frac{dA_y}{dt}, \quad E_z = -\frac{dA_z}{dt}, \quad . \quad . \quad (1)$$

then the vector A, whose components A_x, A_y, A_z are, may be taken to represent the "electromagnetic momentum" at (x, y, z). We assume that

$$\left.\begin{aligned} 4\pi C(E_x - F_x) + K\frac{dE_x}{dt} &= \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}, \\ 4\pi C(E_y - F_y) + K\frac{dE_y}{dt} &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}, \\ 4\pi C(E_z - F_z) + K\frac{dE_z}{dt} &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}; \end{aligned}\right\} \quad . \quad . \quad (2)$$

$$\left.\begin{aligned} \mu H_x &= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \\ \mu H_y &= \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \\ \mu H_z &= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Mr. Heaviside and H. Hertz, it is well known, have constructed the whole of Maxwell's Theory upon two systems of equations, one of which is the system (2) above, whilst the second follows at once from (1) and (3). We shall take A_x, A_y, A_z to be the independent variables; that is the choice

* *Sitz. Berl. Akad.* 12 Mai 1892; *Wiss. Abh.* Bd. iii. p. 476. See also Boltzmann, *Vorlesungen über Maxwell's Theorie*, vol. ii. p. 7.

which Lord Kelvin, Prof. Boltzmann, and other writers adopted when endeavouring to find dynamical analogies for electromagnetic phenomena. The part of the energy, called T, which depends on the quantities dA_x/dt, dA_y/dt, dA_z/dt, will be then the electric energy

$$T = \frac{1}{8\pi}\iiint dx\,dy\,dz\,K(E_x^2 + E_y^2 + E_z^2)\,; \quad . \quad . \quad (4)$$

the other U, connected with the collocation of the variables A_x, A_y, A_z themselves, will be

$$U = \frac{1}{8\pi}\iiint dx\,dy\,dz\,\mu(H_x^2 + H_y^2 + H_z^2). \quad . \quad . \quad (5)$$

Supposing δA_x, δA_y, δA_z to be variations as usual and K, C, μ and $dx\,dy\,dz$ not to be subject to variation, we shall have

$$\frac{1}{4\pi}\int_{t_0}^{t_1} dt \iiint dx\,dy\,dz \left\{ \begin{array}{c} \left[K\frac{dE_x}{dt} - \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) - 4\pi C F_x + 4\pi C E_x\right]\delta A_x \\ + [\,.\,.\,.\,.\,]\delta A_y + [\,.\,.\,.\,.\,]\delta A_z. \end{array} \right\} = 0. \quad (6)$$

This equation is readily transformed. First:

$$\frac{1}{4\pi}\int_{t_0}^{t_1} dt \iiint dx\,dy\,dz\,K\left(\frac{dE_x}{dt}\delta A_x + \frac{dE_y}{dt}\delta A_y + \frac{dE_z}{dt}\delta A_z\right) = \int_{t_0}^{t_1} dt\,\delta T, \quad (7)$$

and

$$-\int_{t_0}^{t_1} dt \iiint dx\,dy\,dz\,C(F_x\delta A_x + F_y\delta A_y + F_z\delta A_z) = \int_{t_0}^{t_1} dt\,\Sigma P_i \delta q_i. \quad . \quad (8)$$

Then, from the well-known Maxwell-Helmholtz principle, on the continuity of properties on surfaces of separation between different media, and from the equations (3), we obtain

$$-\frac{1}{4\pi}\int_{t_0}^{t_1} dt \iiint dx\,dy\,dz \left\{ \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\delta A_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\delta A_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\delta A_z \right\}$$

$$= \frac{1}{4\pi}\int_{t_0}^{t_1} dt \iiint dx\,dy\,dz\,\frac{1}{\mu} \left\{ \begin{array}{l} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\frac{\partial \delta A_x}{\partial y} - \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\frac{\partial \delta A_x}{\partial z} \\ + \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\frac{\partial \delta A_y}{\partial z} - \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\frac{\partial \delta A_y}{\partial x} \\ + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\frac{\partial \delta A_z}{\partial x} - \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\frac{\partial \delta A_z}{\partial y} \end{array} \right\}$$

$$= -\int_{t_0}^{t_1} dt\,\delta U. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If, therefore, the general principle is applicable here, the terms in (6) containing $4\pi CE_x$, &c., should reduce to

$$+\int_{t_0}^{t_1} dt\, \delta' Q. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Now the quantity of energy which becomes absorbed from the æther and converted into heat is, for the time dt and the volume $\iiint dx\, dy\, dz$,

$$-\iiint dx\, dy\, dz\, C(E_x dA_x + E_y dA_y + E_z dA_z)\ ; \quad . \quad (11)$$

hence

$$\delta' Q = \iiint dx\, dy\, dz\, C(E_x \delta A_x + E_y \delta A_y + E_z \delta A_z)\ ; \quad . \quad (12)$$

and thus the principle contained in (I.), or in (III.), is again seen to hold good.

Part II.

§ 14. *Introductory.*—The foregoing naturally raises the question, Does a *general* law exist concerning the infinitesimal expressions $d'Q$ and $\delta' Q$, which have been found to characterize dissipation of energy in the various particular cases discussed? I venture to answer this in the affirmative; but the hypothesis I advance does not profess to be more than a conjecture and an approximation.

Let us consider in every particular case the quantity

$$\frac{d'Q}{dt} = -2F \text{ say.} \quad . \quad . \quad . \quad . \quad . \quad (\text{IV.})$$

In the case of irreversible Dynamics, § 9, the function F is well known, and has been called by Lord Rayleigh the "Dissipation Function;" I should suggest that this term be extended to all cases covered by equation (IV.).

Let us imagine a material (or at any rate partly material) system. Suppose that it is *not* in equilibrium, and observe, in a quantitative manner, the disturbances which its state involves. Let it be isolated so as not to be disturbed by extraneous action. We know from experience that under such circumstances the disturbances in the system must finally subside and tend to disappear. This general behaviour may be called *the coercion* of disturbances, because of the contrast it offers with inertia. (See Phil. Mag. for June 1895, p. 509.) For definiteness let us consider a continuous body. Let $dx\, dy\, dz\, \rho$ be the mass of an element $dx\, dy\, dz$, and let $dx\, dy\, dz\, \rho f$ represent its dissipation function, so that F, the dissipation function for the portion $\iiint dx\, dy\, dz$ of the body,

be $=\iiint dx\,dy\,dz\,\rho f$. Then, generally speaking, F is susceptible of three kinds of variation, and dF/dt is the sum of three terms:—1. A surface-integral relating to the action between the body and the exterior world through the boundary of the body; 2. A volume-integral expressing "action at a distance" between the body and the exterior world; and 3. A volume-integral representing "coercion," *i. e.*, that intimate action whose constant tendency it is to attenuate and finally to efface inequalities and disturbances, if there is no extraneous action to maintain or to excite them; and whose ultimate nature is, of course, unknown to us. It would not be difficult to translate our statement into symbols. Let us adopt, for instance, that general Molecular Theory due to Maxwell, which we have called (on a former occasion) "Kinematical Molecular Theory." Let $u+\xi$, $v+\eta$, $w+\zeta$ denote the components of the velocities of individual molecules, f a function of the $(u+\xi)$, $(v+\eta)$, and $(w+\zeta)$, $\bar{f}$ the mean value of f within an element, and D/Dt the rate of "coercion." Then

$$\rho\frac{d\bar{f}}{dt}=-\left\{\frac{\partial}{\partial x}(\rho\overline{\xi f})+\frac{\partial}{\partial y}(\rho\overline{\eta f})+\frac{\partial}{\partial z}(\rho\overline{\zeta f})\right\}$$

$$+\rho\left(X\frac{\partial\bar{f}}{\partial u}+Y\frac{\partial\bar{f}}{\partial v}+Z\frac{\partial\bar{f}}{\partial w}\right)+\rho\frac{D\bar{f}}{Dt}, \quad . \quad . \quad . \quad (1)$$

hence

$$\frac{dF}{dt}=\frac{d}{dt}\iiint dx\,dy\,dz\,\rho\bar{f}$$

$$=\iiint dS\rho\{\overline{\xi f}\cos(nx)+\overline{\eta f}\cos(ny)+\overline{\zeta f}\cos(nz)\}$$

$$+\iiint dx\,dy\,dz\,\rho\left(X\frac{\partial\bar{f}}{\partial u}+Y\frac{\partial\bar{f}}{\partial v}+Z\frac{\partial\bar{f}}{\partial w}\right)+\iiint dx\,dy\,dz\,\rho\frac{D\bar{f}}{Dt}. \quad (2)$$

The three terms on the right-hand side refer to the three kinds of variation as above stated.

The assumption we propose to examine is that the third, or coercive term DF/Dt is always proportional to the value of F. Thus, writing τ for a constant period of time,

$$\frac{DF}{Dt}=-\frac{2F}{\tau}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (V.)$$

This equation, we shall find, is general; in the neighbourhood of states of equilibrium at least it is exactly fulfilled. The period of time τ was first considered by Clerk-Maxwell;

in an important case it received the name of *the modulus of the time of relaxation** and may, without inconvenience, be called so in other similar cases.

Equation (V.) may be verified in various cases, which we shall take in order.

§ 15. *Irreversible Hydrodynamics.*—From § 10 we have

$$\mathrm{F}=\iiint dx\,dy\,dz\,\mu\left\{\begin{array}{c}\left(\frac{\partial u}{\partial x}-\frac{1}{3}\theta\right)^2+\left(\frac{\partial v}{\partial y}-\frac{1}{3}\theta\right)^2+\left(\frac{\partial w}{\partial z}-\frac{1}{3}\theta\right)^2\\ +\frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\end{array}\right\}. \quad (1)$$

Writing $p_{xx}, \ldots\ldots p_{xy}$ for the usual components, we have

$$p_{xx}-p=-2\mu\left(\frac{\partial u}{\partial x}-\frac{1}{3}\theta\right);\ p_{yz}=-\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right), \quad . \quad (2)$$

and four other equations of the same form. These equations, it is well known, must be fulfilled if the dynamical equations of Navier, Poisson, Stokes, and Maxwell are to be true; they may be described, therefore, as being in agreement with experience, and so also may be equation (1). Hence

$$\mathrm{F}=\iiint dx\,dy\,dz\frac{1}{4\mu}\left\{\begin{array}{c}(p_{xx}-p)^2+(p_{yy}-p)^2+(p_{zz}-p)^2\\ +2p_{yz}^2+2p_{zx}^2+2p_{xy}^2.\end{array}\right\}. \quad (3)$$

Again, if the disturbance is not a very violent one, we have the equations †

$$\frac{\mathrm{D}(p_{xx}-p)}{\mathrm{D}t}=2p\left(\frac{\partial u}{\partial x}-\frac{1}{3}\theta\right);\ \frac{\mathrm{D}p_{yz}}{\mathrm{D}t}=p\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right), \quad . \quad (4)$$

and four other equations, to be written down from symmetry; it may be well to point out that they are "kinematical" equations, therefore independent of any particular molecular hypothesis. Now, if we put

$$\frac{\mu}{p}=\tau, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

we obtain from (2) and (4)

$$\frac{\mathrm{D}(p_{xx}-p)}{\mathrm{D}t}=-\frac{p_{xx}-p}{\tau};\ \frac{\mathrm{D}p_{yz}}{\mathrm{D}t}=-\frac{p_{yz}}{\tau}, \quad . \quad . \quad . \quad (6)$$

* Philosophical Transactions, 1867, p. 82. See also 'Treatise on Electricity and Magnetism,' third edition, vol. i. p. 451.

† Philosophical Transactions, 1867, p. 81.

and four similar equations; and from (3) and (6)

$$F=-\iiint dx\,dy\,dz\,\frac{\tau}{4\mu}\left\{\begin{array}{c}(p_{xx}-p)\dfrac{D(p_{xx}-p)}{Dt}+(p_{yy}-p)\dfrac{D(p_{yy}-p)}{Dt}\\ +(p_{zz}-p)\dfrac{D(p_{zz}-p)}{Dt}\\ +2p_{yz}\dfrac{Dp_{yz}}{Dt}+2p_{zx}\dfrac{Dp_{zx}}{Dt}+2p_{xy}\dfrac{Dp_{xy}}{Dt}\end{array}\right\}, \quad (7)$$

whence, by (3), we infer that $DF/Dt=-2F/\tau$, as stated above.

The value of τ in air, at the temperature 0° C. and normal pressure, is approximately 2.10^{-10} of a second (Maxwell, Phil. Trans. 1867, p. 83). We may also compare the relative values of τ in two fluids. In doing so we may assume, in accordance with Prof. Van der Waals' leading idea, that the values of τ would bear a constant proportion if they were calculated for "corresponding" states of the fluids *. Hence the coefficients of viscosity will likewise, in corresponding states of two fluids, bear a constant numerical ratio †.

§ 16. *Diffusion.*—The signification of the symbols being the same as in § 11, we find the dissipation function of diffusion to be

$$F=\tfrac{1}{2}\iiint dx\,dy\,dz\,A\rho_1\rho_2\{(u_2-u_1)^2+(v_2-v_1)^2+(w_2-w_1)^2\}. \quad (1)$$

The theory of diffusion can be deduced, in the case of two gases, from "kinematical" equations and from the following equations "of coercion," in which D/Dt refers to the total coercive action of both gases:

$$\frac{Du_1}{Dt}=A\rho_2(u_2-u_1)\ ;\ \frac{Du_2}{Dt}=A\rho_1(u_1-u_2), \quad . \quad (2)$$

and four other equations of similar form. If the dynamical equations of Maxwell and Stefan are true, equations (2) must likewise be fulfilled; they may be said therefore to agree with experience. Let us now pass to the usual case of slow and quiet diffusion (Maxwell, Phil. Trans. 1867, pp. 73–74). If we write ϑ for the temperature, R for the gaseous constant, we shall find the value of the coefficient of diffusion, or h say, to be $R\vartheta/A(\rho_1+\rho_2)$; and, if $p=p_1+p_2$, the charac-

* See Kamerlingh Onnes, *Algemeene Theorie der Vloeistoffen*, Tweede Stuk, p. 8, 1881.

† See Kamerlingh Onnes, 'Communications from the Laboratory of Physics at the University of Leiden,' no. 12, p. 11, 1894.

teristic period τ for the coercion of the disturbance will be

$$\tau = \frac{1}{A(\rho_1+\rho_2)} = \frac{(\rho_1+\rho_2)h}{p} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In a system composed of nitrogen and oxygen, at 0° C. and normal pressure, the value of τ (from v. Obermayer's experimental results) is about $4{\cdot}5 \times 10^{-10}$ of a second. Returning to (2) we obtain

$$\frac{D(u_2-u_1)}{Dt} = -\frac{u_2-u_1}{\tau}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and two other equations which may be written down from symmetry; hence (1) reduces to

$$F = -\tfrac{1}{2}\iiint dx\,dy\,dz\,A\rho_1\rho_2\tau\left\{(u_2-u_1)\frac{D(u_2-u_1)}{Dt}\right.$$
$$\left. + (v_2-v_1)\frac{D(v_2-v_1)}{Dt} + (w_2-w_1)\frac{D(w_2-w_1)}{Dt}\right\}, \quad . \quad . \quad (5)$$

and this gives

$$\frac{DF}{Dt} = -\frac{2F}{\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Let us verify that, as stated above, 2F is the rate at which, owing to diffusion, heat is being irreversibly generated. First, from conservation of energy, we have

$$\frac{D}{Dt}\iiint dx\,dy\,dz\,\tfrac{1}{2}\left\{\begin{array}{l}\rho_1(u_1^2+v_1^2+w_1^2+\overline{\xi_1^2}+\overline{\eta_1^2}+\overline{\zeta_1^2})\\ +\rho_2(u_2^2+v_2^2+w_2^2+\overline{\xi_2^2}+\overline{\eta_2^2}+\overline{\zeta_2^2})\end{array}\right\} = 0. \quad (7)$$

Then, from (2) we obtain

$$\frac{D}{Dt}\iiint dx\,dy\,dz\,\tfrac{1}{2}\rho_1(u_1^2+v_1^2+w_1^2) = \iiint dx\,dy\,dz\,A\rho_1\rho_2$$
$$\times\{u_1(u_2-u_1)+v_1(v_2-v_1)+w_1(w_2-w_1)\}, \quad . \quad (8)$$

$$\frac{D}{Dt}\iiint dx\,dy\,dz\,\tfrac{1}{2}\rho_2(u_2^2+v_2^2+w_2^2) = \iiint dx\,dy\,dz\,A\rho_2\rho_1$$
$$\times\{u_2(u_1-u_2)+v_2(v_1-v_2)+w_2(w_1-w_2)\}, \quad . \quad (9)$$

whence by (7) it follows :—

$$\frac{D}{Dt}\iiint dx\,dy\,dz\,\tfrac{1}{2}\{\rho_1(\overline{\xi_1^2}+\overline{\eta_1^2}+\overline{\zeta_1^2})+\rho_2(\overline{\xi_2^2}+\overline{\eta_2^2}+\overline{\zeta_2^2})\}$$
$$= \iiint dx\,dy\,dz\,A\rho_1\rho_2\{(u_2-u_1)^2+(v_2-v_1)^2+(w_2-w_1)^2\}$$
$$= 2F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and this proves the proposition.

§17. *Electromagnetic Dissipation.*—The electromagnetic dissipation function is

$$F=\tfrac{1}{2}\iiint dx\,dy\,dz\,C(E_x^2+E_y^2+E_z^2), \quad . \quad . \quad . \quad (1)$$

the symbols being defined as in § 13. The disturbance settles down obeying the well-known equations

$$K\frac{DE_x}{Dt}=-4\pi CE_x;\ K\frac{DE_y}{Dt}=-4\pi CE_y;\ K\frac{DE_z}{Dt}=-4\pi CE_z; \quad (2)$$

they are therefore the electromagnetic "coercion" equations. If we take $\tau=K/4\pi C$, as has been done by Maxwell and many others, we see that

$$F=-\tfrac{1}{2}\iiint dx\,dy\,dz\,\tau C\left(E_x\frac{DE_x}{Dt}+E_y\frac{DE_y}{Dt}+E_z\frac{DE_z}{Dt}\right), \quad . \quad (3)$$

and

$$\frac{DF}{Dt}=-\frac{2F}{\tau}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Prof. J. J. Thomson has shown* that for water with 8·3 per cent. of H_2SO_4, τ cannot differ much from 2.10^{-11} of a second; and for glass at 200° C. from about 10^{-5} of a second.

§ 18. *Irreversible Dynamics.*—In the case of § 17 the energy we have called T is proportional to the dissipation function F; the same holds in § 16 if we have $\rho_1 u_1+\rho_2 u_2=0$ (see Maxwell, Phil. Trans. 1867, pp. 73–74). Hence, in such cases equation (V.) becomes $DT/Dt=-2T/\tau$. Again, in the Irreversible Dynamics of § 9, if the additional dissipative forces $-R_i'$ be proportional to the corresponding components of momentum, the same proportionality holds. For example, let

$$-R_i'=\frac{1}{\tau}\frac{\partial T}{\partial s_i} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

represent the additional dissipative force acting in the q_i-direction; then $T=\tau F$; and since from (5), § 9, it is easily shown that

$$\frac{DT}{Dt}=-2F, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* 'Notes on Recent Researches in Electricity and Magnetism,' 1893, § 32.

D/Dt being the rate of variation of the kinetic energy arising from the dissipative forces, we see that, in this case,

$$DF/Dt = -2F/\tau \quad \text{and} \quad DT/Dt = -2T/\tau. \quad . \quad (3)$$

Cf. Lord Rayleigh, 'The Theory of Sound,' vol. i. p. 78.

§ 19. *Dissipation Function of Conduction.*—In the Philosophical Magazine for June 1895, p. 506, it was shown that in conduction of heat the dissipation function is of the form

$$2F = -\tfrac{5}{2}\iiint dx\,dy\,dz \left\{ \rho r_x \frac{\partial\theta}{\partial x} + \rho r_y \frac{\partial\theta}{\partial y} + \rho r_z \frac{\partial\theta}{\partial z} \right\}, \quad . \quad (1)$$

the symbol θ being employed to denote $\frac{1}{3}(\overline{\xi^2}+\overline{\eta^2}+\overline{\zeta^2})$. From (12) and (39) in the paper referred to, we have

$$\frac{D\rho r_x}{Dt} = 5p_{xx}\frac{\partial\theta}{\partial x}; \quad \frac{D\rho r_y}{Dt} = 5p_{yy}\frac{\partial\theta}{\partial y}; \quad \frac{D\rho r_z}{Dt} = 5p_{zz}\frac{\partial\theta}{\partial z}; \quad . \quad (2)$$

$$\frac{D\rho r_x}{Dt} = -\frac{5p_{xx}}{k}\rho r_x; \quad \frac{D\rho r_y}{Dt} = -\frac{5p_{yy}}{k}\rho r_y; \quad \frac{D\rho r_z}{Dt} = -\frac{5p_{zz}}{k}\rho r_z; \quad (3)$$

equations (2) are the "kinematical," and equations (3) the "coercive" equations of the problem. They must be fulfilled in order to make the equations hold:

$$\rho r_x = -k\frac{\partial\theta}{\partial x}, \; \rho r_y = -k\frac{\partial\theta}{\partial y}, \; \rho r_z = -k\frac{\partial\theta}{\partial z}, \quad . \quad . \quad (4)$$

and, therefore, to secure applicability for Fourier's equation. The time of relaxation we define as

$$\tau = k/5p, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

neglecting differences $p - p_{xx}$, &c. From (1) and (4) we obtain

$$2F = \tfrac{5}{2}\iiint dx\,dy\,dz \frac{1}{k}\{(\rho r_x)^2 + (\rho r_y)^2 + (\rho r_z)^2\}, \quad . \quad (6)$$

and from (1) and (2) we have:

$$2F = -\frac{D}{Dt}\iiint dx\,dy\,dz \frac{1}{4p}\{(\rho r_x)^2 + (\rho r_y)^2 + (\rho r_z)^2\}, \quad (7)$$

whence, by (5), we find again

$$\frac{DF}{Dt} = -\frac{2F}{\tau}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

§ 20. *Connexion between the periods* τ.—Let τ_F be the

characteristic period τ for conduction of heat in a given gas, and let τ_M denote the period relating, for the same gas, to internal friction. The coefficient of conductivity in Fourier's equation, as usually written, is $\frac{1}{3}c_v k$, k denoting the same quantity as in § 19. Now in the Kinetic Theory of Gases it is shown that this coefficient is equal to

$$\frac{15}{4}(\gamma-1)c_v\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

if γ be written for the ratio c_p/c_v of specific heats, and μ for the coefficient of viscosity [for example, see Prof. Boltzmann's *Vorlesungen über Gastheorie*, equations (238), (54), and (57)]. Hence

$$\tau_F=\tfrac{9}{4}(\gamma-1)\tau_M \quad \text{or} \quad =\tfrac{3}{2}\tau_M, \quad . \quad . \quad . \quad (2)$$

since, strictly speaking, our calculation requires the gas to be monatomic. In a similar manner may all the periods of relaxation, corresponding to the various powers of coercion of a given body, be mutually connected; and every such simple equation, if it holds, is equivalent to a definite physical law.

XLII. *A Mechanical Device for Performing the Temperature Corrections of Barometers. By* John Shields, *D.Sc., Ph.D.**

THE height of the barometer is generally reduced to 0° C. by means of the formula

$$B_0=B_t\frac{1+\gamma t}{1+\beta t},$$

where B_0 is the reduced height, B_t the observed height at the temperature t, and β and γ the coefficients of expansion of the mercury and scale respectively; or, since β and γ are in general both very small, we may write

$$B_0=B_t[1-(\beta-\gamma)t].$$

In order to facilitate the reduction, tables containing the corrections corresponding to definite temperatures and observed heights have been compiled, and in the laboratory it is only necessary to consult such a table, and if necessary perform a simple interpolation in order to find the correction which

* Communicated by the Author.

must be subtracted* from the observed height to reduce it to 0° C.

Graphic methods are sometimes used to obtain the temperature correction, and one of the best of these, which was first brought to my notice by Prof. Ramsay, is described in the Appendix as it seems to be little known.

During the course of an investigation it was necessary for me to read and correct the barometer several times daily, and as this operation became rather tedious I was induced to make a barometer which indicated the height and the correction simultaneously. The construction of the barometer presents no great difficulties, and as it is extremely useful in its new form I now beg to lay a description of it before the scientific world. It can be read with certainty to 0·1 millim., which is sufficiently accurate for most purposes. Whether an improved method of reading and better workmanship than I have been able to bestow upon it would make it suitable for meteorological observations must be left for meteorologists to decide. For ordinary laboratory work, however, it meets all the requirements. It is not necessary to know the temperature at all; and by mentally subtracting the correction as indicated by the correcting instrument from the observed height (also obtained directly by setting the barometer), the observer is enabled in one entry to write the corrected height of the barometer in his note-book.

The most suitable form of barometer to employ with the correcting instrument is that described by Dr. J. Norman Collie †, but an ordinary syphon barometer may also be adapted for the same purpose. The lower end of the barometer is cemented or otherwise securely fitted into a brass cap A, fig. 1, to which is attached a rod B, which moves vertically in a guide in order to prevent the barometer from rotating when it is raised or lowered by means of the screw C. The barometer itself is kept in a vertical position by the guides D D which are attached to the framework. The back of the framework consists of a long, narrow board, the lower end of which is shown at E, and to which the nut carrying the screw is fixed. A plate-glass mirror F, carrying the graduations, is firmly screwed down on the main frame by means of the picture-frame moulding G G, which is planed down at the back to such an extent that the mirror is held tightly clamped in position. The plate-glass mirror is carefully graduated between 700 and 800 millims., and has also

* For temperatures below 0° C. the correction must be added.

† Trans. Chem. Soc. (1895), p. 128.

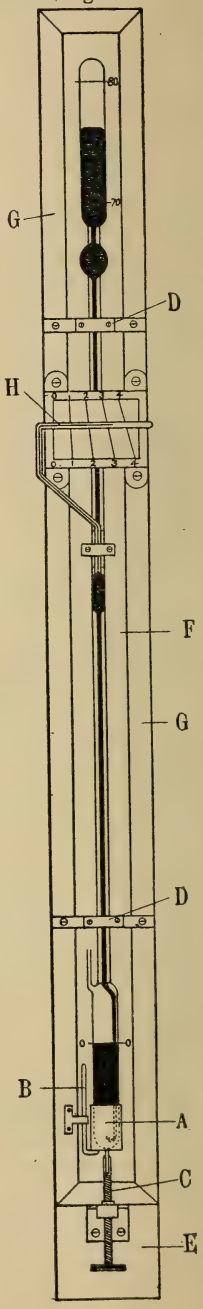
a zero line etched on it, but the space between zero and 700 millims. may be left ungraduated.

This particular form of mounting the barometer, independent of the correcting instrument which has yet to be described, is in itself very useful; as by setting the lower meniscus of the barometer at the zero line by means of the screw at the bottom of the frame, the uncorrected height can be read off *directly*, and this obviates the necessity of taking down the upper and lower readings and adding or subtracting as the case may be.

Before proceeding to show how the device for indicating the amount of the temperature correction can be attached to a barometer mounted in this way, it is well to note that the areas of the upper and lower reservoirs of the barometer are supposed to be equal, and are in fact approximately so, if these reservoirs are cut from adjacent parts of the same piece of tubing. Assuming now that the barometer is accurately set, and that the pressure of the atmosphere then changes, if the pressure rises or falls n millim., then, on again adjusting or setting the barometer, any point on the stem will obviously be raised or lowered $n/2$ millim. Should, however, the cross section of the capillary tube, which connects the main stem of the barometer with the lower cistern, be large when compared with the cross section of the lower cistern, then the above relation will not hold good.

This source of error may be eliminated either by making the cross section of the capillary small, or by selecting a lower cistern with a proportionally larger area. It may also be eliminated, if necessary, in graduating the scale of the correcting instrument, but any slight error introduced in this way has scarcely any appreciable effect on the accuracy of the readings.

Fig. 1.



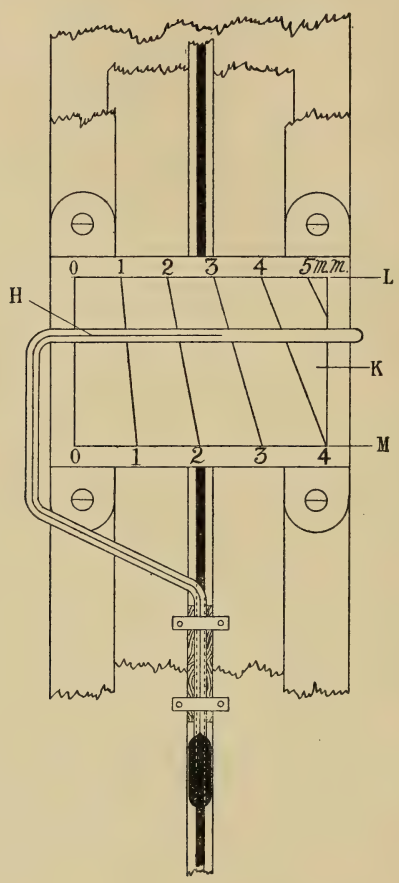
The precision of the complete barometer is limited not so much by the accuracy of the temperature correction, as this can easily be made to read correctly to 0·01 millim., but by the precision with which the barometer can be set and by the accuracy of the graduations. As has already been mentioned, the combined error in setting the barometer and reading the scale at the top should not exceed 0·1 millim. Of course all danger from the above source is removed if the common syphon form of barometer is employed; but as Collie's modification presents other advantages which are clearly set forth in his paper *, his form of barometer is to be preferred, especially as it is only necessary for the purposes of the correcting instrument that the above relation between n and $n/2$ should hold good within a millimetre or two.

In designing the correcting instrument use has been made of the fact that the variation in position of a point on or attached to the stem of the barometer is proportional to the variation of the height of the barometer. The point is represented by a horizontal thread of mercury H (figs. 1 and 2), contained in an ungraduated thermometer which is firmly attached to the stem of the barometer in a horizontal position. Behind the horizontal thread of mercury is fixed a scale or small plate of curves K, in such a position and drawn in such a manner that the position of the horizontal thread of mercury (the ordinates) indicates approximately the height of the barometer. The correcting instrument is shown on a larger scale in fig. 2. The distance from the top to the bottom of the plate of curves, *i.e.* from L to M, is actually 50 millims., but this, from what we have already said, represents altogether 100 millims., the position of L corresponding to a barometric height of 800 millims., whilst M corresponds to 700 millims. The position of the horizontal thread of mercury at the time of setting the barometer thus corresponds approximately to the actual height of the barometer. A series of lines 1, 2, 3, 4, &c. millim. are drawn or engraved on the plate K, so that the position of the meniscus of the horizontal thread of mercury gives the temperature correction directly in millimetres. The method of drawing these lines requires some explanation. It is desired to reduce the height of the barometer to 0° C. Obviously, then, the line of zero correction 0 0 must lie immediately behind the point corresponding to 0° C. on the horizontal thermometer, and it must furthermore be vertical, as no matter what the height of the barometer may be the correction at 0° C. must always remain zero.

* Collie, *loc. cit.*

Before the position of the other lines can be fixed, it is necessary to ascertain the length in millimetres corresponding to 1° C. of the horizontal thermometer. Let this be n millim.

Fig. 2.



We have already seen that the formula for reducing the height of the barometer to 0° C. is

$$B_0 = B_t[1-(\beta-\gamma)t],$$

hence the correction which must be subtracted from the observed height is

$$(\beta-\gamma)B_t t.$$

In order to fix the position of the top of the lines giving the corrections 1, 2, 3, 4 &c. millim., all that we have to do is to make $B_t = 800$, and calculate the temperature t, which would make the above expression equal to 1, 2, 3, 4 &c. millim. Thus, for the top of the line representing a correction of 4 millim., we have

$$(\beta - \gamma) \,.\, 800 \,.\, t = 4.$$

For all ordinary purposes we may make $\beta - \gamma$ for mercury and a glass scale $= 0{\cdot}000181 - 0{\cdot}000009 = 0{\cdot}000172$, hence

$$t = \frac{4}{0{\cdot}000172 \times 800}.$$

But since the thermometer is not graduated, we must multiply the value of t so found by n to get the distance in millimetres of the point 4 from the point 0. Similarly the tops of the other lines may be found, but in general it will be found sufficient to calculate the greatest correction only, and then to divide the distance between it and 0 into the required number of equal parts. In the same way the corrections corresponding to a barometric height of 700 millims. (the lower ends of the lines) may be obtained by making $B_t = 700$. Then, since the correction is proportional to the height of the barometer, straight lines joining 1 and 1, 2 and 2, and so on will represent the corrections between 700 and 800 millims. The spaces between 0 and 1, &c. may also be subdivided into halves or tenths if necessary.

The scale is either drawn or engraved on paper, a plate-glass mirror, or other convenient material, and then mounted on a bridge in front of the stem of the barometer and behind the correcting instrument. The final adjustment is made by moving the correcting thermometer into the proper position before clamping it tightly to the stem of the barometer. A piece of wood or other soft material interposed between the stems of the barometer and thermometer prevents any risk of breakage on screwing up. In adjusting the correcting instrument or thermometer, care must be taken that, firstly, the zero point of the thermometer is precisely in front of the line of zero correction, and, secondly, that the thread of the mercury is truly horizontal, and that its position between the 700 and 800 millim. lines of the correcting scale corresponds as nearly as possible with the actual height of the barometer at the time.

For convenience, the completed barometer should be suspended on a vertical wall with a good light falling on it. In

order to take a reading it is first gently tapped, the lower meniscus is then set exactly at the zero line by means of the screw at the bottom, and the temperature correction as indicated by the correcting instrument at once read off lest the heat of the body should cause any alteration; the height of the barometer is then observed at the top of the instrument, which, after subtraction of the temperature correction, gives the barometric height reduced to 0° C.

By considering the correcting instrument it is obvious that, the temperature remaining constant, the rise or fall of the barometer is accompanied, after setting the lower mercury meniscus to zero again, by an upward or downward displacement of the horizontal thread of mercury, and, consequently, to an increase or decrease in the correction. Similarly, a rise or fall of temperature is accompanied by an increase or decrease in the correction.

The instrument I have just described is one out of four possible modifications. Both scales may be fixed whilst the barometer and thermometer are displaced simultaneously or *vice versâ*. Again, the barometer and correcting scale may be fixed whilst the other parts are adjustable or *vice versâ*.

The first two will probably be found most useful. Distinct advantages might be gained by fixing the barometer with the thermometer above it, and at the same time etching the barometer and correction scales on the same piece of plate-glass mirror, which could be placed behind them and be moved vertically by a set screw at the bottom of the instrument.

University College, London.

Appendix.

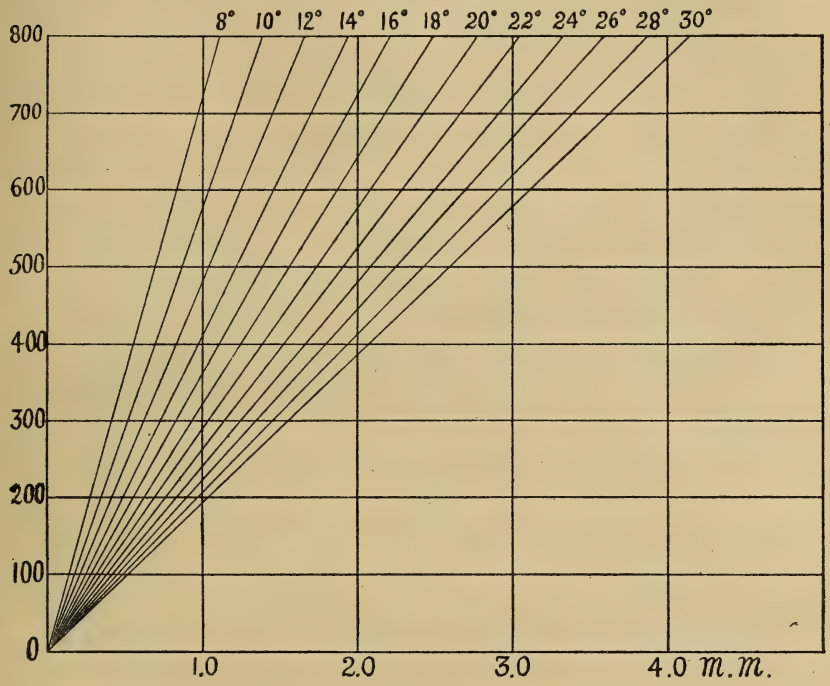
Fig. 3 illustrates a convenient graphic method for obtaining the temperature correction of a barometer or other column of mercury.

The ordinates represent the height of the mercury column* and the abscissæ the correction in millimetres. The correction for any temperature not represented by a diagonal line passing through the origin is easily obtained by a graphic interpolation. As fig. 3 is too much reduced in size to be of any value, a diagram of convenient size may be obtained by plotting the ordinates on a piece of curve paper ruled into

* In this particular case the height is supposed to be measured with a glass scale. If a brass scale is used the correction is always about 60 per cent. less.

inches and tenths of inches, each inch representing 100 millim., whilst each millimetre of correction is represented on the lower horizontal axis by two inches. A series of lines, each

Fig. 3.



representing a difference of 2° C. of temperature, are then ruled through the origin to the points on the upper horizontal axis marked 8°, 10°, 12°, &c. The position of these points is easily obtained from the formula

$$(\beta-\gamma)\mathrm{B}_t t = \text{correction in millim.},$$

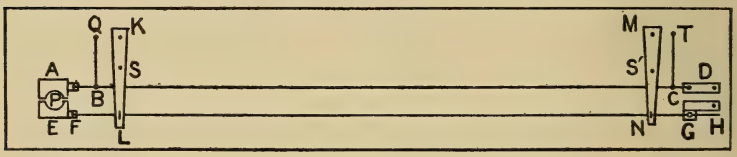
by substituting for t the values 8°, 10°, 12°, &c., for B_t the value 800, and for β and γ the coefficients of expansion of mercury and the material of which the scale is made respectively.

XLIII. *An Addition to the Wheatstone Bridge for the Determination of Low Resistances.* By J. H. Reeves, *M.A., City and Guilds of London Central Technical College* *.

WHILE it can be assumed with fair certainty that even in moderately equipped laboratories there will be found one sensitive galvanometer and a good set of resistance-coils arranged in the form of a Wheatstone bridge, it is by no means so common to find a convenient method for determining low resistances, such as, for example, a Kelvin bridge. The piece of apparatus which forms the subject of this paper is a comparatively cheap addition to the ordinary bridge, and enables the resistances of exact metre-lengths of even thick wires to be directly measured with almost as much ease as larger resistances can be determined with the ordinary Wheatstone bridge. The method also possesses the advantage that all the measurements are made in terms of a standard wire with fixed contacts, and, therefore, not subject to the wear which accompanies the frequent use of a slider; further, in the case of copper wires no temperature-correction is needed.

The apparatus is represented in fig. 1. On a mahogany

Fig. 1.



baseboard are stretched close to one another two wires—one, ABCD, being the standard of comparison, while the other, EFGH, is the wire to be tested.

A and E are two massive pieces of brass which can be joined together by a plug P. D and H are smaller pieces of brass with binding-screws attached.

The standard BC is permanently fixed to A and D, whilst two clamps F and G fixed respectively to E and H form the means of fixing in its place the wire to be tested.

KL and NM are two brass springs which pass over but do not touch BC. They are provided with binding-screws at K and M, and with two knife-edges L, N exactly one metre apart, which press on the wire FG when in position, while

* Communicated by the Physical Society: read March 13, 1896.

two screws S, S′ will raise the springs when required for inserting or removing this wire.

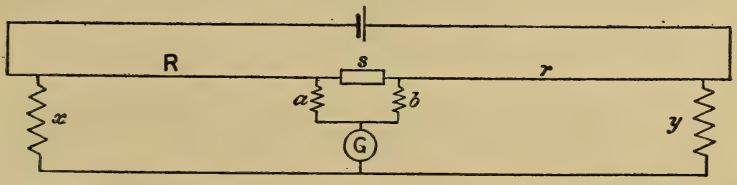
At points B, C in the standard are soldered two short lengths of wire terminating in binding-screws Q and T. The gauge of the standard wire, which is of copper, and the distance apart of B and C are so chosen that the resistance between B and C is 0·01 ohm at the temperature of the wire at the time of its final adjustment, which should be noted. The method of determining these points is given later on.

The arrangement is a variation of the Kelvin bridge, with this difference: that in the latter the measurements are made by varying the length between the knife-edges of sliders which press on the standards, of which several are required: whilst in the former only one standard resistance is employed, and the measurements are effected by the alteration of the other resistances in the arrangement. If the whole apparatus had to be bought, the Kelvin bridge would be the cheaper, but the new Addition utilizes the box of coils, which as beforementioned is sure to be available, and only requires in addition one standard to be made and adjusted.

The theory of the Kelvin bridge is to be found in the text-books. In Gray's 'Absolute Measurements,' for example, vol. i. p. 359, it is proved that if fig. 2 represents the usual arrangement, and, if R, r, x, y, a, b, s are as marked, the following relationship must hold in order to obtain balance:—

$$\frac{R}{r}=\frac{x}{y}=\frac{a}{b}.$$

Fig. 2.



Further, it is shown that when s is small compared with a and b the accuracy of the equation

$$\frac{R}{r}=\frac{x}{y}$$

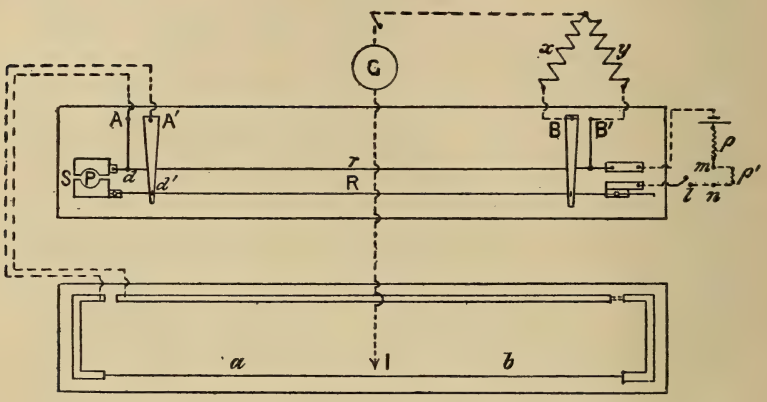
is not affected by a small want of equality between the ratios

$$\frac{a}{b} \text{ and } \frac{x}{y}.$$

In addition to the galvanometer and set of coils there is

required a slide-wire bridge, which may be of a rough description but should have a resistance of an ohm or so. At the end of the paper it is shown how this latter and the Addition can be united in one.

Fig. 3.



The whole arrangement is joined together as shown in fig. 3, the dotted lines indicating the temporary connexions. By comparing this with figs. 1 and 2, no detailed description is necessary, as R, r, x, y, a, b, s are lettered to correspond.

The battery may with advantage be a storage-cell with two resistances in series with it. One, ρ, may be an adjustable carbon resistance, while ρ' may be constructed of wire and should have a considerable resistance. This latter terminates in mercury cups m and n, so that by joining these cups together with a short connector the resistance can be easily short-circuited.

The resistances x and y are resistances unplugged from the box of coils forming the ordinary Wheatstone bridge; y is the 1000-ohm coil of one of the ratio arms, and x is in the adjustable arm. If such a set of coils be not available, y may be any 1000-ohm coil, and x any box of coils containing resistances up to 5000 ohms. The two leads joining x and y to B and B′ respectively should be as stout and short as possible. The resistances from d, d' to I form the $a\,b$ of fig. 2.

To make a measurement, insert the wire whose resistance is required in its proper clamps, taking care that it lies quite straight between them. Its diameter and material being known, the resistance of one metre of it can be approximately

calculated. Calling the resistance R_1, choose x_1 in the box so that

$$\frac{x_1}{1000} = \frac{R_1}{0{\cdot}01}.$$

Remove the plug P, and, including ρ' in the battery circuit, obtain balance by shifting the slider I, the galvanometer being also shunted if necessary. The arrangement is now an ordinary Wheatstone bridge,

$$\therefore \quad \frac{a+R_1}{b+r} = \frac{x_1}{y} = \frac{R_1}{r},$$

or

$$\frac{a}{b} = \frac{R_1}{r} = \frac{R}{r} \text{ approximately.}$$

Next insert the plug P, short-circuit ρ' so that now a strong current passes when the battery circuit is closed, and obtain balance by altering x. Let the new value be x_2. Then, since approximately $\frac{a}{b} = \frac{R}{r}$, we have, by the formula quoted above,

$$\frac{R}{r} = \frac{x_2}{y},$$

i. e.,

$$R = \frac{x_2}{y} \times r \text{ very approximately.}$$

In any case this new value of R is very much nearer the true value than the approximate value R_1. If the value of x has to be but little altered, it may be taken as the true value of R, but if, on the other hand, the first estimate was considerably in error, and, in consequence, a large alteration has to be made in x in order to obtain balance, the much closer value thus obtained must be taken as the first approximation, and both experiments repeated.

Now, since the wires are so close to one another, their temperature can be called the same, and, if the tested wire be of copper, their temperature-corrections are the same. Therefore, if R be a certain fraction of r at one temperature, it will be the same fraction at any other. But in the actual bridge on which the experiments described later on were first made, this temperature was 17°·7 C. Therefore, substituting numerical values in the last equation,

$$R = \frac{x}{1000} \times {\cdot}01 = \frac{x}{100{,}000} \text{ ohms at } 17^\circ{\cdot}7 \text{ C.}$$

Of course, if R be not made of copper, the temperature must be noted and allowed for.

The accuracy and sensibility of this method depend on three conditions :—

(1) The accuracy of the ratio $\frac{x}{y}$;

(2) The accuracy of the value of r;

(3) The sensibility of the galvanometer.

As far as condition (1) is concerned, x and y being taken from a good box have an error of certainly less than 0·1 per cent.

Condition (2) is discussed later, and the effect of condition (3) can be best seen by reference to tests made in the laboratory of the Central Technical College with the apparatus arranged as in fig. 3, except that, at the start, the battery consisted of a single Daniell cell, the galvanometer being a four-coil reflecting one having a resistance of 687 ohms and a sensibility of about 400 scale-divisions per microampere.

TABLE I.

Tests made on high conductivity copper, $R = \frac{x}{100{,}000}$ ohms.

No. of wire. S.W.G.	Diameter. mils.	x.	Observation.	R.	Specific Gravity.	Specific Resistance.	Conductivity. Matthiessen's Standard.
22	28·07	4324	½ div. deflexion left.	0·04325	8·841	1·616	100·3
		4326	Small ,, right.				
		4328	1½ div. ,, right.				
20	35·5	2618	1 ,, ,, left.	0·02621	8·918	1·572	103·1
		2620	Small ,, left.				
		2622	½ div. ,, right.				
18	48·8	1421	½ ,, ,, left.	0·014215	8·912	1·602	101·1
		1422	½ ,, ,, right.				
16	64·2	802	3 ,, ,, left.	0·008032	8·907	1·572	103·1
		803	½ ,, ,, left.				
		804	2 ,, ,, right.				
14	80·1	515	2½ ,, ,, left.	0·005163	8·888	1·566	103·4
		516	Small ,, left.				
		517	1 div. ,, right.				
12	103·2	314	1 , ,, left.	0·003145	8·904	1·585	102·2
		315	1 ,, ,, right.				
—	106·9	291	1 ,, ,, left.	0·002916	8·940	1·580	102·6
		292	½ ,, ,, right.				
—	103·8	308	1½ ,, ,, left.	0·003086	8·870	1·588	102·4
		309	1 ,, ,, right.				
—	104·2	309	2 ,, ,, left.	0·003098	8·891	1·580	102·6
		310	½ ,, ,, right.				

Matthiessen's standard has been taken to be a specific resistance (resistance per cubic centimetre at 0° C.) of 1·620 International Microhms.

Towards the end of this table it will be seen that even a sensitive galvanometer barely allowed an accuracy of 0·1 per cent. The Daniell cell was therefore removed, and a small accumulator with a resistance of 2 ohms in series with

it substituted. The last three wires were then again tested, and a marked improvement was at once seen, for now an alteration of 1 ohm in x produced a difference of deflexion of 10–15 scale-divisions, thus giving the interpolated figure correct to the first place given, *i. e.*, R correct to 1 part in 3000.

To further test the capabilities of this method, a strand cable composed of 7 copper wires, each of No. 16 S.W.G., was taken, the strands being flattened to pass under the springs, which had not been constructed for a wire of such a large diameter. The results, using the same cell and resistance, are shown in the next table.

TABLE II.

x.	Observation.	R.
117	6 divs. deflexion left.	
118	1½ „ „ left.	0·001183
119	3 „ „ right.	

Here only 4½ divisions correspond with a change in x of 1 ohm: thus an error in reading of ½ a division means an error in R of 0·1 per cent. The effect of using a still larger current was next tried, the 2-ohm coil being removed, and the two resistances ρ, ρ' of fig. 3 substituted. An ammeter was also included in the battery-circuit, so that the actual current could also be measured.

Now, on passing a strong current both wires heat, and if the wire r has the smaller diameter it will heat the faster. Therefore its resistance will also increase the faster, and hence the resistance of the wire R will apparently diminish, and *vice versa*. In order to see how much this diminution might amount to, the resistance ρ' was short-circuited, and ρ was adjusted until the battery-current was 6 amperes, and the results are shown in Tables III. and IV.

TABLE III.

With the current only continued long enough to measure the deflexion: current 6 amperes.

x.	Observation.	R.
118	3 divs. deflexion right.	0·0011812
119	23 „ „ left.	

TABLE IV.

After the current had flowed continuously for 3 minutes: 6 amperes as before.

x.	Observation.	R.
119	28 divs. deflexion left.	
118	4 „ „ left.	0·0011784
117	20 „ „ right.	

The difference between these two results is about $\frac{1}{4}$ per cent., while the deflexions are more than sufficient for an accuracy in R of 0·1 per cent.

A few days later these experiments were repeated with a current of 2 amperes. The slight discrepancy in the value of R was due to the fact that in the interval the cable, which was not quite straight, had been removed. On replacing it a slightly less length must have been included between the knife-edges. As, however, these experiments were being made solely to find the *sensibility* of the measuring arrangement, no attempt was made to straighten it. The results are shown in Table V., experiment (*a*) being for the short duration of current, and (*b*) the result after the current had flowed continuously for 3 minutes.

TABLE V.

Current 2 amperes.

Experiment.	x.	Observation.	R.
(*a*)	119	16 divs. deflexion left.	0·001178
	118	3 „ „ left	
	117	10 „ „ right.	
(*b*)	118	4 „ „ left.	0·001177
	117	9 „ „ right.	

From the above results it will be seen, firstly, that 13 divisions correspond with 1 ohm difference in x, thus making the fourth significant correct, in other words, an error in R is less than 0·1 per cent.; and secondly, the passage of a current of 2 amperes only produced in 3 minutes a difference of like amount.

No larger cable was tried, nor was any wire smaller than

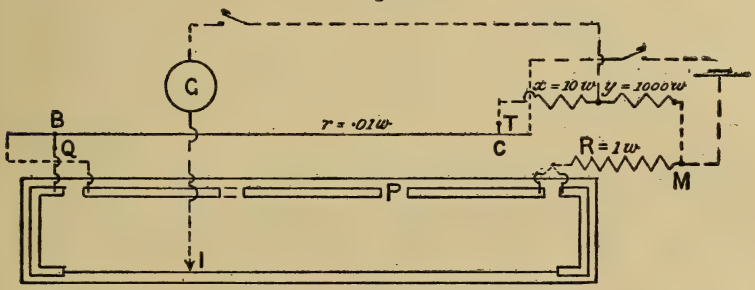
No. 22 S.W.G., but between these limits an accuracy of 0·1 per cent. was easily obtained so far as the *sensibility of the method* was concerned.

The accuracy of the standard now remains to be considered. This can be best seen from a description of the method by which the correct length between the points B, C was determined.

In order that this length might be of a convenient amount to suit the dimensions of the base-board, it should have been made of copper wire No. 17 S.W.G.; but a piece of this gauge which was bought for the purpose proving unsatisfactory, a piece of No. 16 was taken and stretched till of the required diameter.

One, Q B, of the two short arms of fig. 1, was soldered on, the other, T C, being left loose. The whole was then screwed to a rough board, and the wire annealed by passing through it a current of about 50 amperes till it was too hot to touch and then allowing it to cool, the operation being repeated ten times. The arrangement represented in fig. 4 was next made.

Fig. 4.



By following out this figure, it will be seen to be the same arrangement as fig. 3, except that now the place of the plug is taken by the gap P. The standard was a 1-ohm coil of manganin wire, constructed by Messrs. Nalder, Bros. Its terminals rested in mercury-cups from which the various leads were taken, the ends of all wires dipping in the mercury being freshly amalgamated. The leads joining the 10- and 1000-ohm coils together and to the binding-screws T and M respectively were short and stout.

The arm T C was then put approximately in position, and good contact at C was maintained by pressure. To avoid thermoelectric effects this pressure was made with a piece of wood, and in order to see if any such effects were present a commutator, not shown in the figure, was put in the battery-circuit. It was then found that if balance had been obtained

for one direction of the current, a reversal produced no appreciable deflexion, thus showing that this piece of wood had entirely prevented such effects. As a precaution, however, the current was reversed each time balance was obtained, and in no instance was any deflexion caused by the reversal.

The experiments were then conducted in the same order as before. With the gap open, balance was obtained by moving the slider I. The gap was then closed and balance obtained by altering the position of the arm T C. The gap was next opened, and the slider moved till again no deflexion was noted. This motion was very slight, and the alteration, after closing the gap, was not found to have produced any observable want of balance. The correct position of C was thus determined.

In order to see with what degree of accuracy this position had been arrived at, the arm was shifted 1 millim. in both directions, and the want of balance was indicated by a deflexion on either side of zero of one division; hence, as the length between B and C was over one metre, the error in the position of C was well under 0·1 per cent.

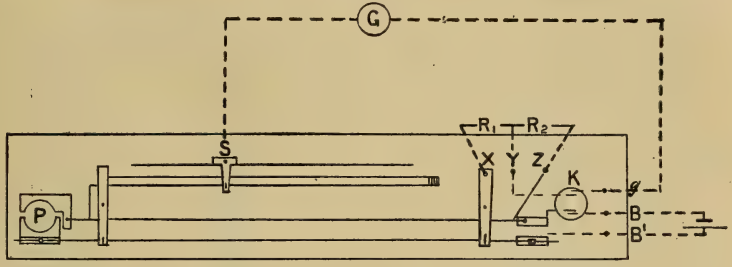
The arm TC was then soldered to the wire, and the tests repeated. No deflexion was observable with the ratio $\frac{10}{1000}$, but on adding a 1-ohm coil to the 1000, making the ratio $\frac{10}{1001}$, the want of balance produced nearly 2 divisions deflexion. The wire was then screwed in its proper place and there tested in a similar fashion with the same result, viz., with the ratio $\frac{10}{1000}$ no deflexion was noted, while with the ratio $\frac{10}{1001}$ a deflexion of over 1 division showed the want of balance. A thermometer lying beside the wire indicated 17°·7 C. Thus the resistance between B and C was 0·01 ohm at 17°·7 C., with an error of less than 0·1 per cent.

Thus on all three conditions an accuracy of 0·1 per cent. was obtained. This simple apparatus is, therefore, capable of measuring the resistances of metre-lengths of wires between the limits of No. 22 S.W.G. and a stranded cable of 7 No. 16's (and probably over a still greater range) with an accuracy *throughout* of 0·1 per cent., an accuracy quite sufficient for all commercial purposes.

Although in the above simple form of The Addition a slide-wire bridge is necessary, it may happen that this latter is not

available. At a slight additional cost this slide-wire can be made an integral part of the apparatus. In fig. 5 is represented a rather more elaborate design. The slide-wire is

Fig. 5.



easily recognized. The contact-piece of the slider S must be made adjustable so as to be able to press on either part of this wire. The circle K represents an ordinary Wheatstone-bridge key.

It will be seen that this design is very self-contained. All the connexions to be made consist of: a battery, with its supplementary resistance, to the binding-screws B, B′, a galvanometer to *g* and S, and two resistances between X, Y and Y, Z. The former, marked R_1, is an adjustable box of coils, and the latter, R_2, is a single coil of 1000 ohms resistance.

XLIV. *On the Absorption Spectrum of Solutions of Iodine and Bromine above the Critical Temperature.* *By* R. W. Wood *.

IN examining solutions of iodine above the critical temperature with a spectroscope, I have found that the fine lines which characterize the absorption spectrum of gaseous iodine may be either present or absent, depending on the amount of the solvent present.

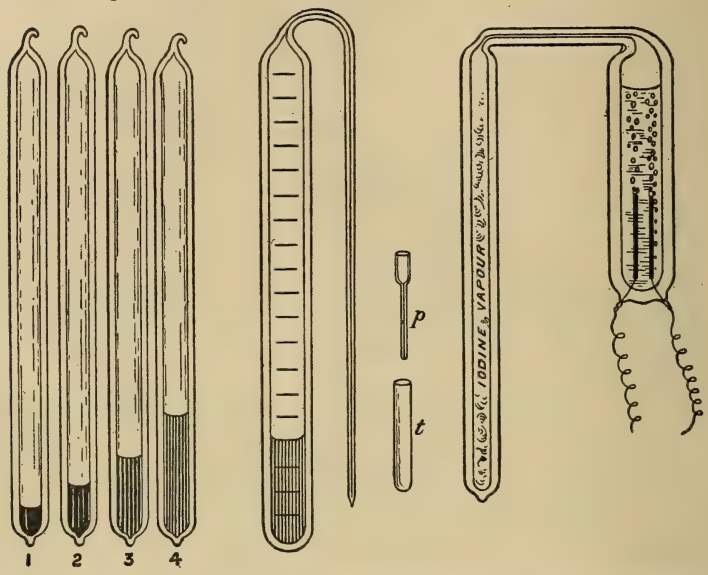
These lines are not present in the spectrum of iodine solutions, and their disappearance under the above-mentioned conditions seemed to be due either to the pressure exerted by the vapour, or to something akin to solution.

The tube containing the liquid was heated in an iron tube provided with two vertical slits, cut opposite each other, for the passage of a light ray, which was subsequently analysed with a large spectroscope of high dispersion.

* From the *Zeitschrift für Phys. Chemie.* Communicated by the Author.

For the preliminary investigation four tubes of similar size were prepared (1, 2, 3, & 4, fig. 1) containing equal amounts of iodine, but successively increasing amounts of bisulphide of carbon.

Fig. 1. Fig. 2. Fig. 3.



These tubes were successively heated until their contents were homogeneous, and their absorption spectra observed. No. 1 showed the lines almost as distinctly as iodine alone: in No. 2 they were fainter but still visible: it was with difficulty that they could be seen in No. 3, while in No. 4 they were entirely absent. No. 4 was then opened, and a little more iodine added. On reheating, the lines appeared. Variations in the temperature had no apparent effect. A mixture which at 300° showed the lines faintly, showed no change when heated to 350°.

A large number of experiments were tried with varying amounts of iodine, and with various solvents such as chloroform, liquefied sulphur dioxide, and water, and all were found to act in the same way. It was difficult to get reliable results with water owing to its action on the glass with formation of iodides.

To determine the effect of a greater pressure with less density, an apparatus (fig. 3) was constructed of glass, in which pressure could be developed by the electrolysis of water. The long arm, which contained the iodine, was heated in the

iron tube, and a current of electricity sent through the water in the short arm. The absorption spectrum was watched for an hour and a half, at the end of which time the apparatus exploded, but up to the very end the fine lines lost nothing in distinctness. The pressure was calculated from the time, the current strength, and the capacity above the liquid, and was found to have been about 250 atmospheres, or more than double the critical pressure of bisulphide of carbon. This indicates that the disappearance of the lines is due to the density of the vapour rather than to its pressure.

The quantitative investigation of these phenomena was next undertaken. A powerful arc light was substituted for the incandescent lamp, and a lens so arranged as to throw an image of the "crater" on the heated tube. By this arrangement, the spectra of much denser solutions could be observed. A tube provided with a long capillary neck, of the form shown in fig. 2, was constructed and carefully graduated. The contents of this tube could be varied without altering its volume by cutting off the tip of the capillary, and by warming or cooling the tube cause the liquid to run out or in; the tip could then be sealed once more. This operation could be repeated about 60 times before using up the capillary, only about 2 mm. being removed at each filling. A certain amount of iodine and bisulphide of carbon being introduced, the tube was sealed, heated, and examined. If the lines were present in the spectrum, a little more of the bisulphide was added, and this was continued until the lines just disappeared, indicating complete solution. The density was determined by noting the amount of fluid as measured by the graduations, since, when the contents are homogeneous, these values are proportional. Just before complete solution the lines are so faint as to be invisible in the *stationary spectrum*, but by moving the telescope to the right and to the left, they could be detected, the eye being more sensitive to a moving faint object than a stationary one. By using this device, the density necessary to just cause the disappearance of the lines could be determined with considerable accuracy.

The iodine was measured in the following manner. A saturated solution in CS_2 was made at 12° and a capillary pipette (*p*, fig. 2) was dipped into it. The fluid rose to a certain height, which was marked. The iodine solution was then washed out of the capillary into the small tube *t* by means of a drop or two of CS_2 put into the wide top of the capillary. This was immediately transferred to the graduated tube in the manner described. The amount held by the capillary

pipette made only a small drop of the saturated solution, and held 0·00031 gr. of iodine. This was determined by filling and emptying the pipette ten times, and determining the iodine volumetrically, with sodium thiosulphite.

The graduated tube was divided into 20 parts, and the amounts of iodine that could be present with amounts of CS_2 varying from 1 to 10 divisions, without showing the iodine-gas spectrum, were determined. With the tube half-full of CS_2, it was possible to add iodine until the tube was quite opaque; consequently densities greater than ·5 that of liquid CS_2 could not be investigated.

The values found directly are not in shape for discussion, since as we increase the density of the vapour we also increase its amount; in other words, we are working with varying amounts of solvent as well as varying densties.

To reduce these varying amounts of solvent to unity is very simple, and the results are given in the following table. For the various densities δ are given the amounts of iodine χ which can be mixed with 1 gram of the CS_2 vapour without the lines appearing in the spectrum; or, in other words, the amounts of iodine which 1 gram of CS_2 will dissolve at different densities.

If the tube contain m grams CS_2 with m' grams of iodine, and if v be the volume of the tube in c.cm., then

$$\delta = \frac{m}{v}, \quad \chi = \frac{m'}{m}.$$

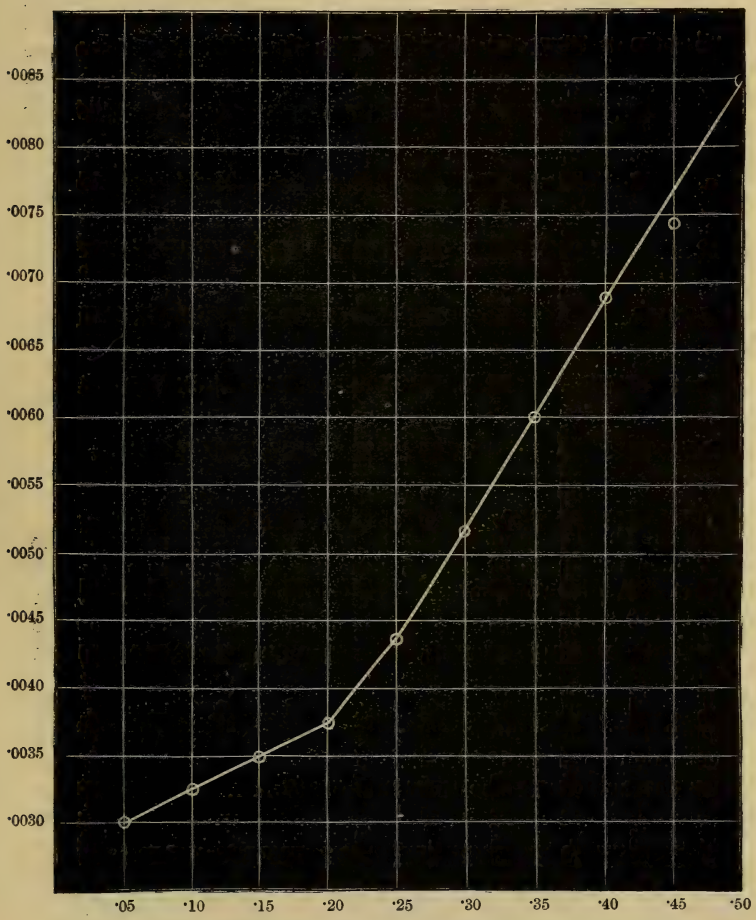
1 Gram CS_2.

δ (H_2O=1).	χ, iodine in grs.
·05	·00300
·10	·00325
·15	·00350
·20	·00375
·25	·00440
·30	·00514
·35	·00600
·40	·00686
·45	·00741
·50	·00851

The values plotted on co-ordinate paper (fig. 4), the δ values as abscissæ, the χ as ordinates, show that the solvent power increases rapidly with the density. Any point on the plane to the left of the curve represents a mixture which shows the iodine lines in the spectrum, any point to the right a mixture in which they are invisible. If the tube holds a

mixture represented by some point on the curve, and a portion be removed, δ is thereby diminished, while χ remains unchanged. The point corresponding to this mixture lies to

Fig. 4.



Iodine in CS_2 vapour.

the left of the curve, and the lines should appear. This was found to be the case.

Similar investigations were made with bromine, the absorption spectrum of which is very similar to that of iodine. A different method of measuring the halogen was adopted, however. Seven drops of fluid bromine were brought into

the tube, and bisulphide of carbon added up to the 11th mark. The absorption spectrum showed no trace of the lines. Half of the contents was then removed, and the amount of bromine in this determined. On sealing and reheating the tube the lines were distinctly visible, as was to be expected, and CS_2 was added little by little until the lines just disappeared. Half of this new quantity was then removed, and the same process repeated. In this way data were obtained from which the following table has been made. The curve is shown in fig. 5.

1 Gram CS_2.

δ ($H_2O=1$).	χ, bromine in grs.	$\frac{\chi \text{ (Bromine)}}{\chi \text{ (Iodine)}}$.
·05	·0182	6·1
·10	·0200	6·1
·15	·0241	6·9
·20	·0263	7·0
·25	·0299	6·8
·30	·0350	6·8
·35	·0415	6·9
·40	·0499	7·3
·45	·0623	8·4
·50	·0802	9·4

In the third column of the table is given the ratio of the χ values for bromine and iodine for the corresponding δ values. A given amount of CS_2 vapour at any density from ·05 to ·40 will dissolve from 6 to 7 times as much bromine as iodine. Liquid bromine and CS_2 appear to be miscible in all proportions.

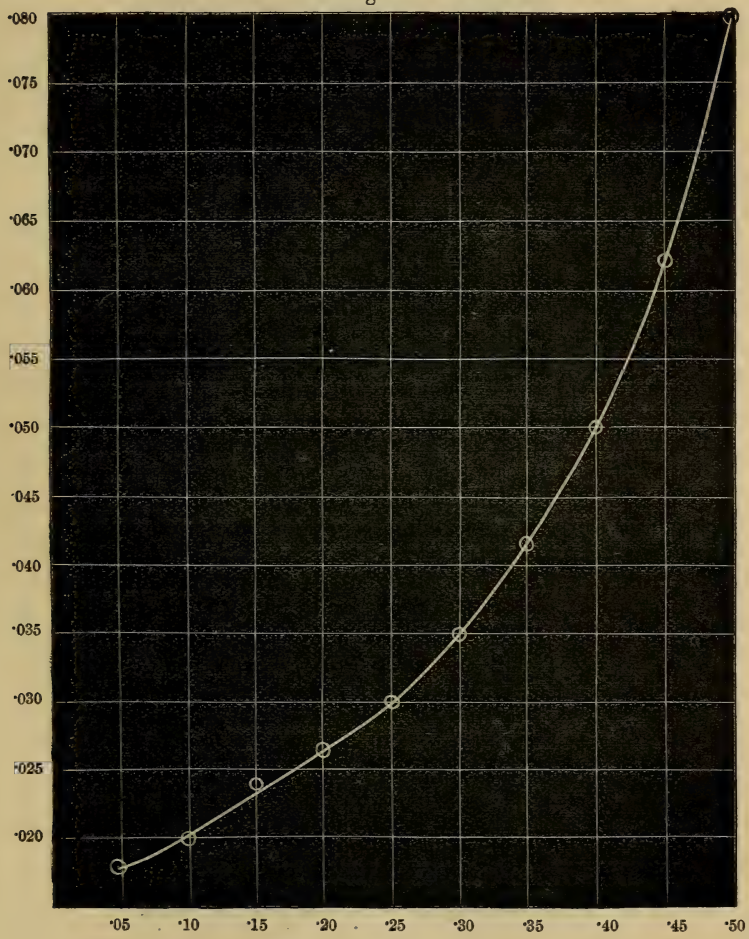
These investigations show that to a certain amount of CS_2 vapour a certain definite amount of iodine vapour (depending on the density of the CS_2) can be added without causing the lines characterizing the spectrum of iodine vapour to appear. If more iodine vapour be added, the lines are at once seen. The conclusion that one naturally draws is that the iodine molecules bind themselves in some way to the CS_2 molecules, and are incapable of exercising the selective absorption peculiar to the molecules of pure iodine gas. On this supposition, we may look on the curves in figs. 4 and 5 as solubility curves, and may consider, in a certain mixture of CS_2 and iodine or bromine vapour, the halogen as existing in two states, one part dissolved in the CS_2 vapour, and the other free.

Hannay and Hogarth have shown (Proc. Roy. Soc. xxx. pp. 178 & 484, 1880) that non-volatile solids in solution are

not precipitated when the solutions are heated above the critical temperature, but remain dissolved in the vapour.

I have made a rather hasty quantitative investigation of this phenomenon, and have obtained curves similar to those obtained for iodine and bromine by the optical method. An

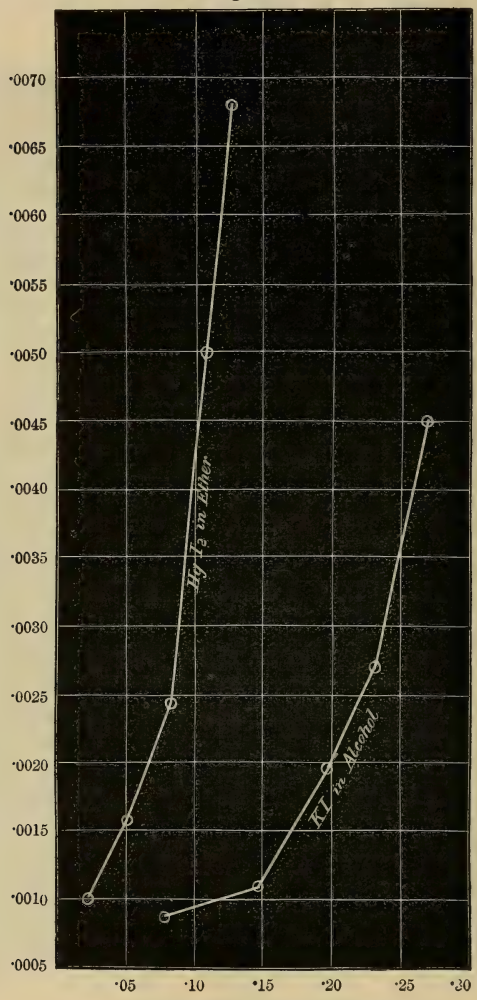
Fig. 5.



ethereal solution of HgI_2, and an alcoholic one of KI, were used. In the first case the tube held a solution so strong, that when the contents became homogeneous above the critical temperature, a portion of the salt was thrown down

on the wall of the tube, which was then inverted quickly, and cooled at the bottom by an air-blast: the vapour condensed here, and the salt remained above on the wall. The

Fig. 6.



tube was then opened, and the amount of dissolved solid determined. In the second case a more accurate method was used; ·02 gr. of the salt (KI) was brought into the tube, and alcohol added little by little until no precipitation occurred above the critical temperature.

The tube was heated in an air-bath and illuminated by a beam from an arc light, so that the faintest crystal film could be easily observed, and the amount of alcohol necessary to completely dissolve the salt in the gaseous state very accurately determined.

Half of the contents was then removed, the tube resealed and heated. A thick film appeared on the wall, which corresponds exactly to the reappearance of the lines in the spectrum, in the experiments with iodine and bromine. More alcohol was added until the vapour had the density requisite for the solution of this amount of salt.

The following values were reckoned for 1 gram of solvent at different densities :—

1 Gram Ether.		1 Gram Alcohol.	
δ (HO_2=1).	χ (HgI_2 in grs.).	δ (H_2O=1).	χ (KI in grs.).
·020	·0010	·084	·00083
·053	·0016	·147	·00106
·080	·0024	·195	·00195
·114	·0050	·231	·00270
·133	·0068	·273	·00450

The curves plotted from these values (fig. 6) are quite similar to the curves for iodine and bromine as determined by the spectroscope, which is not unfavorable to the supposition that the halogen vapour is in part dissolved in the vapour of the bisulphide of carbon. If the dissolved substance is volatile at the temperature used, as is the case with iodine and bromine, the undissolved portion is in the state of a free gas ; if non-volatile, as in the case of HgI_2 and KI, it is precipitated as a crystalline film on the wall.

In conclusion I wish to thank Prof. Warburg for the means of carrying on the investigations which he has placed at my disposal and for the interest that he has taken in the work.

Berlin, Physikalische Institut.

XLV. *The Substitution Groups whose Order is Four.* *By* G. A. Miller, *Ph.D.**

It seems proper to say that Professor Cayley began the enumeration of all the regular substitution groups of a given order since he determined these groups for the first order that presents any difficulties, viz., for the order 8 †.

* Communicated by the Author.

† Phil. Mag. vii. (1854) pp. 40–47 and 408–409; xviii. (1859) pp. 34–37.

Later he gave a list of all the regular groups whose order does not exceed 12 together with a geometrical representation of them*. Kempe had previously given such a list†, but his results were not quite correct.

Since all groups are isomorphic to regular groups ‡ and two distinct regular groups cannot be simply isomorphic §, it is clear that the enumeration of all such groups within certain regions is very important. Complete enumerations for the first part of the two following series of orders have been published: (1) when the order of the groups is the product of a given number of prime factors ‖, and (2) when it does not exceed a given number ¶.

Two more comprehensive enumerations with respect to order may be mentioned, (1) the enumeration of all the transitive groups of given orders, and (2) the enumeration of all the groups of given orders. The latter of these includes the former, and each of them includes the regular groups. It may happen that the transitive groups of a given order are also regular. This is, for instance, the case when the order is a prime number, or the square of a prime number. When the order is a prime number (p) there is one group for every degree which is a multiple of p: *i.e.* there are n groups of order p whose degree does not exceed np, $n-1$ of these are transitive, n being any positive integer. It should be observed that the number of the transitive groups of a finite order is always finite, while that of the intransitive groups of any order is infinite.

When the order of the groups is a composite number, the problem of determining all the possible groups becomes more complex. We shall confine our attention to the groups whose order is four. Since none of the transitive constituents of these groups can be of an odd degree, we see that the degree of such a group must be even and not less than four. We may therefore represent the degree by $2n$.

To find all the cyclical groups of degree $2n$ we have only to construct a 1,1 correspondence between α cyclical transitive groups $\left(\alpha \overset{=}{<} \frac{n}{2}\right)$ and a 2,1 correspondence between

* American Journal of Mathematics, xi. (1889) pp. 139–157.

† Phil. Trans. clxxvii. (1886) pp. 37–43.

‡ Jordon, *Traité des Substitutions*, p. 60.

§ Netto, 'Theory of Substitution Groups' (Cole's edition), p. 110.

‖ Hölder, *Mathematische Annalen*, xliii. (1893) pp. 301–413; Cole and Glover, American Journal of Mathematics, xv. (1893) pp. 191–221; Young, *ibid.* pp. 124–179.

¶ Miller, *Comptes Rendus*, cxxii. (1896) pp. 370–372.

each one of these groups and a group of the second order whose degree is $2n-4\alpha$. The number of such groups for a given value of n is therefore equal to the largest value of α, and the individual groups may be given by assigning to α the successive integers beginning with unity.

To find all the non-cyclical groups of degree $2n$ we may construct a 1,1 correspondence between α four-groups*, and (1) a 2,1 correspondence between each one of these groups and a group of the second order whose degree is $2n-4\alpha$, (2) a 1,1 correspondence between each one of these groups and a group of the fourth order which is of degree $2n-4\alpha$ and contains $n-2\alpha$ systems of intransitivity. The number of groups of the first one of these two types is the same as that of the cyclical groups, and the individual groups may be given in the same way. The number of groups of this and the cyclical type is therefore twice the largest value of α.

The groups of the second type of non-cyclical groups present somewhat greater difficulties. Here α may assume the value zero in addition to its values in the two preceding cases. We shall first determine the groups when α is zero; *i. e.*, we shall first seek all the

Groups which contain 2n Elements and n Systems of Intransitivity.

The average number of elements in the substitutions of such a group is n†, and the number of elements in all of its substitutions is $4n$. The number of systems of two elements is therefore $2n$. These $2n$ systems must occur in three substitutions. If the smallest number of systems in any one of these three substitutions is represented by S, we have

$$S \geqq \frac{2n}{3}.$$

For each value of S which satisfies this relation there must be at least one group, since we have only to use the remaining systems for the second generating substitution in order to construct such a group.

In general we have the following :—

* Bolza, American Journal of Mathematics, xi. (1889), p. 297.
† Frobenius, Crelle's Journal, ci. (1887) p. 287.

Value of S.	Number of Groups.	Number of systems of two elements in the substitutions.		
1	1	1	$n-1$	n
2	2	2	$n-2$	n
		2	$n-1$	$n-1$
3	2	3	$n-3$	n
		3	$n-2$	$n-1$
4	3	4	$n-4$	n
		4	$n-3$	$n-1$
		4	$n-2$	$n-2$
5	3	5	$n-5$	n
		5	$n-4$	$n-1$
		5	$n-3$	$n-2$
⋮	⋮	⋮	⋮	⋮
m	(m even) $\frac{m}{2}+1$	m	$n-m$	n
		m	$n-m+1$	$n-1$
		⋮	⋮	⋮
		m	$n-\frac{m}{2}$	$n-\frac{m}{2}$
	(m odd) $\frac{m+1}{2}$	m	$n-m$	n
		m	$n-m+1$	$n-1$
		⋮	⋮	⋮
		m	$n-\frac{m+1}{2}$	$n-\frac{m-1}{2}$

The groups for the same value of S are all distinct; but it may happen that two groups which correspond to different values of S are identical. This can, however, not occur so long as the value of S satisfies the relation

$$S \leqq \frac{n}{2}.$$

Identical groups can therefore only occur when the value of S is such that

$$\frac{n}{2} < S \leqq \frac{2n}{3}.$$

To the successive values of S which satisfy this relation there corresponds one of the following series of identical groups:—

$$(A) \begin{cases} 2,\ 4,\ 6,\ 8, \ldots & \text{If } n \text{ is even.} \\ 1,\ 3,\ 5,\ 7, \ldots & \text{,, ,, odd.} \end{cases}$$

To find the number of all the given groups which corre-

spond to a particular value of n, we may therefore first find the sum of the number of these groups which correspond to the different values of S that satisfy the relation

$$S \leqq \frac{2n}{3},$$

and deduct from this sum the sum of $\left[\frac{2n}{3}\right] - \left[\frac{n}{2}\right]$* terms of the corresponding series (A). These operations are indicated in the following formula:—

$$\left.\begin{array}{ll}\left(\left[\frac{2n}{3}\right]\text{even}\right) & \left[\frac{n}{3}\right]\left(\left[\frac{n}{3}\right]+2\right) \\ \left(\left[\frac{2n}{3}\right]\text{odd}\right) & \left(\left[\frac{n}{3}\right]+1\right)\left(\left[\frac{n}{3}\right]+2\right)-1\end{array}\right\} - \left\{\begin{array}{l}(n\text{ even})\left(\left[\frac{2n}{3}\right]-\frac{n}{2}\right)\left(\left[\frac{2n}{3}\right]-\frac{n}{2}+1\right) \\ (n\text{ odd})\left(\left[\frac{2n}{3}\right]-\left[\frac{n}{2}\right]\right)^2.\end{array}\right.$$

By means of this formula we can readily determine the number of groups which contain $2n$ elements and n systems of intransitivity for any particular small value of n. The individual groups may be found by assigning to S the successive integers from 1 to $\left[\frac{2n}{3}\right]$ and rejecting the identical groups according to series (A) when the value of S satisfies the relation

$$\frac{n}{2} < S \leqq \frac{2n}{3}.$$

The groups of the second type of non-cyclical groups, which correspond to the other values of α, are found in exactly the same way. Their number may therefore be found by means of the given formula provided we use instead of n the following series in order,

$$n-2,\ n-4,\ n-6,\ \ldots,\ 2 \text{ or } 3.$$

By adding the double of the largest value of α to the sum of the numbers of these groups corresponding to the different possible values of α, we obtain the number of groups whose order is four and whose degree is $2n$. Two of the groups are transitive when n is 2. For the other values of n all the groups are intransitive.

Example.

It is required to find all the groups whose degree and order are 14 and 4 respectively.

* The brackets indicate that the largest integer which does not exceed the inclosed fraction is to be used.

To find the number of these groups we observe that the largest value of α is 3. Hence there are 6 groups of the first two types. To find the number of groups of the third type, *i.e.* of the second type of non-cyclical groups, we assign the following three values to n:—

$$7,\ 5,\ 3.$$

Hence we have for

$$n=7,\quad 2.4-(4-3)^2=7,$$
$$n=5,\quad 2.3-1-(3-2)^2=4,$$
$$n=3,\quad 1.3-(2-1)^2=2.$$

The total number of groups is $6+7+4+2=19$. The individual groups are given in the following list*:—

Number.	Groups.
1.	$\{(abcd.efgh.ijkl)cyc.(mn)\}dim.$
2.	$\{(abcd.efgh)cyc.(ij.kl.mn)\}dim.$
3.	$\{(abcd)cyc.(ef.gh.ij.kl.mn)\}dim.$
4.	$\{(abcd.efgh.ijkl)_4(mn)\}dim.$
5.	$\{(abcd.efgh)_4(ij.kl.mn)\}dim.$
6.	$\{(abcd)_4(ef.gh.ij.kl.mn)\}dim.$
7.	$(ab)(cd.ef.gh.ij.kl.mn).$
8.	$(ab.cd)(ef.gh.ij.kl.mn).$
9.	$(ab.cd)(cd.ef.gh.ij.kl.mn).$
10.	$(ab.cd.ef)(gh.ij.kl.mn).$
11.	$(ab.cd.ef)(ef.gh.ij.kl.mn).$
12.	$(ab.cd.ef.gh)(gh.ij.kl.mn).$
13.	$(ab.cd.ef.gh)(ef.gh.ij.kl.mn).$
14.	$\{(abcd)_4[(ef)(gh.ij.kl.mn)]\}_{1,1}.$
15.	$\{(abcd)_4[(ef.gh)(ij.kl.mn)]\}_{1,1}.$
16.	$\{(abcd)_4[(ef.gh)(gh.ij.kl.mn)]\}_{1,1}.$
17.	$\{(abcd)_4[ef.gh.ij)(ij.kl.mn)]\}_{1,1}.$
18.	$\{(abcd.efgh)_4[(ij)(kl.mn)]\}_{1,1}.$
19.	$\{(abcd.efgh)_4[(ij.kl)(kl.mn)]\}_{1\,1}.$

* The notation is that which Professor Cayley used and explained in his articles in the Quarterly Journal of Mathematics, vol. xxv.

When only the number of the possible groups for a given degree is required, and when n is a large number, it is very desirable to avoid assigning so many different values to n as are necessary if we employ the given formula. By observing that all the fractions in this formula are increased by integers when n is increased by 6, we may readily find the following formula. By means of it we can find the number of groups (N) directly for any value of n. m represents any positive integer, and α_1 represents the largest value of α, *i. e.* the largest integral value of x which satisfies the relation

$$x \overset{=}{<} \frac{n}{2}.$$

$$\left.\begin{array}{lll}
\text{When } n=6m, & \mathrm{N}=m(3m^2+6m+1) & \\
\quad\text{,,} \quad n=6m+1, & \mathrm{N}=\dfrac{m(6m^2+15m+5)}{2} & \\
\quad\text{,,} \quad n=6m+2, & \mathrm{N}=3m(m+1)(m+2)+1 & \\
\quad\text{,,} \quad n=6m+3, & \mathrm{N}=\dfrac{(2m+1)(3m^2+9m+4)}{2} & \\
\quad\text{,,} \quad n=6m+4, & \mathrm{N}=(m+1)(3m^2+9m+4) & \\
\quad\text{,,} \quad n=6m+5, & \mathrm{N}=\dfrac{3(m+1)(2m^2+7m+4)}{2} &
\end{array}\right\} +2\alpha_1.$$

Hence there are

$$\frac{4(96+60+5)}{2}+24 = 346 \text{ groups of degree } 50,$$

$$249.84.85+1+500=1{,}778{,}361 \text{ ,,} \qquad \text{,,} \quad 1000, \text{ \&c.}$$

Zürich, Switzerland, March 1896.

XLVI. *On the alleged Scattering of Positive Electricity by Light.* *By* J. Elster *and* H. Geitel *.

THE question whether light which facilitates the passage of negative electricity from a conductor into the surrounding gas can, in like manner, accelerate the discharge of positive electricity is not without significance for the proper apprehension of the photoelectric process.

* Translated from the *Ann. der Physik und Chemie*, Bd. lvii. (1896); from a separate impression communicated by the Authors.

If, in fact, this action of light can be shown to take place, then it is no longer possible to believe that we have here to do with a specific phenomenon of the kathode, and, moreover, the view that the photoelectric process depends upon the discharge of the one (the gaseous) coating of an electric double layer which is continually renewed at the surface of contact between the conductor and the gas must be put to experimental proof, since the nature of the electricity escaping in light must always be the same as that which the gas in contact with the conductor itself takes. From this point of view, therefore, it would not be intelligible that one and the same conductor in the same atmosphere should give off both electricities more easily in light than in darkness. Now experiment shows that the illumination of a negatively charged surface, with proper choice of light and of substance illuminated, causes an active discharge of electricity into the surrounding gas, whilst the corresponding phenomenon for positive electricity—if it takes place at all—must be much more insignificant. Thus Hrn. Stoletow and Righi have not been able certainly to recognize the action of ultra-violet light upon positively charged surfaces, and we ourselves have so far not been able to observe any loss of positive electricity in light which was not sufficiently well accounted for by the usual loss of electricity or by the sources of error to be more definitely spoken of in what follows.

A paper by Herr E. Branly has recently appeared *, in which the acceleration of the electric discharge by ultra-violet light is maintained to hold good also for positive electricity. On account of the importance of the subject we have repeated the experiments described in this paper, and with the arrangements which seemed to us best suited to exclude sources of error, and following the method of Herr Branly as closely as possible in essential points. After we had failed in obtaining the same result as Herr Branly, we tried whether the alkali metals, which are so sensitive to ordinary light with negative electrification, would show a photoelectric discharge also with positive electricity. In what follows we venture to report upon the results obtained in these experiments.

The most obvious method of observing the scattering of electricity in light, which method was also employed by Herr Branly, is to connect the electrified surface to be examined with an electroscope, and to judge of the loss of electricity produced by the light in a given time from the decrease in the divergence of the leaves. This method has the disadvantage that, on account of the high tension employed, the whole of the

* *Compt. Rend.* cxx. p. 829 (1895).

electrified system of conductors must be extremely well insulated so that a feeble action of the light may not be hidden by the loss of electricity not connected with the action of the light. But apart from this, there is a disturbing cause arising from the fact that each time the observer changes the sign of the electric charge, there is a return current from the insulating supports of *that* electricity which had passed to them during the previous electric condition.

Much less exposed to these sources of error is the arrangement which Herr A. Righi and we ourselves have often employed in photoelectric experiments, especially if it is necessary to recognize feeble action. In this method the electrical measuring apparatus, together with the conductor to be illuminated, are at the commencement of the experiment at zero potential, and the strength of the action of the light is measured by the velocity with which the potential becomes equal to that of a conductor kept at constant potential, which stands opposite to the illuminated surface at a small distance from it.

The arrangement of the experiment was as follows :—The ultra-violet light was furnished by the spark of a condenser, which was connected with the poles of an induction-coil actuated by 4 to 8 large Bunsen elements, the spark having a maximum length of 18 centim. The current was broken by means of a Wagner's hammer with platinum contacts, and the spark of the condenser was taken between two aluminium wires at a distance of 2 millim. The galvanic battery, the induction-coil, spark-space, and all the necessary connexions were placed in the open air in front of the closed window of the observing room. One of the panes of the window was replaced by a plate of thin iron connected to earth, which was provided with a circular opening, in which was inserted a quartz lens of 50 millim. diameter. Since the focus of this lens coincided with the spark, a parallel beam of ultra-violet light was formed by the lens within the room, whilst at the same time the electrostatic action of the induction-coil and of the electrified air from the spark was shut off from the room. Within the room, at a distance of about 25 centim. from the window and at right angles to the beam of light, was placed a piece of iron-wire gauze with a mesh of about 1 millim., and parallel to this, at a distance of 2 to 4 millim., the insulated plate of the substance to be examined. From this a wire went to the quadrant electrometer (sensitiveness, 1 volt=23 divisions), whilst the wire gauze was charged to a potential of about 525 volts by a battery of several hundred Leclanché cells. According as the wire gauze was

charged with positive or negative electricity, the plate parallel to it must become charged with negative or positive electricity. If now the earth connexion of the electrometer was removed, then as soon as a passage of electricity took place between the gauze and the plate, the change of potential in the latter could be read off on the electrometer. It is to be observed that with a positive charge of the gauze, the plate to be tested has negative electricity on its surface, and that therefore from this clean metallic surface in ultra-violet light a free discharge towards the gauze was to be expected and a consequent positive charge of the plate. We used for the experiment a disk of amalgamated zinc, also similar pieces of zinc covered with a thin layer of paraffin or tallow, and also a plate of wood covered with tallow. According to Herr Branly, such surfaces covered with paraffin or tallow suffer a greater loss of positive than of negative electricity when exposed to light.

We observed the deflexion of the electrometer-needle which took place in one minute, both with positive and with negative charge of the gauze, and both in the dark and when illuminated with light from the spark. The Wagner's hammer was so arranged that it came into action of itself upon closing the current, it was therefore only necessary to keep the current closed for one minute. The results of a series of measurements are brought together in the following table. The numbers give the change of potential of the plate, measured in volts, which took place in one minute: each number is the mean of two readings :—

	Amalgamated Zinc Plate.		Paraffined Zinc Plate.		Zinc Plate covered with Tallow.		Wood Plate covered with Tallow.	
	Dark.	Illuminated.	Dark.	Illuminated.	Dark.	Illuminated.	Dark.	Illuminated.
Gauze positive	+0·40	+138 (+123)	+0·54	+0·69	+0·79	+0·38	+0·58	+0·52
Gauze negative	−0·16	−0·40	−0·22	−0·16	−0·09	−0·06	−0·52	−0·52

As was to be expected from what has been said, we have in this series of observations evidence of the great photoelectric dispersion from a plate of amalgamated zinc charged with negative electricity.

We were not able to expose the plate for a full minute to

the light, as the deflexion of the electrometer on the scale could not then be read off. Therefore the plate was exposed only five seconds, and the deflexion—reduced to volts—was multiplied by 12*. But beside this action of the light—undoubted and already known—the numbers show no other. There are, it is true, deflexions of the electrometer-needle in the dark as well as in the light, which, however, in no case reach the amount of one volt, and which on account of their inconstancy are to be referred to an irregular passage of electricity from the gauze to the plate, probably caused by the dust of the air.

Only in two cases is this feeble transference of electricity slightly greater in light than in the dark, viz. with a negatively charged gauze opposed to a plate of amalgamated zinc, and with the gauze positive and the plate of paraffined zinc. If one wishes to find in this a proof of an action of light, then only the first case can be taken to show a photoelectric dispersion of positive electricity. But here also a sufficient explanation is to be found in the fact that the ultra-violet light reflected from the polished surface of the amalgamated zinc strikes the side of the gauze turned towards it and produces a passage of negative electricity from it to the plate, so that in this case the photoelectric discharge is not from the positively charged plate but from the negatively charged gauze.

We have now to describe an experimental arrangement in which this action of the reflected light appears perfectly clearly. All the observations show that the paraffined or greased surfaces are not photoelectrically sensitive; in no case is the scattering of electricity from these found to be greater with a positive charge than with a negative charge or in the dark.

The small deflexions of the electrometer observed in the

* A more accurate calculation of the change of potential during an exposure of one minute would be obtained by use of the formula

$$V - v_1 = V \cdot e^{-kJt_1},$$

in which V denotes the potential of the charging battery, V_1 that of the illuminated plate, k a constant, J the intensity of the light, and t_1 the time of exposure. From this we should have for two times of exposure t_1 and t_2 and the corresponding potentials v_1 and v_2:

$$\log\left(\frac{V-v_2}{V}\right) = \frac{t_2}{t_1} \cdot \log\left(\frac{V-v_1}{V}\right),$$

from which v_2 can be easily calculated. In the foregoing case we obtain for v_2 the value shown in brackets, +123 volts.

dark, which always indicated a loss in the charge of the gauze, show that the instrument is too delicate for experiments such as these, in which a thin plate of air is exposed to a fall of potential of more than 100 volts per millimetre. We have therefore repeated these experiments with the much less sensitive aluminium-leaf electroscope, and were able to make the charge of the gauze and the time of exposure twice as great as before. But then also the photoelectric discharge took place only when the gauze had a negative charge. Thus the charge of an amalgamated zinc plate rose in five seconds to 400 volts, of an oxidized zinc plate in two minutes to 190 volts, and with greased or paraffined zinc or wood plates the potential remained at zero, irrespective of the sign of the charge of the gauze.

From the result of this experiment we concluded that in the experiments of Herr Branly some unsuspected source of error must have existed. In order to discover what this may have been, we have repeated the experiment of Herr Branly, essentially according to his arrangement as far as was possible from the data which he gives. The sparks of the induction-coil were taken within a box of sheet-iron connected to earth, in the side of which there was a quartz window. Opposite to this was the insulated and electrified plate connected with an aluminium-leaf electroscope. Since there was no gauze placed in the way of the light rays, any electricity escaping from the plate must be lost in the air or partly pass to the side of the iron box, and from there pass to the earth.

So long as the plate was some distance (about 50 centim.) from the quartz window, we also observed with this arrangement an increase of electric dispersion in light with a negative charge. But if the plate is brought to within a few centimetres of the window, and, consequently, near to the box, it may happen, if the surface of the plate is covered with tallow or with paraffin, that a positive charge decreases in light more rapidly than a negative charge. But here, as in the above discussed analogous case, it is to be remembered that the positive charge of the plate collects negative electricity upon the side of the box turned towards it by induction; if therefore this surface is struck by the ultra-violet light reflected from the surface of the layer of fat, a passage of negative electricity from it to the plate must result, and give the same effect as if a photoelectric dispersion of positive electricity from it had taken place.

This suspicion was converted into certainty by the observation that the phenomenon is dependent upon the nature of

the surface of the *box*. If this is covered with bright metal—as tin-foil—the transference of electricity largely increases, and becomes strikingly great if a piece of amalgamated zinc is placed upon it.

Since Herr Branly gives no information as to the distance of the illuminated plate from the quartz window of the metal box, we may consider it not improbable that this was chosen too small, and that the dispersion of positive electricity observed by him in ultra-violet light was caused by the deceptive action of the light reflected from the surface of the electrified disk.

We believe that we are justified, by the results of the experiments described, in asserting that an increase in the dispersion of positive electricity by illumination of the electrified surface by ultra-violet light has not been proved.

The striking inability of surfaces of alkali-metal to retain a charge of negative electricity in ordinary light might suggest that a possible action of light with a positive charge might be expected to take place most readily with such surfaces. As we have mentioned in a previous paper*, exhausted glass globes of which the one electrode is formed by an alkali-metal, the other by platinum, also allow a photoelectric current to be observed in more or less distinct manner when they are reversed, *i. e.* when the alkali-metal forms the positive pole. But we had also arrived at the conclusion that in this case the photoelectric action had its seat not at the surface of the alkali-metal, but at the platinum electrode. There is, in fact, formed upon the platinum a superficial layer by condensation of the vapours of the alkaline metal from which in light negative electricity passes to the anode. By heating the platinum wire with a galvanic current this layer is volatilized, and the photoelectric cell becomes—in its reversed arrangement—for a short time insensitive to light.

When we recently repeated this experiment with better arrangements and greater care, we found that after the wire had been heated there remained a small amount of sensitiveness to light, which, perhaps, had its origin in a scattering of positive electricity from the surface of the alkali-metal. It seemed important to determine the seat of this action without doubt, whether anode or kathode.

We started from the observation that the current as usually produced by illumination of the kathode is dependent upon the position of the plane of polarization of the incident light with respect to the surface of the kathode†. It was to be

* Elster and Geitel, Wied. *Ann.* xliii. p. 236 (1891).

† ? *Cf.* Elster and Geitel, Wied. *Ann.* lii. p. 440 (1894).

expected that any discharge possibly produced by light at the anode would also be in some way dependent upon the direction of the light-vibrations toward the surface of the anode. The experiments made in this direction gave, however, a negative result: if we allowed a ray of light to fall through a Nicol's prism upon the fluid surface of the sodium-potassium alloy which formed the anode, and altered the position of the plane of polarization by turning the Nicol, we found the photoelectric dispersion to be independent of the azimuth of the light.

It must be remarked that in the "reversed" arrangement of the cell here employed the current strength, even in strong light, is far too small to give a measurable deflexion, even on the very sensitive galvanometer which we employed to measure the photoelectric currents. We therefore employed the same method which we had used in the experiments upon ultra-violet light, *i.e.* we connected the alkali-metal surface with the positive pole of the above-described battery, and the opposed platinum electrode with the quadrant electrometer. The passage of electricity through the cell betrayed itself then by the constant increase in the deflexion of the electrometer-needle. A constant condition of the instrument may be attained by making an earth connexion through a very large resistance (a pencil mark on an insulating surface) to the wire leading to the electrometer.

Not only does the fact that the transference of electricity is not affected by change in the direction of the light-vibrations with respect to the plane of the anode prove that the seat of photoelectric action is not at the anode, but we have the further evidence that this action is perceptibly increased if the platinum wire which serves as kathode is exposed to the direct action of the light. It even continues of the same intensity when by inclining the bulb the alkali-metal is made to flow over into the side bulb and is thus removed from the cell*.

Since a clean platinum-wire in a vacuum shows no photoelectric action in ordinary light, its sensitiveness can only have been communicated to it by contact with the alkali-metal or its vapour. As we see, the result of this experiment also leads us to the conclusion that the light has acted not on the *anode* of alkali-metal but on the platinum *kathode* made sensitive by its superficial coating, and we might expect that the "reversed" cell would lose its sensitiveness upon ignition of the platinum wire. But, as we have said, this expectation was, curiously enough, not verified by experiment. There

* The form of the cell is shown in fig., Wied. *Ann.* xlii. p. 564 (1891); see also Phil. Mag. 1896, xli. p. 220.

remained then only the supposition that the inner glass wall of the bulb had become covered with a layer, by contact with the alkali-metal, from which negative electricity escaped when the light fell upon it. In order to remove this source of error also the whole wall of the cell must be maintained at the same potential as the anode of alkali-metal, so that there could be no fall of potential from it to the wall. We attained this result by covering the outside of the bulb with silver by precipitation; with the exception of a small space where the kathode-wire was melted into the bulb and a "window" for the entrance of the light.

If now the alkali-metal surface and, with it, the glass wall of the bulb was charged with positive electricity, and the kathode-wire was connected to earth, then immediately after ignition of the wire no loss of electricity occurred in light, not even when a beam of sunlight entered through the window in the silver coating. Not until after some time, when the wire on cooling had again covered itself with a coating of alkali-metal, could the photoelectric discharge be again observed with increasing distinctness.

Thus the experiments with ordinary light on surfaces of alkali-metal in a vacuum lead to the same result as those with ultra-violet light, namely, that the photoelectric action is limited to the kathode.

We have pleasure in gratefully acknowledging the assistance we have received in this work from the Elizabeth Thompson Science Fund in Boston.

XLVII. *The Tinfoil Grating Detector for Electric Waves.* By T. Mizuno, *Rigakushi, Professor of Physics, First Higher Schools, Tokio**.

§ 1. In a paper †, which not long since I communicated to this Journal, I suggested that the change of the resistance of the grating might be due to a mechanical effect exerted upon it by impinging trains of electric waves. In other words, electric waves might give impulses to some of the strips of the grating in such a way as to let leaflets on their margins come in contact with one another, thereby causing a diminution of resistance. In order to confirm this view, further inquiries were carried out soon after the communication of the above-mentioned paper.

* From a separate impression from the Journal of the College of Science, Imperial University, Tokio, Japan, vol. ix. part 2. Communicated by the Author.

† "Note on Tinfoil Grating as a Detector for Electric Waves," Phil. Mag. vol. xl. p. 497 (1895).

§ 2. Having constructed about forty gratings and tested their action, I found to my surprise that while some were extremely sensitive, others were not, being even utterly indifferent to the impulses of electric waves, although they had all been prepared with the same care and apparently with the same success.

This led me to undertake a closer examination of such gratings, which gave results that throw much light upon their nature. But before these results can be stated, it is necessary to describe in detail my way of preparing the gratings, because upon that their sensibility wholly depends.

§ 3. The face of a flat wooden block of convenient size, say 10 centim. on a side, was pasted over with very fine tinfoil, as described in my former paper.

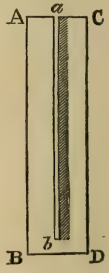
Then came cutting lines into the tinfoil, to which particular attention was given. Along the edge of a bamboo ruler a sharp knife, held always inclined away from the ruler, was drawn lightly across the surface of the tinfoil. In this way many fine parallel slits were cut in the tinfoil, so as to make one continuous, regular, zigzag line, as shown in fig. 1.

Fig. 1.

A few of the gratings, thus carefully prepared, were found to be sensitive. But experience has taught me that success in preparing good detectors depends, to a large extent, upon the nature of the wood block on which the tinfoil is pasted in the first place, and next upon the degree of adhesion of the foil to the wood. A soft wood is preferable to a hard one, and the paste used should not be thick enough to make the foil adhere too firmly.

§ 4. The majority of the slits of the sensitive gratings, when examined under a microscope, presented such an appearance as that shown in fig. 2. A B and C D represent two strips of foil with the very narrow slit or gap *a b* between them that has been formed by the knife. The shaded portion indicates the slope of the tinfoil found at one edge of each strip.

Fig. 2.



For the sake of clearness, there is shown in fig. 3 an end view, that is, a section of the two strips perpendicular to their lengths. The shaded portions indicate the tinfoil strips, A B and C D in fig. 2, of which the edge of one strip, C D, extends some distrance into the gap, *a b*, and forms the slope mentioned above. Along this slope the tinfoil presents many folds or wrinkles, which seems

to show that the tinfoil strip was somewhat stretched along its edge by the act of cutting it. Non-sensitive gratings showed none of these characteristic appearances, but had the gap between the strips much wider, with no decided slope and no appreciable folds along the edges of the strips. Hence for a grating to be sensitive, it appears to be necessary that the gaps should be narrow and their margins sloped and in folds.

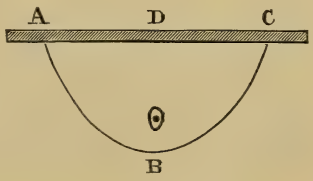
Fig. 3.

§ 5. Although I have been unable to see clearly the interior of a gap, yet it is quite reasonable to assume that in sensitive gratings there will be numbers of leaflets along the margins of adjacent tinfoil strips; and the existence of such leaflets once admitted, the explanation of the action of the gratings becomes clear. For in a properly constructed grating some of the leaflets may easily come in contact with one another under the action of the electric waves, because of the extremely small distance between any two opposite leaflets in the narrow gap. Then, too, it seems to me that these leaflets must be of various dimensions and, accordingly, some of them will be extremely sensitive, others less so but still highly sensitive, others again only moderately so. This being the case, the amount of change in the resistance of the grating must depend upon the intensity of energy of the impinging electric oscillations, for, when it is not great enough, only the most sensitive leaflets will come into play, but when it is sufficiently great all the effective leaflets will be brought into action. All the experiments I have yet made are in agreement with this representation of the matter.

§ 6. A grating, well prepared so as to fulfil the conditions mentioned above, proves to be an extremely sensitive detector for electric waves, as will be seen from the experiments which I now describe.

Experiment 1. A Hertzian parabolic vibrator, A B C, was placed horizontally with aperture turned upwards, as shown in fig. 4. The aperture was covered with a sufficiently large wooden plate, A D C, entirely coated with tinfoil. A grating, whose initial resistance was about 71 ohms, was placed at about 5 centim. from the plate and in a vertical line with the primary conductor, O, radiating electric waves of 60 centim. wave-length. Then, exciting the primary oscillations, I always found that

Fig. 4.

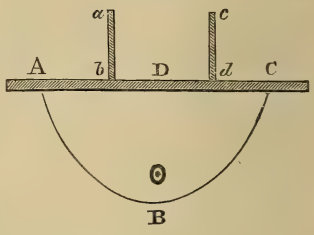


the resistance of the grating was diminished by from 1 to nearly 2 ohms.

The experiment was repeated after raising the plate, A D C, parallel to itself and keeping it at some height from the aperture, A C. Similar changes of resistance were also observed in this case. This phenomenon may of course be understood by considering the fact that some electric waves, which pass out of the uncovered portions of the parabolic vibrator, will, after going through the room and being reflected from the surrounding walls, ceilings, &c., come back ultimately to the grating in a much enfeebled state.

Experiment 2. The above experiment was modified by placing on the plate, A D C, a zinc box, *a b c d*, 17 centim. by 27 centim., without top or bottom and putting the grating inside it. In this case also, a change of resistance was observed, though smaller. It is then certain that although the side effects were got rid of, the top effect still remained, through which traces of waves might affect the resistance of the grating. The fact that we can annul the change of resistance by completely closing the top of the box with a metallic plate, seems especially to favour the above explanation.

Fig. 5.



Experiment 3. The grating was connected with the Wheatstone bridge by means of two leading wires, and at the same time placed inside the zinc box, just as in Exp. 2. After balance had been well established and the top of the box closed, the primary oscillations were excited. This time, the balance was at once destroyed and the resistance of the grating showed an appreciable diminution, in spite of the fact of the grating being wholly enclosed in a metallic box. Taking away one of the leading wires the phenomenon yet remained the same, though the change of resistance seemed somewhat smaller than in the former case. The leading wires thus appeared to catch up electric oscillations and guide them to the grating. Hence in experiments with electric waves it is necessary to keep the grating free from any exposed wires, which might easily take up electric disturbances. Such effects due to leading wires were observed also by Herr Aschkinass during his researches with these gratings.

§ 7. To what extent the sensibility of the grating reaches

will now be quite clear from the results of the above experiments. It is next of great importance to describe some experiments as to the variation of the sensibility. In its primitive state, the grating properly constructed is so sensitive that it can detect even the smallest electric oscillations. But after having been used a few times, its sensibility undergoes a sudden and decided diminution, and then remains nearly constant. At first, when the grating is exposed to electric waves and its resistance consequently diminished, a single tap given to it is almost enough to restore the resistance to its initial or primitive value. But when we have used the grating repeatedly, we find it necessary to give it a greater number of taps to effect this restoration. Later on, when the sensibility has diminished to a certain value, it seems to retain that value without any decided further change for a long time. This variation in the sensibility may be accounted for in the following way:—As mentioned in § 5, the effective leaflets along the margins of the several tinfoil strips may be of different sizes, and some of them possibly very small. The smaller the leaflets the more sensitive to electric disturbances, and consequently the more liable to fatigue will they be. Hence in the primitive state such leaflets are easily affected by even very weak electric impulses, but soon lose this sensibility as a result both of the repeated electric disturbances and of the mechanical taps given to them each time.

§ 8. Though the sensibility of the grating thus always diminishes to a certain extent by a little use, still it is even in such a state far superior to that of an ordinary Hertzian resonator. Even where the latter fails, the grating always shows the presence of electric waves if there be any. Experiments on the nature of electric waves, namely, on rectilinear propagation, reflexion, refraction, diffraction, polarization, &c., can all be easily carried on by means of a properly constructed grating. Moreover, such a grating gives not only qualitative, but also quantitative results, to a certain extent, because the amount of diminution of the resistance depends upon the quantity of energy of the impinging waves. Hence, I believe, it may prove to be of great advantage to make use of such gratings in all lecture experiments as well as in laboratory researches on electric waves.

In conclusion I wish to express my thanks to Mr. U. Takashima for the kind and earnest assistance he has given me in the preparation of many of these gratings and in carrying out researches upon them.

XLVIII. *Carbon and Oxygen in the Sun.* *By* John Trowbridge*.

In 1887 Professor Hutchins, of Bowdoin College, and myself brought forward evidence to show that the peculiar bands of the voltaic-arc spectrum of carbon can be detected in the sun's spectrum. They are, however, almost obliterated by the overlying absorption-lines of other metals, especially by the lines due to iron. In order to form an idea of the amount of iron in the atmosphere of the sun which would be necessary to obliterate the banded spectra of carbon, I have compared the spectrum of carbon with that of carbon dust, and a definite proportion of iron distributed uniformly through it. The carbon dust and iron reduced by hydrogen was formed into pencils suitable for forming the arc†. Chemical analysis showed that the iron was uniformly mixed with the carbon dust; specimens taken from different sections of the terminals showed in the carbons which I burned in the electric arc 28 per cent. of iron and 72 per cent. of carbon.

The method of experimenting was as follows:—That portion of the spectrum of the sun which contains traces of the peculiar carbon band lying at wave-length 3883·7, and which had been almost obliterated by the lines of absorption of other metals, among them those of iron, was photographed. The pure carbon banded spectrum was photographed on the same plate immediately below the solar spectrum, and the spectrum of the mixture of iron and carbon immediately below this. The sun's spectrum can be regarded as a composite photograph, and the iron and carbon can also be regarded as a composite photograph. It was speedily seen that from 28 to 30 per cent. of iron in combination with 72 to 70 per cent. of carbon almost completely obliterated the peculiar banded spectrum of carbon. This proportion, therefore, of iron in the atmosphere of the sun, were there no vapours of other metals present, would be sufficient to prevent our seeing the full spectrum of carbon.

The iron in the carbon terminals which I employed greatly increased the conductivity, as will be seen from Table I., which was obtained in the following manner.

The carbons were separated by means of a micrometer-screw, and the current and difference of potential were mea-

* Communicated by the Author.

† I am indebted to Mr. John Lee, of the American Bell Telephone Co., for his skill in making the carbons and for analysis of the composite carbons.

sured with different lengths of arc. Table I. gives the results for pure carbon; Table II. for 28 per cent. of iron and 72 per cent. of carbon.

TABLE I.

Length of Arc in millims.	Amperes.	Volts.
1	27	25
2	23	24
3	22·5	20
4	20	18
5	16·5	15

TABLE II.

Length of Arc in millims.	Amperes.	Volts.
1	30·5	30
2	29	30
3	27·5	28
4	24	25
5	22	20
6	20	20
7	18	19
8	16	18

The length of the arc could be nearly doubled with the same current and the same voltage by the admixture of 28 per cent. of iron. The light was apparently greatly increased, but the difference in colour between the pure carbon light and the iron-carbon light made measurements unreliable.

Moissan* has shown that the carbon in an electric oven through which powerful electric currents have flowed is free from foreign admixtures. Deslandres has confirmed this, and finds only a trace of calcium present. The self purification comes from a species of distillation of the volatile impurities. The purest carbon is found at the negative pole. The light of the electric furnace is due to the combustion of carbon. Can we conclude that the sun is a vast electric furnace?

If the voltaic arc is formed in rarefied air or under water, its

* *Comptes Rendus*, cxx. pp. 1259–1260 (1895).

brilliancy diminishes greatly. On the other hand, an atmosphere of oxygen greatly augments its vividness. The question therefore whether oxygen exists in the sun is closely related to questions in regard to the presence of carbon, when we consider the temperature and light of the sun.

If suppositions also are made in regard to the magnetic condition of the atmosphere of the sun, it is of great interest to determine whether oxygen exists there, for oxygen has been shown by Faraday, and later by Professor Dewar, to be strongly magnetic.

Professor Henry Draper brought forward evidence to prove the existence of bright oxygen lines in the solar spectrum. Professor Hutchins, of Bowdoin College, and myself examined this evidence, and after a long study of the oxygen spectrum in comparison with the solar spectrum, came to the conclusion that the bright lines of oxygen could not be distinguished in the solar spectrum. We published our paper in 1885. I have lately studied the subject from another standpoint; having carefully examined the regions in the solar spectrum where the bright lines of oxygen should occur, if they manifest themselves, in order to see if any of the fine absorption-lines of iron in the spectrum of iron were absent, for it is reasonable to suppose that the bright nebulous lines of oxygen would obliterate the faintest lines of iron.

The method adopted by Draper for obtaining the spectrum of oxygen consisted in the employment of a powerful spark in ordinary air. To obtain this spark, the current from a dynamo running through the primary of a Ruhmkorf coil was suitably interrupted. By the use of an alternating machine and a step-up transformer, powerful sparks can be more readily obtained. Since the time of exposure with a grating of large dispersion is long, considerable heat is developed in the transformer from the strong currents which are necessary to produce a spark of sufficient brilliancy. I have therefore modified the method in the following manner. The spark-gap is enclosed in a suitable chamber, which can be exhausted. When the exhaustion is pushed to a certain point, the length of the spark can be increased ten or twenty times over its length in air, and a suitable spark for photographic purposes can therefore be obtained by the employment of far less energy in the transformer. A pressure of eight to ten inches of mercury in the exhausted vessel is sufficient. A quartz lens inserted in the wall of the exhausted chamber serves to focus the light of the spark on the slit of the spectroscope.

The following table gives the certain oxygen lines and iron lines in the same region of the spectrum :—

O.	Fe in Sun.	Intensity.
4631	4629·44	1
	4730·22	4
	4630·91	1
	4631·61	1
4656	4654·7	
	4657·71	
4683	4683·04	1
	4683·93	2
4601·5	4600·09	1*n*
	4601·8	1
	4602·11	4
4607	4604·84	1*n*
	4605·52	1
	4606·32	
	4607·79	6
4613	4611·38	8
	4613·35	4
	4614·29	1
4693·5	4691·52	6
	4694·97	1

The faintest iron lines are therefore not obliterated in the spaces where the oxygen lines should occur.

If we examine the great absorption region about the K-line, we find that between wave-lengths 3930·29 and 3938·55 Rowland gives eight lines which coincide with iron lines.

From the table of wave-lengths of iron lines in the arc spectrum given in the Report of the British Association for 1891, I find the following lines given between these limits :—

3930·37 *
3931·22 *
3932·71 *
3933·01 *
3933·75
3934·47
3934·81 *
3935·40 *
3935·92 *
3937·42 *
3938·16
3938·59

The starred lines are probably the iron lines given by Rowland in his list of standard solar lines. The iron lines that are not starred apparently are obliterated in the great absorption-band near the calcium line K.

Lord Salisbury, in his address before the British Association at Oxford, 1894, remarks :—" Oxygen constitutes the largest portion of the solid and liquid substance of our planet, so far as we know it ; and nitrogen is very far the predominant constituent of our atmosphere. If the earth is a detached bit whirled off the mass of the sun, as cosmogonists love to tell us, how comes it that in leaving the sun we cleaned him out so completely of his nitrogen and oxygen that not a trace of these gases remains behind to be discovered even by the sensitive vision of the spectroscope?"

Although we have not succeeded in detecting oxygen in the sun, it seems to me that the character of its light, the fact of the combustion of carbon in its mass, the conditions for the incandescence of the oxides of the rare earths which exist, would prevent the detection of oxygen in its uncombined state. Notwithstanding the negative evidence which I have brought forward, I cannot help feeling strongly that oxygen is present in the sun and that the sun's light is due to carbon vapour in an atmosphere of oxygen.

Jefferson Physical Laboratory,
Harvard University, Cambridge, Mass., U.S.

XLIX. *On the Effects of Magnetic Stress in Magnetostriction.* *By* H. Nagaoka *and* E. Taylor Jones*.

MATHEMATICAL expressions for electric and magnetic stresses were given first by Maxwell in his 'Electricity and Magnetism.' In Art. 105 of this treatise the values are found of stresses in a dielectric medium, which may be regarded as producing the observed electric action between two systems ; in Art. 644 corresponding expressions are found for a magnetic medium. The expressions in the two cases are, however, not similar ; *i.e.* the electric stress in a dielectric cannot be deduced from the magnetic stress in a magnetic substance simply by substituting specific inductive capacity for magnetic permeability. It does not seem quite clear whether Maxwell intended his expressions to apply to the case of induced as well as rigid magnetization.

In 1881, v. Helmholtz † published expressions for the stresses which are more general than those of Maxwell, since they contain terms depending on possible changes of density in the medium. These expressions were assumed to have the same form for a dielectric as for a temporarily magnetized substance, but not necessarily for a permanent magnet ; and if

* Communicated by the Authors.
† Wied. *Ann.* xiii. p. 400 (1881) ; or *Abh.* i. p. 798.

the terms depending on changes of density are neglected, the expressions reduce to those given by Maxwell in Art. 105.

In 1884, Kirchhoff * gave a still more general theory, including in his equations terms depending on elongations in the direction of the electric or magnetic force, as well as terms depending on the change of density. The fundamental relations between intensity of magnetization I (A, B, C) and resultant magnetizing force H (α, β, γ) are thus :—

$$A = \left\{ K - K'\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - K''\frac{\partial u}{\partial x} \right\} \alpha,$$

$$B = \left\{ K - K'\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - K''\frac{\partial v}{\partial y} \right\} \beta,$$

$$C = \left\{ K - K'\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - K''\frac{\partial w}{\partial z} \right\} \gamma,$$

where u, v, w are the component displacements of a particle of the medium at (x, y, z) in the directions of the axes of coordinates, and K, K′, K″ are coefficients depending on the nature of the medium. It is to be noticed that the quantities in the brackets are nearly equal to the susceptibility, because the elongations and change of volume are considered to be very small. From these and the principle of Conservation of Energy, Kirchhoff deduces his general expressions (in Maxwellian notation) for the stresses in a substance magnetized by induction :—

$$P_{xx} = -\left(\frac{1}{4\pi} + K + \frac{K''}{2}\right)\alpha^2 + \tfrac{1}{2}\left(\frac{1}{4\pi} + K - K'\right)(\alpha^2 + \beta^2 + \gamma^2),$$

$$P_{yy} = -\left(\frac{1}{4\pi} + K + \frac{K''}{2}\right)\beta^2 + \tfrac{1}{2}\left(\frac{1}{4\pi} + K - K'\right)(\alpha^2 + \beta^2 + \gamma^2),$$

$$P_{zz} = -\left(\frac{1}{4\pi} + K + \frac{K''}{2}\right)\gamma^2 + \tfrac{1}{2}\left(\frac{1}{4\pi} + K - K'\right)(\alpha^2 + \beta^2 + \gamma^2),$$

$$P_{yz} = P_{zy} = -\left(\frac{1}{4\pi} + K + \frac{K''}{2}\right)\beta\gamma,$$

$$P_{zx} = P_{xz} = -\left(\frac{1}{4\pi} + K + \frac{K''}{2}\right)\gamma\alpha,$$

$$P_{xy} = P_{yx} = -\left(\frac{1}{4\pi} + K + \frac{K''}{2}\right)\alpha\beta.$$

These expressions reduce to those of v. Helmholtz if it be assumed that $K''=0$, and to those of Maxwell for electric stress (Art. 106) if we put $K'=K''=0$, and $1+4\pi K=$ specific inductive capacity of the medium.

It will be seen from these expressions that Kirchhoff's

* Wied. *Ann.* xxiv. p. 52 (1885); *Ges. Abh., Nachtrag,* p. 91 (1891).

theory makes the stress depend principally on the coefficients K′ and K″. The experiments of Villari, Lord Kelvin, and Ewing on the effects of longitudinal stress on magnetization show that

$$-(3K'+K'')+K'.\frac{\text{Young's Modulus}}{\text{Rigidity}} \doteqdot -\left(\frac{K'}{2}+K''\right)$$

in iron is a quantity which may amount to 10^5 for moderate magnetizations, so that in this case K is quite negligible in comparison with one or both of the two other coefficients.

The preponderating influence exercised by these latter factors will, perhaps, explain the existence of a maximum elongation in iron and the continual contraction in nickel. Maxwell's expressions for magnetic stress (Art. 644), in the case when B and H have the same direction, are, however, apparently by a coincidence deduced from the above expressions of Kirchhoff by putting $K''=0$, $K'=K=$susceptibility.

About the same time Lorberg* and J. J. Thomson† discussed the present problem in a manner similar to that of Kirchhoff. More recently Hertz‡ arrived at expressions of precisely the same form as those given by v. Helmholtz. In comparing his expressions with those given by Maxwell for the general case of a magnet in which the induction and magnetic force have different directions, Hertz says (p. 281) :—

"A difference of far greater importance (*i. e.* than the effect of a change of density by electromagnetic strain) is that in Maxwell's theory the tangential stresses P_{xy} and P_{yx} have different values, while in our theory they are identical. Under our system of stresses every material element, when left to itself, will only change its shape; under that of Maxwell it will also experience a rotation as a whole. The Maxwellian stresses cannot therefore arise from processes in the interior of the element, and can have no place in the present theory. They are, however, admissible on the assumption that in the interior of the body in motion, the æther remains at rest and furnishes the necessary fulcrum for the rotation which takes place."

In attempting to calculate the change of dimensions of a body due to magnetization we are at once placed face to face with the question as to whether these stresses actually

* Wied. *Ann.* xxi. p. 300 (1884).

† Application of Dynamics, §§ 35–37 (1886).

‡ *Ausbreitung der electrischen Kraft*, p. 275 (Leipzig, 1892).

Hertz, speaking of equation (6*c*), p. 284, which is obtained as a simplified form of stress agreeing in the general case with that found by v. Helmholtz, says that the stress is the same as that given by Maxwell in Art. 642 of the treatise. His expression, however, is not that given in Art. 642 but in Art. 105.

exist in the body, or whether they, existing in a medium in which the particles of the body are imbedded, produce modified stresses in the body.

Kirchhoff supposed the surface tractions acting on a piece of magnetized soft iron to be the same as if the stresses P_{xx} actually existed in the iron and in the surrounding air, where K, K', K'' are put $=0$, and proceeded to calculate the changes in the dimensions of a soft iron sphere placed in a uniform magnetic field, due to a system of stresses which satisfy the ordinary equations of an elastic solid and at the surface of the sphere have the above given values.

Having obtained the general solution for the strain of a sphere, Kirchhoff gives a numerical example, neglecting the terms affected with $(K-K')$ and K'', supposing these quantities to be very small in comparison with K^2. Kirchhoff's final value for the elongation of a soft iron sphere is therefore precisely the same as that which would be given by Maxwell's system of stresses (Art. 644).

Proceeding exactly on the lines of Kirchhoff, Cantone* has calculated the variations δl and δv of length and volume of a soft iron ellipsoid of revolution, and finds that for an ellipsoid of great eccentricity and length l,

$$\frac{\delta l}{l}\dagger = \frac{I^2}{E}\left\{\pi + \frac{K-K'}{4K^2} - \frac{K''}{2K^2}\right\},$$

$$\frac{\delta v}{v} = \frac{I^2}{E}\left\{\pi + \frac{3(K-K')}{4K^2} - \frac{K''}{4K^2}\right\} \text{approximately,}$$

where $E=$ Young's Modulus for the iron (on the supposition that the Poisson ratio $=\frac{1}{4}$). He also measured experimentally the changes of length and volume of a soft iron ellipsoid due to uniform magnetization, and, assuming that these were due entirely to Kirchhoff's system of stresses, deduced the values of K' and K''. He found the change of volume to be negligibly small, and for K' and K'' the values 44,000 and $-92{,}000$ for the mean field-intensity ($H=33$ C.G.S., H (in iron)$=3{\cdot}5$, $I=250$) which he employed.

* *Mem. R. Acc. Linc.* ser. 4, vol. vi. p. 487 (1890); Wied. *Electr.* iii. p. 740.

† More exactly,

$$\frac{\delta l}{l} = \frac{4\pi I^2}{3E}\left(\frac{1+\theta}{1+2\theta}\right) + \frac{IH}{2E(1+2\theta)} - \frac{K'H^2}{2(1+2\theta)E} - \frac{K''H^2}{2E},$$

where θ is a constant defined by the equation

$$\frac{E}{2}\left(\frac{1+2\theta}{1+3\theta}\right) = \text{Rigidity}.$$

The change of length of a soft-iron ellipsoid has, however, been investigated over a much greater range of field-intensities by one of us*, and the results represented by curves as functions of magnetizing force, of the square of magnetization, and of the former, respectively. If the change of length due to Kirchhoff's system of stresses be calculated from Cantone's formula, and the result subtracted from the observed change of length, these curves (K' and K'' being provisionally disregarded) assume the forms shown in figs. 1, 2, 3.

Since the coefficients K' and K'' are not known with any accuracy, for the iron used, the correction due to them cannot at present be calculated. The ordinates of the full-line curves represent therefore the changes of length produced by uniform magnetization, corrected on Kirchhoff's theory for the partial stresses represented by terms containing H and K, which were measured at the same time. The residual change of length, represented by the ordinates in figs. 1, 2, 3, must be due to the other unknown terms in the expressions for the stresses, or to some change of molecular arrangement accompanying the magnetization.

A similar process was followed by More†, who measured the change of length of part of an iron wire magnetized by a coil. It was assumed, however, that there existed in the wire a contracting stress $\frac{B^2}{8\pi}$, the reason given for this assumption being that if the wire were cut in two, the two ends would be held together by a force $\frac{B^2}{8\pi}$ per unit area. Now, according both to Maxwell's theory and to that of Kirchhoff, the tension in a narrow air-gap in the wire is $\frac{B^2}{8\pi}$, and this has been experimentally verified by one of us over a large range of magnetizing field‡; but both theories give different values for the stress inside the iron. Even if the wire were cut in two, the force holding the parts together would not be $\frac{B^2}{8\pi}$ per unit area unless the coil were also cut into two parts, into which the pieces of wire were rigidly fixed, so that part of the total tractive force is due to the mutual attraction of the two coils.

* H. Nagaoka, Wied. *Ann.* liii. p. 487 (1894).

† Phil. Mag. Oct. 1895.

‡ See du Bois, *Magnetische Kreise*, § 104; E. T. Jones, Wied. *Ann.* lvii. p. 271 (1896); Phil. Mag. March 1896; Wiedemann's *Electr.* iii. p. 640.

Further, as has been already pointed out by Chree*, the longitudinal stress in iron is, according to both theories, a

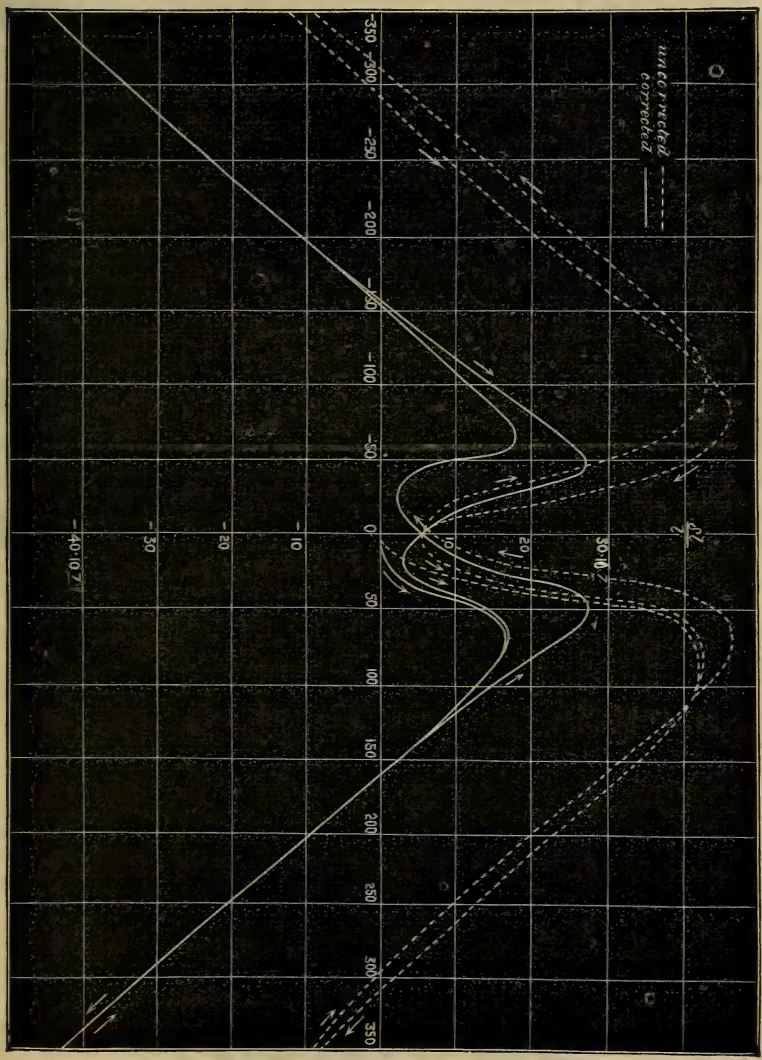


Fig. 1.

tension, not a pressure; its effect is therefore to *lengthen*, not to shorten a piece of soft iron along the lines of force.

* 'Nature,' Jan. 23, 1896.

Fig. 2.

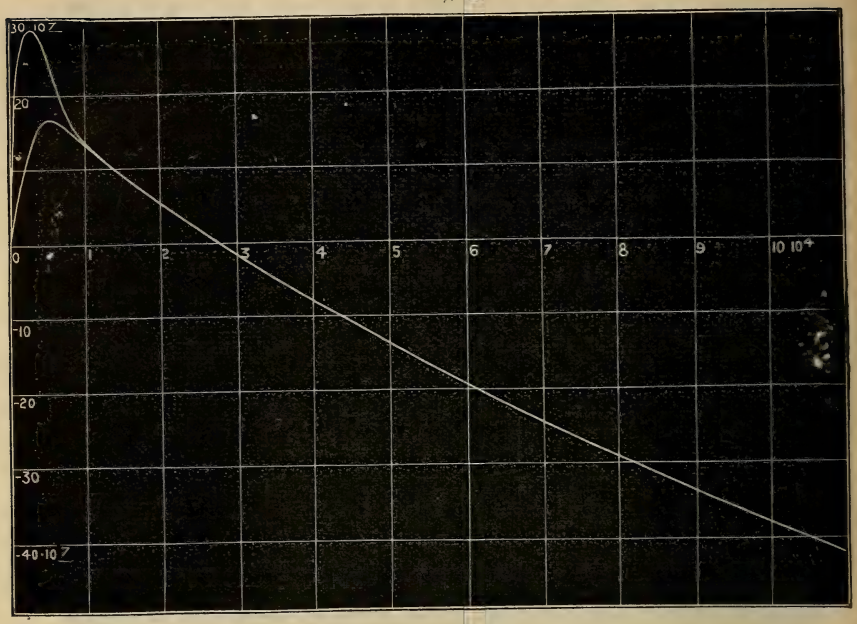
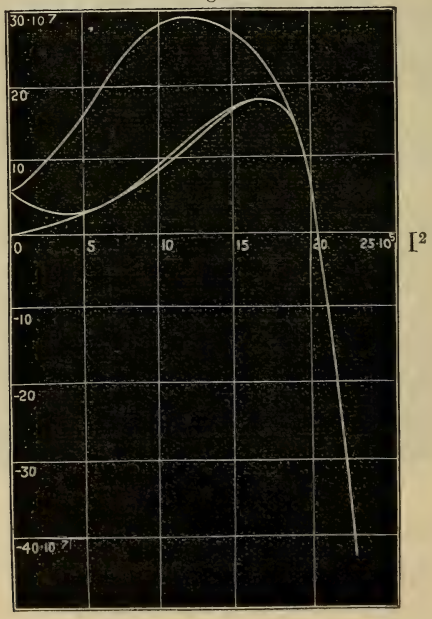

30·10 7
20
0
1
2
3
4
5
6
7
8
9
10 10 4
H²
-10
-20
-30
-40·10 7

Fig. 3.


30·10 7
20
10
0
5
10
15
20
25·10 5
I²
-10
-20
-30
-40·10 7

This use of the expression $\frac{B^2}{8\pi}$, incorrect according to existing theories, appears to have been first made by S. Bidwell*.

As to the stresses inside the iron, Maxwell's theory (Art. 642) gives "a hydrostatic pressure $\frac{H^2}{8\pi}$ combined with a longitudinal tension $\frac{BH}{4\pi}$ along the lines of force;" while Kirchhoff's theory gives a hydrostatic pressure

$$\frac{1}{2}\left\{\frac{1}{4\pi}+K-K'\right\}H^2=\frac{BH}{8\pi}-\frac{K'H^2}{2}$$

combined with a tension

$$\left\{\frac{1}{4\pi}+K+\frac{K''}{2}\right\}H^2=\frac{BH}{4\pi}+\frac{K''H^2}{2}$$

along the lines of force.

Other systems of stresses in an isotropic elastic medium, which are equivalent to gravitational and electrostatic forces in certain cases, are discussed by Chree†, and shown to be essentially different from that given by Maxwell (Art. 105).

So far there does not appear to be sufficient experimental evidence to enable one to decide between these theories. Calculating the strain of an anchor-ring according to Kirchhoff's theory, we easily find that

$$\frac{\delta l}{l}=\frac{H}{4n(1+3\theta)}(I-K'H)$$
$$=\frac{1}{3}\frac{\delta v}{v},$$

n denoting the rigidity.

As the principal factor K' can be found from the measurement of effect of pressure on magnetization, the easiest method for making the crucial test would be to try experiments on an anchor-ring. Moreover, it is worth while to notice that the volume-change would in this case amount to the order $10^{-5}.v$ in iron and to $-10^{-4}.v$ in nickel, calculating the value of K' from the experiments of Bidwell on rings of these two metals.

In conclusion, we would express our thanks to Dr. H. du Bois, at whose suggestion the preparation of this communication was undertaken.

Bangor, Berlin, } March 25, 1896.

* Phil. Trans. clxxix. p. 217 (1888).
† Proc. Edin. Math. Soc. xi. p. 107 (1892–93).

L. *Notices respecting New Books.*

Index of Spectra. Appendix G. By W. Marshall Watts, *D.Sc., F.I.C.* Manchester: Abel Heywood and Son, 1896.

THE present Appendix to Dr. Watts's 'Index of Spectra' brings the record of spectroscopic work down to the present time, and contains results of observations published within the past three years. It opens with Rowland's table of standard wave-lengths, which is followed by an account of the researches of Eder and Valenta. These include the spark-spectra of sodium, potassium, and cadmium, and the line and band spectra of mercury. The oxyhydrogen-flame spectra of several metals and oxides, observed by Hartley, are next tabulated, and the record is brought to a close with an account of the recent work of Runge and Paschen on helium and par-helium, the two constituents of cleveite gas.—J. L. H.

LI. *Intelligence and Miscellaneous Articles.*

ON AN ELECTROCHEMICAL ACTION OF THE RÖNTGEN RAYS ON SILVER BROMIDE. BY PROF. DR. FRANZ STREINITZ.

TO Röntgen we are indebted for his great discovery of the property of the x-rays of exciting fluorescence and producing chemical reductions on a photographic plate. According to his previous experiments, these properties are the only ones which the rays have in common with those of light. Now light alters not only the electromotive deportment, but also the conductivity of the silver haloids. The proofs of this were furnished by Becquerel and Svante Arrhenius. It can therefore scarcely be doubted that electrochemical changes will be produced by the Röntgen rays; of course it is a different question whether they will be accessible to observation.

Experiments were made in both directions. In order to establish the fact of a change in the conductivity, a method used by Arrhenius (*Wiener Berichte*, vol. xcvi. p. 831, 1887) was adopted. On a glass tube a silver wire was wound bifilar, and then coated with ammoniacal solution of silver chloride. After the ammonia and water had been evaporated, the glass tube was then placed in a light-tight box, out of which the ends of the wire projected. These ends were connected with a source of electricity and a very sensitive galvanometer (Thomson-Carpentier). The deflexion in the galvanometer showed variations when discharges were passed through a Hittorf's tube near to the box. These, however, were manifestly due to inductive actions on the galvanometer circuit. For when the induced circuit was open, an increase in the deflexion could not with certainty be established—as ought to be the case, since there was an increase of the conducting power—in comparison with the deflexion which was obtained before the discharge was set up.

Experiments on the influence on electromotive behaviour were attended with better success. A square platinum-foil, 2 cm. in the side, was coated electrolytically with an exceedingly thin layer of

silver bromide. Combined in dilute potassium-bromide solutions with a standard electrode, the electrode in question showed sensitiveness to light. This is easily proved with the help of a quadrant electrometer (Luggin, Ostwald's *Zeitschrift für phys. Chemie*, xiv. p. 387, 1894). A candle placed at a distance of 25 cm. from the electrode produced in half an hour a diminution of 0·022 V. in the electromotive force of the combination

$$Zn/Zn\,SO_4\,aq + K_2\,SO_4\,aq + BrK\,aq + Br\,Ag/Pt.$$

At the same time it is to be remembered that the light only struck one side of the platinum-foil, while the other, which was also sensitized, remained dark. If now a carefully enclosed discharge-tube through which induction-sparks passed was substituted for the candle, a diminution in the electromotive force could also be observed. With a small induction-coil the change amounted to 0·017 volt in the course of 45 minutes, and in another experiment with a larger induction-coil to 0·019 in 40 minutes.

By a corresponding increase in the delicacy of the method, the electrochemical deportment of Röntgen rays may possibly furnish a more convenient method of investigating them than that with the help of photography.—*Wiener Berichte*, Feb. 6, 1896.

TRIANGULATION BY MEANS OF THE CATHODE PHOTOGRAPHY.

BY JOHN TROWBRIDGE.

Photography by means of the Röntgen rays seems already to be of great importance in examining certain portions of the human body to determine the presence of metallic bodies, calcareous formations, and fragments of glass. The shadow pictures as they are taken at present, however, do not give the approximate position of the shots, for instance, embedded in the flesh. They indicate only the line in which they are situated. It occurred to me that the principles of triangulation could be applied with success to determine more exactly the position of the metallic particles. I was led to this conclusion by considering Rumford's photometer. This instrument, it is well known, consists merely of a vertical rod placed opposite a suitable screen of white paper. The two lights, the intensities of which are to be compared, are placed in a fixed position, and throw two shadows of the rod on the screen. From a measurement of the positions of the lights when shadows of equal intensity are thrown on the screen, an extinction of the brightness of the lights can be obtained. Moreover, by measuring the distance between the shadows, and by drawing lines from them to the lights, the position of the rod throwing the shadows can be determined. This position is evidently at the intersection of these lines.

I have used two Crookes' tubes with two terminals making an angle with each other, and have employed a to-and-fro excitation by means of a Tesla coil. A suitable screen of glass shielded the sensitive plate first from one cathode and then from the other. From the distance between the shadow pictures of a shot, for instance, on the back of the hand and from the position of the

terminals, the height of the shot above the sensitive plate could be estimated. It seems to me that this method promises to be of importance in the surgery of the extremities of the body; for the question whether to make an incision from the palm of a child's hand or from the back of the hand is an important one. Stereoscopic pictures can also be obtained.

The use of a Tesla coil in obtaining shadow pictures is advantageous in certain respects, for by changing the size of the spark-gap in the primary circuit of the Tesla coil one has a great range of electrical energy at command. This range can be still further increased by putting the spark-gap in a magnetic field. I have taken such pictures in less than a minute, showing the bones in the fingers. The tubes were, at first, destroyed by disruptive sparks over the surface of the tube which apparently penetrated the glass between the platinum terminals and the glass. I have lately discovered, however, that if the terminals of the tube are placed in a vessel filled with paraffin oil, and if the oil is kept cool by an outside vessel filled with snow or ice, the entire energy developed by the Tesla coil can be employed, and the tubes are not destroyed.

I have tried wooden lenses, both double convex and double concave, in order to see whether the rays travel slower or faster in wood than in air, but my results are negative. A copper ring placed on a double convex lens of wood of approximately six inches focus, and one also on a concave lens of the same radius as the surfaces of the double convex lens, gave shadow pictures of the ring which were of the same size and character as those of an equal copper ring placed in air at the same distance from the sensitive plate.

We naturally turn to Maxwell's great treatise on Electricity and Magnetism, to see if a hint of this new phenomenon cannot be found there: for I believe there is no manifestation of electromagnetism since the death of Maxwell which has not been predicted or treated by him in one form or another in his remarkable book. In section 792, vol. ii. of the treatise on Electricity and Magnetism, he says:—"Hence the combined effect of the electrostatic and the electrokinetic stresses is a pressure equal to 2ρ in the direction of the propagation of the wave. Now 2ρ also expresses the whole energy in unit of volume. Hence in a medium in which waves are propagated there is a pressure in the direction normal to the waves, and numerically equal to the energy in unit of volume. Thus, if in strong sunlight the energy of the light which falls on one square foot is 83·4 foot-pounds per second, the mean energy in one cubic foot of sunlight is about 0·0000000882 of a foot-pound, and the mean pressure on a square foot is 0·0000000882 of a pound weight. A flat body exposed to sunlight would experience this pressure on its illuminated side only, and would therefore be repelled from the side on which light falls. It is probable that a much greater energy of radiation might be obtained by means of the concentrated rays of the electric lamp. Such rays falling on a thin metallic disk, delicately suspended in a vacuum, might perhaps produce an observable mechanical effect."—*American Journal of Science*, March, 1896.

THE

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[FIFTH SERIES.]

JUNE 1896.

LII. *Thermo-Electric Interpolation Formulæ.* *By* SILAS W. HOLMAN.*

IN this paper are collected the several well-known types of formulæ for expressing the thermal electromotive force of a couple as a function of the temperature of its junctions. Two new formulæ are also proposed. All then are tested against the most reliable experimental data upon the subject, and their relative merits discussed.

The Existing Formulæ.

Consider a simple closed electric circuit composed of two different metals, each homogeneous in matter and temper, the metals being in contact at two points. For simplicity, assume the metals to be in the form of wires joined at their ends. Let one junction be at a temperature of $h°$, the other of $c°$, on the ordinary Centigrade scale. Let $\Sigma_c^h e$ be employed as a suggestive symbol to denote the resultant electromotive force in the circuit, induction being excluded from consideration. Then $\Sigma_c^h e$ is a function of h and c which involves constants dependent upon the nature of the metals, and which may be represented by

$$\Sigma_c^h e = f(h, c).$$

The discovery of the natural expression for $f(h, c)$ is not only

* From an advance proof of the 'Proceedings of the American Academy,' vol. xxxi. (n. s. xxiii.) p. 193. Communicated by the Author.

of scientific importance, but is urgently needed in the development of the art of pyrometry. At present even a satisfactory empirical formula for interpolation is lacking, the best still being probably that of Avenarius and Tait.

The existing formulæ are the five following :—

Ordinary or parabolic :

$$\Sigma_0^t e = at + bt^2 + ct^3 + \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is, of course, merely a series in ascending powers of t, where one junction is at any temperature $t°$ C., and the other at 0° C., a, b, and c being constants. A more general form for the case where the cold junction is at any constant temperature, $t_1°$, is

$$\Sigma_{t1}^t e = a(t-t_1) + b(t^2-t_1^2) + c(t^3-t_1^3) + \ldots$$

These expressions may, of course, be inverted, giving t as a function of Σe.

Avenarius :

$$\Sigma_c^h e = (h-c)\{a+b(h+c)\} \quad . \quad . \quad . \quad . \quad (2)$$

in accordance with the foregoing notation.

Thomson :

$$\Sigma_c^h e = a(\tau_h - \tau_c)\left\{\tau_n - \frac{\tau_h+\tau_c}{2}\right\} \quad . \quad . \quad . \quad (3)$$

where τ is the absolute temperature, τ_n being that of the "neutral point."

Tait :

$$\Sigma_0^h e = (k'-k)(\tau_h - \tau_c)\left\{\tau_n - \frac{\tau_h+\tau_c}{2}\right\} \quad . \quad . \quad . \quad (4)$$

Both of the last two, by the substitution of $t+273$ for τ, obviously reduce to the Avenarius form.

Barus :

$$e_h + e_c = 10^{P+Qh} + 10^{P'+Q'c}, \quad . \quad . \quad . \quad . \quad (5)$$

where e_h represents the thermal E.M.F. of the hot junction and e_c that of the cold junction. In view, however, of the existence of the Thomson effect, these symbols can strictly be interpreted only as having the meaning that $e_h - e_c = \Sigma_c^h e$.

Note.—With regard to the Avenarius, Thomson, and Tait expressions, it may be remarked that they are not only mutually equivalent, but that if t_c or τ_c becomes 0° C. they reduce at once to the ordinary parabolic form of two terms :

$$\Sigma_0^t e = at + bt^2.$$

They are all, therefore, forms which must apply if the latter

purely empirical expression for the same temperature ranges applies, and with the same closeness, so that it is unnecessary to test more than one of the first four expressions against any one set of data. Also the fact that the Avenarius and Tait equations approximately conform to the observed data does not necessarily in any material degree strengthen the hypotheses which are adduced to show that these equations are a natural expression of the law.

Without attempting here a further analysis of the components making up the resultant E.M.F. $\Sigma_c^h e$, which is the measured E.M.F. of the thermo-couple, the proposed interpolation formulæ will be merely developed and applied. It may, however, be suggested in passing, that there seems to the writer to be little hope of arriving at a close approximation to the natural law except through an expression which shall contain separate terms representing the temperature function of the component arising at the contact of the dissimilar metals, and that arising from the inequality of temperature of the ends of each (homogeneous) element (Thomson E.M.F.). The parabolic and Avenarius formulæ would comply in part with this requirement on the supposition that the E.M.F. at contact varied as the first power, and the Thomson E.M.F. in both wires as the square of the temperature. And looked at from that point of view, the neutral point would seem to have an explanation materially different from that usually accorded to it.

The Proposed Formulæ.

Exponential Equation.—The significance of this proposed expression may be thus stated. Suppose the cold junction of the couple be maintained at the absolute zero of temperature, $\tau = 0°$, and its E.M.F. to be consequently zero. Let the other (hot) junction be at any temperature $\tau_h{}^\circ$ absolute. The proposed equation is based on the assumption that the total E.M.F of the couple would then be representable by

$$e' = m\tau_h^n.$$

where m and n are numerical constants. If then the cold junction were raised to any temperature $\tau_c{}^\circ$, there would be introduced an opposing E.M.F. e'', which would be expressible by

$$e'' = m\tau^n.$$

The resultant E.M.F $\Sigma_c^h e$ would then be $e' - e''$, and therefore expressible by

$$\Sigma_c^h e = m\tau_h^n - m\tau_c^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If in any instance, as is frequently the case in measurements, the temperature of the cold junction is maintained constant while that of the hot junction varies, then $m\tau_c^n$ becomes a constant, and it will be convenient to denote this constant by β when $\tau=273°$ abs. $=0°$ C. So that for this special case where the cold junction is at 0° C, and the hot junction at $t°$ C., we have

$$\Sigma_0^t e = m\tau^n - \beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This expression is not advanced as a possible natural form of the function $f(h, c)$. It is essentially empirical, and is not designed to account separately for the several distinct components entering into Σe. The fact that it closely fits the experimental data arises chiefly from the well known adaptability of the exponential equation to represent limited portions of curved lines. The equation also leads to certain inferences which appear inconsistent with the known thermo-electric laws, and fails to explain some known phenomena.

The evaluation of the constants m, n, and β is unfortunately attended by considerable labour. No application of the method of least squares readily presents itself, but by a method of successive approximations the values can be obtained with any desired degree of exactitude. Only two measured pairs of values of $\Sigma_0^t e$ and t are necessary for this approximation method, the third required pair being furnished by $\Sigma_0^0 e=0$ and $t=0$; although, of course, by the employment of three pairs of values well distributed in the data, a more closely fitting equation might frequently be obtained. The calibration of a thermo-couple for pyrometric work can thus be effected by the employment of but two known temperatures, and this, on account of the uncertainty of our knowledge of high melting-points, is of great importance in high temperature work.

Let $t_0=0°$ C., t', and t'' be the selected observed temperatures from which to compute the constant, so that $\tau_0=273°$, $\tau'=t'+273°$, $\tau''=t''+273°$ abs. And let $\Sigma_0^{t0} e=0$, $\Sigma_0^{t'} e$, $\Sigma_0^{t''} e$, be the corresponding observed E.M.F's. of the couple. Then, by substituting these in equation (7), and combining the three expressions, or their logarithms, we easily deduce

$$\beta = \frac{\Sigma_0^{t'} e}{\left(\frac{\tau'}{\tau_0}\right)^n - 1}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$n=\frac{\log(\Sigma_0^{t''}e+\beta)-\log(\Sigma_0^{t'}e+\beta)}{\log\tau''-\log\tau'}\ ; \quad . \quad . \quad (9)$$

$$m=\frac{\Sigma_0^{t''}e+\beta}{(\tau'')^n} \quad \text{or} \quad \frac{\Sigma_0^{t'}e+\beta}{(\tau')^n}. \quad . \quad . \quad . \quad . \quad (10)$$

By means of these the numerical values of the constants may be calculated from those of τ', τ'', $\Sigma_0^{t'}e$, &c., as follows :—

1. Assume as a first approximation some value of n, say $n=1$, unless some better approximation is in some way suggested. Substituting this value in (8), compute the corresponding value of β.

2. Using this as a first approximation, substitute it in (9) and compute the corresponding value of n.

3. Using this value as a second approximation to n, insert it in (8), and compute a second approximation to β.

4. With this compute a third approximation to n, and so continue until consistent values of β and n are found to the desired number of figures. Then compute m by (10).

The rate of convergence is not rapid, but after one or two approximations have been made an inspection of the rate will enable the computer to estimate values of n which will be nearer than the preceding approximation, and thus hasten the computation.

Where an equation is to be computed to best represent a progressive series of observed values of t and $\Sigma\, e$, this method is of course open to some objections, since it incorporates in the constants the accidental errors of the selected observations from which the constants are deduced. This difficulty can be sufficiently overcome by computing residuals between the equation and the data, and amending the equation if necessary to give them a better distribution.

Logarithmic Formula.—A very simple expression for interpolation is of the general form

$$\Sigma_0^t e=mt^n,$$

when m and n are constants. This serves fairly well for a short range, $t''-t'$, when $t'-0°$ is not less than one third of $t''-t'$.

The convenience of the expression arises from two facts : first, that its two constants are very easily evaluated either by computation or graphically from the logarithmic expression (whence the name)

$$\log\Sigma_0^t e=n\log t+\log m.$$

second, that its logarithmic plot is a straight line, since this

expression is the equation to a straight line if we regard $\log \Sigma_0^t e$ and $\log t$ as the variables. If, therefore, a series of values of Σe and t are known for a given couple, points obtained by plotting $\log t$ as abscissas and $\log \Sigma e$ as ordinates should lie along a straight line. Thus a couple may be completely "calibrated" for all temperatures by measuring Σe and t for any two values of t (suitably disposed). The constants m and n may be computed, or a plot of $\log \Sigma e$ and $\log t$ may be made, and a straight line be drawn through them. Graphical interpolation on this line will then of course yield the values of $\log t$ and hence of t corresponding to observed values of Σe, and *vice versa*, and, if desired, the constants m and n. The expression for t as a function of Σe is of course

$$t = m' (\Sigma_0^t e)^{n'}, \quad \text{or} \quad t = \left(\frac{\Sigma_0^t e}{m}\right)^{\frac{1}{n}}.$$

This formula is well adapted to pyrometric work not of the very highest grade of accuracy, and has been advantageously employed in connexion with the Le Chatelier thermo-electric pyrometer in a method to be described in a later article.

Test of Formulæ.

This will be made by applying the several formulæ to the experimental data of Barus, Holborn and Wien, Chassagny and Abraham, and Noll. These investigators employed modern methods of thermometry and of electrical measurement. Temperatures are either made in or reduced to the scale of the hydrogen (C. & A.), or of the air thermometer (B., H. & W., N.). Constants for the formulæ will be deduced, and the residuals or deviations of the data from the equations (*i.e.* δ = data − equation) will be computed for the observed points. For discussion these deviations will be expressed in percentages, viz. $100\frac{\delta}{e}$, rather than in microvolts or degrees. This is preferable because the process of measurement of the E.M.F., and to some extent at least of the temperature, is such as to yield results of a nearly constant fractional or percentage precision at all temperatures rather than of a constant number of microvolts or degrees. Thus by comparing percentages we eliminate a complication arising otherwise from the increasing value of δ as t increases. Incidentally there are also other well recognized advantages frequently attending the comparison of percentages rather than of absolute quantities.

The Barus Data.—Taking the data in the order of priority,

those of Barus will be first employed. The measurements to be used consist of very elaborate and painstaking direct comparisons of several 20 per cent. irido-platinum thermo-couples with several porcelain bulb air-thermometers used under the constant-pressure method.

Quotations of, or rather interpolations in, his original data* are given by Barus† later, as a basis from which to deduce constants for his proposed equation

$$e+e_0=10^{P+Qh}+10^{P'+Q'c}.$$

Barus's numerical values for the constants are:

$e_0=45680$ microvolts.
$P=4{\cdot}6515$ „ $Q=1{\cdot}106\,.\,10^{-4}$.
$P'=2{\cdot}849$ „ $Q'=3{\cdot}01\,.\,10^{-3}$.

These constitute his "equation 3," for which e_0 corresponds to 20° C. The data and the deviations which I have computed for it, viz. $\delta=\text{data}-\text{equation}$, are given in Table I. The last column gives the deviations expressed in percentages, viz., $100\frac{\delta}{E}$, where $E=e+e_0+1880$. This value of E is adopted to make the percentages comparable with those in subsequent discussions. The number 1880 is $1730+150$, which are the values of e_0 and $\Sigma_0^{20}\,e$ of the next two pages.

TABLE I.

Barus's American Journal of Science Data.

t^0	$e+e_0$ mv. observed.	$e+e_0$ computed from "Equ. 3."	δ mv.	$100\frac{\delta}{E}$ Per cent.
° C.				
0	−150			
100	+680	653	+27	+1·11
200	1650	1657	−7	−0·20
300	2760	2788	−28	−0·60
400	3950	3994	−44	−0·80
600	6560	6551	+9	+0·11
800	9310	9273	+37	+0·34
1000	12200	12140	+60	+0·43

* Barus, C., U.S. Geol. Surv. Bull. no. 54 (1889); Phil. Mag. xxxiv. p. 1 (1892).

† Amer. Jour. Sci. xlviii. p. 332 (1894). See also xlvii. p. 366 (1894).

The lines AB and CD on the diagram (page 483), constructed with percentage deviations as ordinates and temperature as abscissas, show clearly that the deviations are systematic. Upon inspection of this plot it appears that the data may be separated into two groups, one including 0°–300°, the other 400°–1000°, which appear to have entirely distinct forms of systematic error. This division corresponds to two distinct groups of data, one extending from 0° to 300°, the other including the second group and extending from 350° to 1075°. The latter were given in the Bulletin as the final results of the high temperature comparisons of the iridoplatinum couple with porcelain bulb air thermometers. The detailed statement of the 0°–300° comparison I have not seen. Although the discrepancy between the two sets of systematic deviations is not extremely large, yet it has seemed to me that it is beyond the limits of concordance in the higher temperature work, and that it would be better for the present purpose to deal solely with the 350°–1075° data. Two points regarding Barus's work should be noticed: one the strikingly high degree of concordance between individual observations even with different thermometer bulbs and different thermocouples; the other the remark in which Barus notes a possibility of being able still further to reduce the "stem error" entering into the result, which so far as I am aware has not yet been done.

The high temperature air-thermometer comparisons (Bulletin, Series I., II., III., IV., and V.) of Barus are so numerous (108) and so distributed that the labour of utilizing them simply for deducing constants and testing an equation would be excessive. Also they are too concordant to permit interpolation on a direct plot without a sacrifice of some of their precision. For the purposes of discussion, therefore, I averaged them in nine groups. The first group contained all where the E.M.F. lay between 3000 and 4000 microvolts; the second group between 4000 and 5000 mv.; and so on by steps of 1000 microvolts, except that the seventh group covered 2000 mv. from 9000 to 11000. These groups were not exactly equal in number of observations, and therefore in weight, nor is the arithmetical average a strictly legitimate value where the function is not linear; but, as easily seen by inspection of the originals, the errors thus introduced are negligible. In Table II. columns 1 and 2 give the direct values of the averages. Column 3 reduces $\Sigma_{20}^{t}\, e$ to $\Sigma_{0}^{t}\, e$ by adding 150 microvolts, the value of $\Sigma_{20}^{0}\, e$ being elsewhere given by Barus as -150 microvolts.

TABLE II.

Barus's Air-Thermometer Comparisons, Series I.–V.

t^0.	$\Sigma_{20}^{t} e$.	$\Sigma_{0}^{t} e$.	Avenarius.		Exponential.		Logarithmic.	
			δ, Da.–Eq.	Per cent. $=100\frac{\delta}{e}$	δ, Da.–Eq.	Per cent. $=100\frac{\delta}{e}$	δ, Da.–Eq.	Per cent. $=100\frac{\delta}{e}$
° C.	mv.	mv.	mv.		mv.		mv.	
0·0	(−150)	0	0	0·0	+23	+1·3		
378·5	3679	3829	−84	−4·0	−66	−1·2	−33	−0·60
440·3	4508	4658	+18	+0·28	+18	+0·28	+30	+0·47
522·0	5486	5636	−4	−0·07	−25	−0·34	−36	−0·50
588·4	6404	6554	+70	+1·1	+33	+0·40	+9	+0·12
672·1	7550	7700	+110	+1·5	+60	+0·64	+26	+0·28
745·6	8530	8680	+82	+0·92	+26	+0·25	−9	−0·10
840·1	9898	10048	+101	+1·0	+49	+0·41	+26	+0·22
946·6	11396	11546	+9	+0·07	−19	−0·14	−13	−0·10
1019·7	12475	12625	−45	−0·32	−45	−0·32	−10	−0·07
Average percentage deviations				0·93		0·53		

The Avenarius equation applied to these data yields

$$\Sigma_0^t e = 9{\cdot}104t + 3{\cdot}249.10^{-3}\, t^2$$

microvolts. Range 350° to 1075° C.

Computing from this equation values of $\Sigma_0^t e$ for the successive values of t in column 1 and subtracting them from the data in column 3 gives the deviations between data and equation. These are expressed in microvolts in column 4, and in percentages in column 5, the percentage being reckoned in terms of e_t as deduced by the exponential formula. Objections may be felt to this use of e_0 (here as throughout the subsequent tables) as a basis, since e_t involves e_0 which is an extrapolated value, certainly not exact, and possibly wide of the truth. Such a criticism is valid, but inasmuch as the values of e_0 employed are nearly equal, and as the percentage deviations are used merely for purposes of expressing relative accuracy, the possible error involved is nearly annulled. Hence, although it would be better to compute δt, and express this as a percentage of the absolute temperature τ, the added labour did not seem justified by the small gain.

The exponential equation applied to the Barus data yields

$$\Sigma_0^t e = 0{\cdot}7691\ \tau^{1{\cdot}373} - 1730, \text{ or}$$
$$e_\tau = 0{\cdot}7691\ \tau^{1{\cdot}373}, \text{ and } \beta = 1730 \text{ mv.}$$

Range of data 350° to 1075° C.

[N.B. This equation was deduced with the value 0°C. = 273°·7 absolute, whereas in all subsequent tables 0°C. = 273°·0 absolute is employed as a more probable value. The numerical values of the constants are therefore subject to a slight modification, but as for the present purpose we are concerned only with δ, which would not be sensibly changed, the recomputation is not worth while.]

Columns 6 and 7, Table II., give δ and its percentage value for the exponential equation.

The Barus Equation.—The excessive labour involved in the evaluation of the constants P, Q, P′, and Q′ of Barus's proposed equation detracts so seriously from its usefulness that I have also allowed it to deter me from computing them for the above tabulated values. The comparison of the values of δ for his "equation 3," and for an approximate exponential of my own based on the same data, is, however, decidedly in favour of the latter.

The logarithmic equation applied to the Barus data yields

$$\Sigma_0^t e = 2{\cdot}665\ t^{1{\cdot}220},$$

or its equivalent,

$$\log \Sigma_0^t e = 1{\cdot}220 \log t + 0{\cdot}42570.$$

The deviations are given in the last two columns of Table II.

Holborn and Wien Data.—This important comparison * of the rhodo-platinum thermo-couple with the porcelain bulb air thermometer up to high temperatures was performed under the auspices of the Reichsanstalt at Berlin, and appears to be on the whole the most important and reliable contribution to this subject in recent years. The experimental work was evidently conducted with great care, and although not showing the concordance of results, nor the multiplication of observations of Barus's work, yet in respect to stem-exposure correction, to exposure of the thermal junction, and to direct measurement of the coefficient of expansion of the bulb, it is probably more free from systematic error. It is to be regretted that the results were not more thoroughly discussed, and that neither a chemical analysis nor even a

* Holborn and Wien., *Zeit. f. Instk.* xii. pp. 257, 296 (1892); also in full in Wied. *Ann.* xlvii. p. 107 (1892).

statement was given to indicate the reliability of the stated percentage composition of the various alloys used. For when closely examined, the data seem to indicate a definite relation between the composition and the E.M.F., as was shown by a relation discovered between the constants in my exponential equations for the various alloys. The deviations were only such as might be attributed to uncertainty of composition, but as no measure of the latter was given, a statement of the relations and interesting inferences from them is not warranted. It is also unfortunate that an analysis, or at least a definite statement of the percentage purity, was not given for the gold, copper, and silver whose melting-points were observed. The assertion that the gold showed on qualitative analysis only a trace ("Spur") of copper, and the silver a "trace" of iron, is hardly definite. The value of the whole work would have been enhanced by these additions far more than in proportion to the comparatively small labour demanded by them, and such completeness is naturally to be expected in work emanating from this source. It is to be hoped that a continuation of this research is in progress, and that additional high melting-points may be measured.

Table III., columns 1 and 2, quotes the interpolated mean values of several comparisons expressed in international microvolts and degrees centigrade. With regard to these data it should be stated that below about 400° they were not supposed to be of as high accuracy as above that point. Also, that owing to unavoidable circumstances the data below 300° were obtained with only a single air-thermometer bulb, and similarly those above about 1300° with one bulb only, but a different one, while the data intermediate between 400° and 1300° are the mean of observations with the two bulbs. This fact may partially account for the erratic character of the residuals above 1300°, where the deviations are so great and so distributed (see diagram, page 483) as to render these observations of very little service. Direct comparison with the air-thermometer was made with one 10 per cent rhodo-platinum couple "A" only.

The parabolic formula applied to these by Holborn and Wien, when corrected as to decimal points *, is

$$t = 1{\cdot}376.10^{-1}(\Sigma_0^t e) - 4{\cdot}841.10^{-6}(\Sigma_0^t e)^2 + 1{\cdot}378.10^{-10}(\Sigma_0^t e)^3.$$

Range $-80°$ C. to $+1445°$ C.

The residuals are given in Table III., columns 3 and 4.

* The equation at both references, and stated to be in microvolts and degrees, is erroneously printed as

"$t = f(e) = 13{\cdot}76e - 0{\cdot}004841e^2 + 0{\cdot}000001378e^3$."

Table III.

Holborn and Wien.—Air-Thermometer Comparisons, Alloy A.

t C.	$\Sigma_0^t e$ mv.	H. and W. Eq.		Avenarius.		Exponential.		Logarithmic.	
		δ = Data − Eq. mv.	Per Cent. $=100\frac{\delta}{e}$	δ = Data − Eq. mv.	Per Cent. $=100\frac{\delta}{e}$	δ = Data − Eq. mv.	Per Cent. $=100\frac{\delta}{e}$	δ = Data − Eq. mv.	Per Cent. $=100\frac{\delta}{e}$
−80°	−361	—		—		—		—	
0	0	0	0	0	0	0	0	0	0
+82	+500	−84	−4·6	−107	−5·1	−69	−3·6	+40	+2·20
154	1000	−147	−6·1	−166	−7·2	−122	−4·8	+11	+0·50
220	1500	−135	−5·0	−199	−7·1	−140	−4·7	−27	−1·00
273	2000	−150	−4·5	−142	−4·3	−85	−2·6	+16	+0·50
329	2500	−130	−3·6	−124	−3·3	−73	−1·9	+11	+0·30
379	3000	−60	−1·4	−66	−1·5	−24	−0·57	+45	+1·00
431	3500	−30	−0·60	−41	−0·90	−6	−0·12	+45	+0·90
482	4000	0	0·00	−14	−0·22	+8	+0·15	+41	+0·80
533	4500	+10	+0·17	−1	0	+9	+0·15	+27	+0·50
584	5000	0	0·00	0	0	−2	−0·03	+1	+0·01
633	5500	0	0·00	+9	+0·13	−4	−0·06	−13	−0·20
680	6000	+10	+0·11	+28	+0·38	+5	+0·07	−15	−0·20
725	6500	+30	+0·40	+58	+0·74	+26	+0·33	−1	−0·01
774	7000	−10	−0·12	+35	+0·42	−5	−0·06	−39	−0·47
816	7500	+20	+0·22	+78	+0·90	+33	+0·37	−8	−0·10
862	8000	0	0·00	+69	+0·74	+19	+0·20	−24	−0·26
906	8500	−20	−0·20	+72	+0·74	+19	+0·20	−25	−0·25
952	9000	−55	−0·55	+44	+0·43	−10	−0·10	−53	−0·52
996	9500	−88	−0·80	+29	+0·26	−24	−0·22	−67	−0·60
1038	10000	−88	−0·80	+29	+0·25	−20	−0·18	−57	−0·50
1080	10500	−100	−0·85	+22	+0·20	−23	−0·20	−54	−0·43
1120	11000	−100	−0·80	+31	+0·25	−7	−0·06	−22	−0·19
1163	11500	−140	−1·10	−6	−0·05	−33	−0·26	−51	−0·40
1200	12000	−96	−0·74	+26	+0·20	+9	+0·07	+2	+0·02
1241	12500	−96	−0·70	0	0	−3	−0·03	−8	−0·06
1273	13000	0	0·00	+84	+0·60	+94	+0·67	+111	+0·80
1311	13500	+36	+0·24	+84	+0·57	+110	+0·80	+140	+0·90
1354	14000	+24	−0·17	+10	+0·07	+58	+0·38	+107	+0·70
1402	14500	−60	−0·40	−141	−0·90	−65	−0·41	+2	+0·01
1445	15000	−72	−0·45	−231	−1·40	−128	−0·80	−38	−0·23
a. d. for 0 to 1445.			1·12		1·15		0·77		1·46
a. d. for 431 to 1445.			0·39		0·43		0·25		0·38
a. d. for 431 to 1241.			0·43		0·36		0·15		0·34

The Avenarius Formula applied to the Holborn and Wien data with constants deduced from $t=584°$ and $1273°$ becomes

$$\Sigma_0^t e=(t-t_0)\{7{\cdot}2188+0{\cdot}0022994(t+t_0)\}, \text{ or}$$
$$=7{\cdot}2188t+2{\cdot}2994.10^{-3}\,t^2.$$

Range 0° to 1445° C.

The deviations in microvolts and percentages from this equation are given in columns 5 and 6, Table III.

The exponential equation fitting these data most closely, and coinciding with them at nearly the same points as the others, viz. at about 584° and 1250°, is

$$\Sigma_0^t e = 0{\cdot}57674\ \tau^{1{\cdot}377} - 1310, \text{ or}$$

$$e = 0{\cdot}57674\ \tau^{1{\cdot}377},\ \beta = 1310.$$

Range 0° to 1445° C.

The deviations in microvolts and percentages are in columns 7 and 8, Table III.

The logarithmic equation applied to the Holborn and Wien data on A yields

$$\Sigma_0^t\ e = 2{\cdot}1682\ t^{1{\cdot}2156}, \text{ or}$$

$$\log \Sigma_0^t\ e = 1{\cdot}2156\ \log t + 0{\cdot}36610.$$

The deviations are given in the last two columns of Table III.

Holborn and Wien not only compared the ten per cent. rhodo-platinum couple A directly with the air thermometer, but compared with A seven other couples in which one element was platinum, and the other a rhodo-platinum alloy, the percentage of rhodium being stated respectively as, for C_1 and C_2, 10 per cent. (these two I have combined under C), D, 9 per cent., E, 11 per cent., F, 20 per cent., G, 30 per cent., H, 40 per cent. For the present purpose I have combined these data, which were differences of E.M.F. between C and A, D and A, &c., with the corresponding E.M.F. of A, and thence have deduced the exponential equations for each

TABLE IV.

Designation of the Alloy.	Nominal Percentage of Rhodium.	Expon. Eq. Constants.		
		m.	*n.*	β *mv.*
D	9	1·3671	1·250	1517
C	10	0·95596	1·310	1485
E	11	0·81734	1·336	1469
A	10	0·57689	1·377	1305
F	20	0·22865	1·522	1167
G	30	0·065990	1·708	956
H	40	0·063034	1·720	977

of the alloys. Table V. gives the percentage deviations of these alloys from the exponential equation (data—equation), and Table IV. shows the values of m, n, and β for those equations.

TABLE V.

Holborn and Wien.—Comparison of Alloys.

t.	D.	A.	C.	E.	F.	G.	H.	Average.
°C.								
154	...	−4·8	...	−4·0	...	−5·0	−4·4	−4·5
273	−0·9	−2·6	−2·7	−2·2	−2·3	−3·5	−2·3	−2·6
379	+2·1	−0·57	−0·80	−0·60	−1·3	−0·60	−0·25	−0·67
482	+2·6	+0·15	−0·03	0	−0·60	+0·22	+0·46	−0·03
584	+0·9	−0·03	+0·19	+0·08	+0·13	+0·30	−0·03	+0·02
680	−0·30	+0·07	+0·13	+0·13	+0·12	−0·16	−0·26	0
774	−0·19	−0·06	−0·08	−0·02	−0·12	−0·42	−0·46	−0·19
862	+0·23	+0·20	+0·20	+0·20	+0·09	+0·15	−0·11	+0·12
952	+0·20	−0·10	0	−0·11	−0·15	+0·09	−0·24	−0·08
1038	+0·12	−0·18	−0·10	−0·21	−0·27	+0·04	−0·14	−0·16
1120	−0·15	−0·06	−0·09	−0·02	+0·10	+0·01	+0·22	−0·01
1200	+0·02	+0·07	0	+0·10	0	+0·04	0	+0·03
1273	...	+0·67	+0·70	+0·80	+0·80	+0·43	+1·00	+0·73
1354	...	+0·38	+0·60	+0·67	+0·51	+0·30	+1·00	+0·58
1445	...	−0·80	−0·38	−0·50	...	−0·90	−0·40	−0·60
a. d. 400–1200	0·52	0·15	0·09	0·10	0·12	0·16	0·21	0·07
a. d. 400–1445	...	0·25 Direct from Air Th.	0·21	0·24	(0·24)	0·23	0·36	

*Chassagny and Abraham Data**.—The apparently very careful measurements of these observers cover a range of 0° to 100° C. with observations at 25°, 50°, and 75° only. The range is too short and the intervals are too great to render the work of much service in testing a general formula, but if

TABLE VI.

Couple.	Σ_0^{100} *e*.	Σ_0^{75} *e*.	Σ_0^{50} *e*.	Σ_0^{25} *e*.
Fe–Cu	1093·3	864·9	604·8	315·5
Fe–Pt Rh.....	895·1	708·9	496·1	259·1
Fe–Ag	1123·0	885·6	617·4	321·1
Fe–Pt	1685·1	1278·9	859·9	432·1

* Chassagny et Abraham, *Ann. de Chim. et de Phys.* xxvii. p. 355 (1892).

its accuracy is as high as about 0°·01, as it appears to be, this in part offsets the disadvantage. Measurements of $\Sigma_0^t e$ and t were made with four thermo-couples, with the results shown in Table VI. (international microvolts and degrees centigrade on hydrogen scale).

The Avenarius equation was applied to these data by Chassagny and Abraham in the form

$$\Sigma_0^t e = at + bt^2.$$

They evaluated the constants from the 50° and 100° data. With these they computed the temperatures which the equation would yield by insertion of the observed values $\Sigma_0^{25} e$ and $\Sigma_0^{75} e$. These values are given in Table VII., columns 2 and 3.

The exponential equation applied to these data for Fe—Pt becomes

$$\Sigma_0^t e = 105{\cdot}096\ t^{0{\cdot}7360} - 6525{\cdot}3\ [\text{Range } 0° \text{ to } 100° \text{ C.}].$$

The values of t corresponding to the observed values $\Sigma_0^{25} e$ and $\Sigma_0^{75} e$ are given in Table VII. It has not seemed for the present purpose worth while to make similar computations for the other couples, as they would not materially affect the inferences to be drawn.

The logarithmic equation yields

$$\Sigma_0^t e = 19{\cdot}2946\ t^{0{\cdot}970595};$$

$$\log \Sigma_0^t e = 0{\cdot}970595 \log t + 1{\cdot}285436.$$

The deviations are given in the Table.

TABLE VII.

Couple.	Avenarius.		Exponential.				Logarithmic.	
	t.	δt.	t.	δt.	δ mv.	100 δ/e per cent.	δt.	δ mv.
	°	°	°	°			°	
Fe–Cu	24·88	+0·12						
Fe–Pt Rh ...	24·885	0·115						
Fe–Ag	24·87	0·13						
Fe–Pt	24·87	0·13	24·80	+0·20	−2·6	−0·037	+0·52	−6·7
Fe–Cu	75·13	−0·13						
Fe–Pt Rh ...	75·135	0·135						
Fe–Ag	75·135	0·135						
Fe–Pt	75·135	0·135	75·15	−0·15	+2·5	+0·032	−0·26	+4·3

The Noll Data.—A contribution of much permanent value to the data on thermo-electrics has recently been made by Noll *, who has measured $\Sigma_0^t e$ and t for thirty-two couples over a range in most cases of 0° to 218° C. The metals employed (including carbon) were usually of a high and stated degree of purity, and consisted of eighteen different substances, two of which were alloys (german silver and brass), and the remainder samples of different degrees of purity or hardness of the pure substances. The couples contained, as one element, for the most part, either copper or mercury. Temperatures were reduced to the air-thermometer scale.

The Avenarius formula was applied to fourteen of the more important of them by Noll. The deviations are given in Table VIII.

TABLE VIII.

Noll's Data on Pure Metals.

Couple.	m.	n.	β. mv.	Av. Pct. Deviation.	
				Avenarius.	Expon.
Au–Hg	$4{\cdot}6954 . 10^{-3}$	2·136	750·4	±0·27	±0·17
Ag–Hg	$2{\cdot}8637 . 10^{-3}$	2·206	677·8	0·33	0·15
Ni–Cu	$8{\cdot}2333 . 10^{-1}$	1·511	3950·2	0·30	0·17
(Cd–Cu	$3{\cdot}7617 . 10^{-11}$	4·94	40·7	0·48	3·40)
Br–Cu	$2{\cdot}4969 . 10^{-1}$	1·366	531·1	0·14	0·13
Zn–Hg	$8{\cdot}2890 . 10^{-4}$	2·420	651·6	0·15	0·18
Pb–Cu	$1{\cdot}7674 . 10^{-2}$	1·800	429·0	0·05	0·07
Cu_1–Hg.....	$4{\cdot}6726 . 10^{-3}$	2·130	768·4	0·12	0·11
[Fe–Hg	$1{\cdot}0913 . 10^{+2}$	0·7220	6264·2]		
Co–Hg	$8{\cdot}3295 . 10^{-3}$	2·166	1575·2	0·26	0·22
Pt_1–Cu	$2{\cdot}1475 . 10^{-3}$	2·266	711·1	0·08	0·12
Pt_2–Cu	$1{\cdot}1095 . 10^{-3}$	2·353	599·0	0·19	0·25
Sn_1–Cu	$4{\cdot}2021 . 10^{-2}$	1·667	482·8	0·21	0·09
Mg–Cu	$2{\cdot}0449 . 10^{-2}$	1·782	448·7	0·15	0·17
Al–Cu	$7{\cdot}5643 . 10^{-2}$	1·590	565·3	0·11	0·12
G.s.–Cu.....	$2{\cdot}0454 . 10^{-1}$	1·684	2589·9	0·06	0·05
Average omitting Cd–Cu and Fe–Hg				±0·17	±0·14

The exponential equation I have applied to the same data. It has not seemed essential to reproduce here the entire series of data, and the deviations of both equations. They are therefore, presented in a somewhat more digested form.

* Noll, *Wied. Ann.* iii. p. 874 (1894).

Table VIII. gives the constants for the exponential equation (those for the Avenarius may be found in Noll's article), the mean deviations (=data−equation) for each series, and the mean percentage deviations ($=100\ \delta/e$). (See remark as to use of e under "Barus Data.") Table IX. groups the percentage deviations under their nearest values of t for exhibiting their systematic character. The fact that the experimental method brought the observations all very nearly to the respective temperatures t given in the table renders this grouping possible. I have taken the liberty of correcting a few obvious numerical errors, and of dropping a very few values evidently containing a mistake.

It may, perhaps, not be out of place here to caution those who would make use of Noll's data to their full accuracy that his original, and not his interpolated, numbers should be resorted to. The approximate linear interpolation which he has employed is not as accurate as his experimental data demand.

The logarithmic equation applied to the Cu–Hg couple as typical of the Noll data yields

$$\Sigma_0^t\, e = 2{\cdot}57434\ t^{1{\cdot}2250};$$

or

$$\log \Sigma_0^t\, e = 1{\cdot}2250 \log t + 0{\cdot}410665.$$

The residuals to this expression are given in Table IX.

TABLE IX.

Avenarius Equation.—Data minus Equation in Per Cent.

	15°.	57°.	100°.	138°.	181°.	198°.	217°.
Au–Hg	−0·10	−0·40	0	+0·05	+0·60	+0·90	0
Ag–Hg	−0·13	+0·20	0	+0·21	−0·50	−1·30	0
Ni–Cu	−0·26	−0·15	0	0	−0·22	−0·31	−1·17
Br–Cu	...	−0·10	0	+0·20	+0·17	+0·35	0
Zn–Hg	...	−0·03	0	+0·09	+0·41	+0·40	0
Pb–Cu	...	−0·03	0	−0·11	+0·12	−0·04	0
Cu_1–Hg	+0·02	−0·16	0	+0·01	+0·15	+0·38	0
Co–Hg	...	...	0	+0·42	0	...	−0·56
Pt_1–Cu	...	−0·12	0	−0·06	+0·09	+0·21	0
Pt_2–Cu	...	−0·13	0	−0·11	0	+0·67	−0·21
Sn_1–Cu	...	−0·20	0	0	+0·34	+0·40	+0·30
Mg–Cu	...	+0·11	0	+0·22	−0·37	+0·11	0
Al–Cu	...	+0·16	0	+0·18	−0·18	+0·15	0
G. s.–Cu	...	−0·01	0	+0·16	+0·11	...	0
Average	−0·12	−0·07	0	+0·09	+0·05	+0·16	−0·12

Table IX.—*Continued.*

Exponential Equation.—Data minus Equation in Per Cent.

	15°.	57°.	100°.	138°.	181°.	198°.	217°.
Au–Hg	+0·01	−0·12	−0·06	−0·06	+0·09	+0·12	−0·80
Ag–Hg	−0·18	+0·22	0	+0·05	−0·03	0	−0·50
Ni–Cu	−0·26	−0·16	0	+0·07	0·00	0	−0·72
Br–Cu	...	−0·06	−0·01	+0·12	+0·20	−0·26	−0·13
Zn–Hg	...	−0·30	+0·02	+0·14	+0·10	+0·65	−0·04
Pb–Cu	...	−0·10	0	−0·25	+0·04	0	0·00
Cu_1–Hg	−0·19	−0·27	+0·01	...	−0·03	0	−0·25
Co–Hg	...	−0·05	+0·01	+0·48	−0·01	...	−1·00
Pt_1–Cu	...	−0·15	+0·03	−0·06	+0·17	0	−0·31
Pt_2–Cu	...	−0·56	0	−0·10	−0·16	+0·70	0
Sn_1–Cu	...	−0·04	+0·04	−0·29	+0·12	+0·06	0
Mg–Cu	...	+0·18	0	+0·15	−0·52	+0·17	0
Al–Cu	...	+0·16	0	+0·12	−0·24	+0·18	0
G. s.–Cu	...	+0·07	−0·01	+0·08	...	...	+0·05
Average	−0·16	+0·08	+0·01	+0·03	−0·02	+0·14	−0·27

Logarithmic Equation.—Data minus Equation in Per Cent.

Cu_1–Hg	+2·3	+1·5	0	−0·40	0	+0·50	+0·80

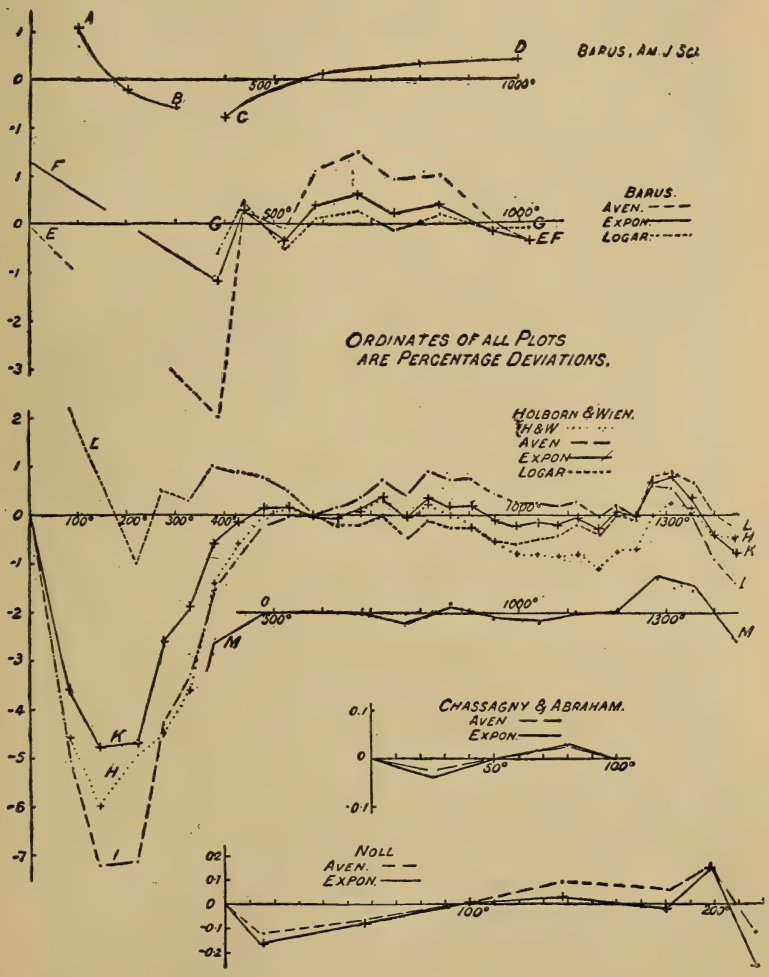
Discussion of the Deviations.

Plots are given in the following diagram with temperatures as abscissas and percentage deviations between the data and the sundry equations as ordinates, *i. e.* $100\ \delta/e$ where δ = data − equation. Inspection will show that with one exception (viz. the logarithmic equation applied to the Barus data) these plots, whether the equation is the ordinary parabolic, the Avenarius, the Barus, the exponential, or the logarithmic, have the same general form, which may be imperfectly described as follows. If the equation be made to conform to the data at 0° C. and at two higher points, *a* and *b*, then the deviation will be of the negative sign from 0 to *a*, positive from *a* to *b*, and negative above *b*. The slight departures from this general form are clearly due either to accidental errors, or to failure to make the equation conform to the data at all three points, or at suitable ones. The evidence is therefore conclusive

that *for all of the expressions the deviations are systematic and not purely "accidental" in character.*

One of two inferences is therefore warranted :—

1. *That neither the parabolic, Avenarius, Barus, exponential, nor logarithmic equation is the natural expression of the function.*



2. Or that the scale of temperature to which the values of t are referred in the foregoing investigations departs from the

normal scale by an amount and system roughly indicated by the above residual plots.

The latter inference, suggested by Chassagny and Abraham in the interpretation of their results, does not seem to possess much weight, notwithstanding the urgent need of renewed elaborate experimental investigation of the relation between the hydrogen, air, and thermodynamic scales of temperature.

As to the relative usefulness of the various expressions for purposes of interpolation and extrapolation some further inspection is necessary. The Barus equation 3, line C D, shows slightly smaller deviations on the plot than do the Avenarius and exponential, lines E E and F F. This, however, is due to the fact that the data against which 3 is tested are mean interpolated values, and hence have a sensibly less variable error than those against which the other equations are tested. An approximate exponential equation showed less deviations than 3 against the same data. There seems, therefore, to be no advantage in this equation sufficient to offset the difficulty of evaluation of its constants.

Applied to the Barus data from 350° to 1250°, the exponential equation shows deviations considerably less than one half as great as those of the Avenarius, while those of the logarithmic equation are so small as to lie far within the range of the variable errors, and they moreover show no clear evidence of systematic error between these limits of temperature. *For interpolation in the Barus data, therefore, the logarithmic equation is far preferable,* and must be conceded to be representative of the data. *For extrapolation* it is undoubtedly better than the Avenarius, which (as would the exponential in less degree) would certainly give above 1000° extrapolated values of Σe too large, or of t too small. The advantage due to its simplicity is also to be noted.

Applied to the Holborn and Wien data from 400° to 1450° the exponential equation shows (line K K) the same sort of superiority to both logarithmic (line L L) and Avenarius (line I I) that the logarithmic shows to the others with the Barus data, but in a still more marked degree. Within the limits 450° to 1450°, in fact, the distribution of the residuals to the exponential is such as not to warrant of itself alone any inference of systematic departure, especially when the mean line M M from all the couples is considered. It will be noted as an important confirmation of both the exactness of the electrical measurements in the investigation and the applicability of the exponential formula through a considerable range of alloys (and therefore of values of m and n) that this

mean line M M is almost identical in form with the line K K for alloy A. Relatively to the Holborn and Wien formula (line H H), the exponential possesses a similar advantage, with also the merit of greater simplicity of form.

It may therefore be affirmed that *for interpolation between* 450° *and* 1450° *in the H. and W. data the exponential equation is abundantly exact. For extrapolation* above 1450° it would not be entirely safe, although presumably better than the others, since the departure between 0° and 450°, and the similarity of the form to others, make a systematic departure sufficiently certain.

Applied to the Chassagny and Abraham data, 0°–100°, and to the Noll data, 0°–218° (see diagram), the Avenarius and exponential formulæ show about equal deviations, but with the advantage slightly on the side of the former. In the case of the Noll data, the line indicates that the systematic error is slightly greater for the exponential than for the Avenarius expression. The average deviations in Table IX., on the contrary, show that for each individual equation the concordance is greater for the exponential than the Avenarius. This discrepancy is due to the fact that, in order to eliminate local accidental errors, the equations (both Avenarius and exponential) are not all made to coincide with the data at the same temperatures, so that the process of averaging by which the data for the Noll plots is obtained is not numerically rigid. This does not, however, sensibly affect the general form of the curve. The greater ease of computation of the numerical constants of the Avenarius expression, and its applicability where both t and t_0 change, ought not to be overlooked. For extrapolation the exponential would be safer, for the reason that it has been shown above that for long ranges its systematic error is less.

The logarithmic equation fits the Noll data very badly, as shown by the deviation in Table IX. (not plotted), and also is much less close to the Chassagny and Abraham data than are the others.

The general conclusion as to applicability, then, seems to be that, while the *Avenarius* expression may be equally good or better than the exponential *for interpolation over short ranges*, yet *for interpolation over long ranges and for extrapolation above the observation limits the exponential is decidedly preferable.* The exponential form is also preferable to the remaining expressions with the exception noted.

The logarithmic form, although closely applicable to the Barus data, is of more doubtful general value, yet on account

of its great convenience it may find application in industrial pyrometry, as will be elsewhere indicated. Although failing below 300° or 400°, it may probably be applied to the irido- or rhodo-platinum couple between 400° and 1200° C. with a maximum error not exceeding about 5°. If extended to cover 400° to 2000° the error might rise to 15° or 20°.

More in detail it may be briefly noted by way of summary :—

That the logarithmic equation fits the Barus data between 400° and 1250° with scarcely sensible systematic error, and within the limits of variable errors of the data.

That the exponential equation similarly fits the Holborn and Wien data within the limits 400° to 1445°.

That when made to coincide with the data at about 450° and 1200° the systematic deviations of the exponential equation from the Barus data, and of the logarithmic equation from the Holborn and Wien data, are in general of opposite sign and of roughly equal magnitude.

Barus Melting and Boiling-Point Data.

From the foregoing demonstration of its applicability, it seems proper to apply the logarithmic formula to the Barus thermo-electric data on melting-points *.

Whether the extrapolation above 1000° by the logarithmic formula is entitled to any great weight may be questioned, but there is no obvious reason why it is not more reliable than by any of the others. I have employed the equation given on page 474, which represents very closely Barus's high temperature air-thermometer comparisons, calculating thence the temperatures t corresponding to the values of $\Sigma_{20}^{t} e$ given by Barus for the various points, assuming Barus's value $\Sigma_{0}^{20} e = 150$ mv. The results are given in column 3 of Table X. Column 4 quotes the most reliable previous determinations of the same points by other observers. As to which of the two columns of results best represents Barus's work, there can be little doubt from the above evidence that below 1000° it is the second, that is, the one computed from the logarithmic equation. These combine both his own air-thermometer and melting-point work. Above 1000° the logarithmic values are probably slightly too high.

* Amer. Journ. Sci. xlviii. p. 332 (1894).

TABLE X.

Barus Melting and Boiling Points.

	Computed by Eq. 3.	Computed by Log. Eq.	Data by other Observers.	
	°	°	°	
Mercury (B. Pt.)	357	359	356·76	Callendar and Griffiths.
Zinc	420	423	417·57	,, ,,
Sulphur (B. Pt.)	446	449	444·53	,, ,,
Aluminium	638	641	635	Le Chatelier.
Selenium (B. Pt.)	694	697		
Cadmium (B. Pt.)	782	782		
Zinc (B. Pt.)	929	926	930	Deville and Troost.
Silver	986	985	968	Holborn and Wien.
			954	Violle.
Gold	1091	1090	1072	Holborn and Wien.
			1035	Violle.
Copper	1096	1095	1082	Holborn and Wien.
			1054	Violle.
Bismuth.....	1435	1441		
Nickel	1476	1485	1450	Carnelly and Williams.
Palladium	1585	1597	1500	Violle.
Platinum	1757	1783	1775	Violle.

Remark.

Review of the laborious researches which have been devoted to the direct comparison of thermo-electric elements with the air thermometer, mainly for the purpose of advancing the art of pyrometry, has enforced the conviction that, at least for the immediate future, this end would be better served by accurate gas-thermometer measurements of melting-points of metals. Each such determination made upon a reducible metal of known high purity under proper reproducible conditions fixes an enduring and reproducible reference-point, a pyrometric "bench mark." And there are enough inexpensive metals, together with a possible system of simple alloys, to give points of sufficient frequency. These would then afford a convenient means of obtaining accurately known high temperatures for purposes of study of all high temperature phenomena, and particularly for calibrating thermo-electric, electrical resistance, optical, or other secondary pyrometric interpolation apparatus,—for it must be remembered that all such apparatus is necessarily secondary, the gas thermometer being inevitably the primary.

On the other hand, comparison with the air thermometer of a thermo-couple, or of a resistance pyrometer, or the study of

any progressive thermal phenomenon, while it possibly may result in the eduction of a natural law, is very unlikely to lead to anything more than the establishment of an approximate equation with constants characteristic only of the individual materials actually employed, and not transferable to other, although similar materials. Such results are obviously of a much more ephemeral character than the melting-point measurements. Even when any pyrometer thus tested is applied to the establishment of melting-points, it must at best yield results inferior to direct application of the gas thermometer, except in cases where the latter is hampered by want of sufficient quantity of the metal to be experimented upon,—a condition which need only affect such costly substances as gold and platinum.

Stated broadly, the great need of the art of pyrometry is convenient methods of producing, or of recognizing when produced, a series of accurately known high temperatures. The analogous problem has been partially solved for thermometry at temperatures up to 300° C. by the investigation of boiling-points of certain chemically pure substances under controlled pressures.

Rogers Laboratory of Physics,
Massachusetts Institute of Technology,
Boston, September, 1895.

LIII. *Notes on the Electro-Magnetic Theory of Moving Charges.* *By* W. B. Morton, *B.A.**

1. THIS subject has been brought into prominence recently by the use which Mr. Larmor has made of moving electrons in his dynamical theory of the æther. The matter was investigated in 1881 by Prof. J. J. Thomson†, who showed that a point charge moving so slowly that the electric displacement it carries is not sensibly disturbed generates magnetic force like a current element according to Ampère's rule; and by Mr. Heaviside‡, who investigated the matter more generally in 1889, and showed that in steady rectilinear motion at any speed less than that of light, the lines of displacement continue to be radial but are concentrated towards the plane perpendicular to the direction of motion. The displacement at distance r, in a direction

* Communicated by the Physical Society: read March 27, 1896.

† Phil. Mag. April 1881, July 1889; Recent Researches, pp. 16–23.

‡ Electrical Papers, ii. pp. 504–518; Electromagnetic Theory, i. pp. 269–274.

making an angle θ with the line of motion is proportional to

$$\frac{1}{r^2\left(1-\frac{u^2}{V^2}\sin^2\theta\right)^{\frac{3}{2}}},$$

where u is the velocity of the moving charge and V the velocity of light. The lines of magnetic force are circles round the line of motion.

2. This solution of course represents the state of affairs at a great distance from a small charged conductor of any shape. It would also give us the distribution of charge on a moving sphere if it were correct to assume that the lines of displacement meet the charged surface at right angles. This assumption was made by Prof. Thomson and, at first, by Mr. Heaviside, but the latter, quoting a suggestion of Mr. G. F. C. Searle, subsequently pointed out that when there is motion the electric force is no longer derived from a potential function, and as a consequence does not meet the equilibrium surface at right angles. Substituting the correct surface condition, he showed that the charged conductor, whose motion would give at all points the radial distribution found for a point charge, was not a sphere but a spheroid of certain ellipticity.

3. It seemed of some interest to inquire what the distribution of charge on a moving sphere would be. The surface-density at a point of the surface is now the normal component of the displacement at that point. By carrying the investigation a step further I have found that, if the conductor be a sphere or any ellipsoid, the ordinary static arrangement of charge is unaltered by the motion; *i.e.* the number of tubes of displacement leaving each element of the surface is unchanged, but the tubes no longer leave the surface at right angles. We may imagine that the motion has the effect of deforming the tubes, keeping their ends on the conductor fixed. The proof of this, involving a consideration of the general case, is here given and is followed by a note on the energy of a moving charge in a magnetic field.

4. Suppose we have any distribution of charge moving with uniform velocity u parallel to the axis of z, and that the field has assumed its steady configuration. We shall denote $1-\frac{u^2}{V^2}$ by k^2, V being the velocity of light. Then since we have a steady state,

$$\frac{d}{dt}=-u\frac{d}{dz}.$$

Also, since each element of charge produces a magnetic field with no z-component, we have $\gamma=0$ in the general case also. Using these two data, the equations connecting the displacement (f, g, h) and the magnetic force (α, β, γ) become

$$-\frac{d\beta}{dz}=-4\pi u\frac{df}{dz},$$

$$\frac{d\alpha}{dz}=-4\pi u\frac{dg}{dz}$$

$$\frac{d\beta}{dx}-\frac{d\alpha}{dy}=-4\pi u\frac{dh}{dz}$$

$$\frac{dg}{dz}-\frac{dh}{dy}=-\frac{u}{4\pi V^2}\frac{d\alpha}{dz},$$

$$\frac{dh}{dx}-\frac{df}{dz}=-\frac{u}{4\pi V^2}\frac{d\beta}{dz},$$

$$\frac{df}{dy}-\frac{dg}{dx}=0.$$

These equations together with

$$\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0$$

are satisfied by

$$f=-\frac{d\phi}{dx},\quad g=-\frac{d\phi}{dy},\quad h=-k^2\frac{d\phi}{dz},$$

$$\alpha=4\pi u\frac{d\phi}{dy},\quad \beta=-4\pi\frac{d\phi}{dx},$$

where ϕ is any function satisfying

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+k^2\frac{d^2\phi}{dz^2}=0.$$

These results have been obtained by Prof. Thomson and Mr. Heaviside. The particular case of a point charge, e, is got by putting

$$\phi=\frac{e}{4\pi\sqrt{k^2(x^2+y^2)+z^2}}.$$

Evidently in the general case ϕ must vanish at infinity.

5. Mr. Heaviside points out that $\phi=$constant is the condition holding at a surface of equilibrium. The matter may be stated thus :—If we suppose the field to terminate at the surface of a conductor, inside which the vectors vanish, we must see that the "curl" relations of the field are not violated

for circuits which lie partly inside the empty space enclosed by the conductor. In particular, if there is a vector whose line integral round every circuit in the field vanishes, the lines of this vector must meet the surface at right angles. Otherwise we should have a finite value for the integral round a circuit drawn close to the surface outside and completed inside. In other words, if a vector is derived from a potential function, this function must be constant over the surface. In the ordinary static case it is the electric force (X, Y, Z) which is so derived; but in the case of a steadily moving field it is the vector $\left(X, Y, \frac{Z}{k^2}\right)$ which meets the surface at right angles.

6. Let $F(x\,y\,z)=C$ be the equation of the charged surface. Then $\phi(x\,y\,z)$ has to be constant over this surface and satisfy

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+k^2\frac{d^2\phi}{dz^2}=0.$$

Put $z=k\zeta$, then ϕ is a function of x, y, ζ, which is constant when $F(x, y, k\zeta)=C$, and which satisfies

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{d\zeta^2}=0.$$

Therefore if we regard $(x\ y\ \zeta)$ as Cartesian coordinates of a point, ϕ is the potential at external points of an electrostatic free distribution on the surface $F(x, y, k\zeta)=C$. The components of electric force due to this distribution, at a point $(x\ y\ \zeta)$ on the surface, are

$$-\frac{d\phi}{dx},\quad -\frac{d\phi}{dy},\quad -\frac{d\phi}{d\zeta}.$$

This force acts in the normal to the surface, and is proportional to the surface-density at $(x\ y\ \zeta)$, which we shall call σ'. Therefore

$$\left(\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{d\zeta}\right)=-A\sigma'\left(\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{d\zeta}\right)\Big/\sqrt{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{d\zeta}\right)^2}$$

But

$$\frac{d}{d\zeta}=k\frac{d}{dz};$$

h erefore, denoting differentiation with respect to $x\ y\ z$ by ubscripts 1 2 3,

$$(\phi_1, \phi_2, \phi_3)=-A\sigma'(F_1, F_2, F_3)/\sqrt{F_1^2+F_2^2+k^2F_3^2}.$$

Now let σ be the surface-density at $(x\ y\ z)$ on the moving conductor $F(x\ y\ z)=C$, then equating σ to the normal com-

ponent of $(f\,g\,h)$

$$\sigma=\frac{f\mathrm{F}_1+g\mathrm{F}_2+h\mathrm{F}_3}{\sqrt{\mathrm{F}_1^2+\mathrm{F}_2^2+\mathrm{F}_3^2}};$$

or putting in the values we have found for $(f\,g\,h)$ in terms of ϕ,

$$\sigma=-\frac{\phi_1\mathrm{F}_1+\phi_2\mathrm{F}_2+k^2\phi_3\mathrm{F}_3}{\sqrt{\mathrm{F}_1^2+\mathrm{F}_2^2+\mathrm{F}_3^2}}$$

$$=\mathrm{A}\sigma'\sqrt{\frac{\mathrm{F}_1^2+\mathrm{F}_2^2+k^2\mathrm{F}_3^2}{\mathrm{F}_1^2+\mathrm{F}_2^2+\mathrm{F}_3^2}}.$$

Now the perpendicular from the origin on the tangent plane to $\mathrm{F}(x\,y\,z)=\mathrm{C}$ at the point $(x\,y\,z)$ is

$$p=\frac{x\mathrm{F}_1+y\mathrm{F}_2+z\mathrm{F}_3}{\sqrt{\mathrm{F}_1^2+\mathrm{F}_2^2+\mathrm{F}_3^2}},$$

and the perpendicular from the origin on the tangent plane to $\mathrm{F}(x,\,y,\,k\zeta)=\mathrm{C}$ at $(x\,y\,\zeta)$ is

$$p'=\frac{x\dfrac{d\mathrm{F}}{dx}+y\dfrac{d\mathrm{F}}{dy}+\zeta\dfrac{d\mathrm{F}}{d\zeta}}{\sqrt{\left(\dfrac{d\mathrm{F}}{dx}\right)^2+\left(\dfrac{d\mathrm{F}}{dy}\right)^2+\left(\dfrac{d\mathrm{F}}{d\zeta}\right)^2}}$$

$$=\frac{x\mathrm{F}_1+y\mathrm{F}_2+k\zeta\mathrm{F}_3}{\sqrt{\mathrm{F}_1^2+\mathrm{F}_2^2+k^2\mathrm{F}_3^2}}$$

$$=\frac{x\mathrm{F}_1+y\mathrm{F}_2+z\mathrm{F}_3}{\sqrt{\mathrm{F}_1^2+\mathrm{F}_2^2+k^2\mathrm{F}_3^2}};$$

$$\therefore\ \frac{\sigma}{\sigma'}=\frac{\mathrm{A}p}{p'}.$$

If now $\mathrm{F}(x\,y\,z)=\mathrm{C}$ is an ellipsoid, then we know that $\sigma'\propto p'$, therefore also $\sigma\propto p$, that is the arrangement of charge on the moving ellipsoid is the same as if it were at rest.

7. Applying the above to the ellipsoid $(a\,b\,c)$, we find that ϕ as a function of $(x\,y\,\zeta)$ is the potential of a free distribution on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{k^2\zeta^2}{c^2}=1;$$

$$\therefore\ \phi = C\int_{\mu}^{\infty} \frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)\left(\frac{c^2}{k^2}+\lambda\right)}}$$

$$= Ck\int_{\mu}^{\infty} \frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+k^2\lambda)}},$$

where μ is given by

$$\frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} + \frac{\zeta^2}{\frac{c^2}{k^2}+\mu} = 1,$$

or

$$\frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} + \frac{z^2}{c^2+k^2\mu} = 1.$$

Determining the value of the constant C so that the density at a point shall be $\frac{ep}{4\pi abc}$, we get

$$\phi = \frac{e}{8\pi}\int_{\mu}^{\infty} \frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+k^2\lambda)}}.$$

Putting $b=a$, $c=ka$, we get

$$\phi = \frac{e}{4\pi\sqrt{k^2(x^2+y^2)+z^2}}.$$

Showing that, as Mr. Heaviside pointed out, the field of a point charge is given when the conductor is an oblate spheroid whose axes have the ratio $1:k$.

For a sphere the integral becomes

$$\phi = \frac{e}{8\pi k'a}\log\frac{\theta+k'a}{\theta-k'a}$$

where $$k' = \sqrt{1-k^2} = \frac{u}{V},$$

and θ is given by

$$\frac{k^2(x^2+y^2)}{\theta^2-k'^2a^2} + \frac{z^2}{\theta^2} = 1.$$

To test the value of ϕ let us make k' approach zero, *i.e.* the motion becomes infinitely slow. θ is then $=r$.

Then $$\phi = \frac{e}{8\pi a}\ \mathrm{Lt}_{k'=0}\ \log\frac{(r+k'a)-\log(r-k'a)}{k'}$$

$$= \frac{e}{8\pi a}\cdot\frac{2a}{r} = \frac{e}{4\pi r}.$$

8. The mutual energy of a moving charge and external magnetic field has been given by Mr. Heaviside for the case of motion which is very slow compared with the velocity of radiation. It is $eu\,\mathrm{A}\,.\cos(u\mathrm{A})$, where A is the circuital vector potential of the external field. Mr. Larmor, in the second part of his "Dynamical Theory" (Phil. Trans. 1895, p. 717), concludes that the same expression holds good for motion at any speed. He seems, however, to overlook the fact that in the general case the displacement-currents in the medium—being no longer derivable from a potential function—will make their appearance in the result as well as the convection-current eu.

If (F G H) is the vector potential, the part of the energy corresponding to the displacement currents will be

$$\int(\mathrm{F}\dot{f}+\mathrm{G}\dot{g}+\mathrm{H}\dot{h})\,d\tau,$$

which in the case we have been considering becomes

$$-u\int\left(\mathrm{F}\frac{df}{dz}+\mathrm{G}\frac{dg}{dz}+\mathrm{H}\frac{dh}{dz}\right)d\tau$$

$$=u\int\left(\mathrm{F}\frac{d^2\phi}{dx\,dz}+\mathrm{G}\frac{d^2\phi}{dy\,dz}+k^2\mathrm{H}\frac{d^2\phi}{dz^2}\right)d\tau.$$

But by a well-known transformation, when we take the integral through all space, we have

$$\int\left(\mathrm{F}\frac{d^2\phi}{dx\,dz}+\mathrm{G}\frac{d^2\phi}{dy\,dz}+\mathrm{H}\frac{d^2\phi}{dz^2}\right)d\tau$$

$$=\int\frac{d\phi}{dz}\left(\frac{d\mathrm{F}}{dx}+\frac{d\mathrm{G}}{dy}+\frac{d\mathrm{H}}{dz}\right)d\tau$$

$$=0 \text{ since (F G H) is circuital.}$$

∴ The expression for this part of the energy reduces to

$$-u(1-k^2)\int\mathrm{H}\frac{d^2\phi}{dz^2}\,d\tau=-\frac{u^3}{\mathrm{V}^2}\int\mathrm{H}\frac{d^2\phi}{dz^2}\,d\tau.$$

Therefore if the velocity u ceases to be negligible in comparison with V, we have a correction of the *second* order in the ratio $\frac{u}{\mathrm{V}}$ in addition to the expression involving the convection-current simply. It also appears from the above that the force on the moving charge cannot, unless this term be neglected, be expressed in terms of the magnetic intensity at the charge, but will depend on the entire field.

LIV. *On a Simple Apparatus for determining the Thermal Conductivities of Cements and other Substances used in the Arts.* By CHARLES H. LEES, *D.Sc.*, and J. D. CHORLTON, *B.Sc., Joule Scholar of the Royal Society**.

THE following method of determining the thermal conductivities of bad conductors has been designed with the object of simplifying as much as possible the apparatus and observations required in making determinations which are not required to be of the highest order of accuracy, but in which errors of more than 2 per cent. are to be avoided. When the constants of the apparatus have once been determined, the only observations necessary to determine a conductivity are the thickness of a sheet of the substance used and the temperatures of three thermometers. It is hoped that this simplification will lead those who require bad conductors of heat for structural purposes to carry out their own tests of the materials they have available.

Method.—The apparatus consists of a flat cylindrical metal box, of 11·4 cms. diameter and 3 cms. depth, through which steam can be passed. The bottom of the box consists of a circular brass plate 1·3 cm. thick in which a radial hole reaching to the centre is bored. In this hole a thermometer is placed with its bulb at the centre of the plate. The top and sides of the box are covered with green baize to prevent loss of heat as far as possible. This vessel is supported on a circular plate of the material to be tested, which in its turn is supported on a brass disk similar to the one forming the base of the heating vessel, and like it provided with a radial hole and thermometer.

The lower disk is suspended horizontally from a support by three strings attached to three short pegs projecting from the edge of the disk, and hangs about 30 centims. above the table. About 10 centims. above the table a thermometer is placed horizontally, with its bulb under the centre of the lower plate to give the temperature of the air ascending to the disks. The bulb of the thermometer is protected against radiation from the lower disk by a bright metal screen.

The two surfaces of the disks which come into contact with the material to be experimented on are amalgamated, so that if the material is a solid, contact over its entire surface may be obtained by using a thin mercury film between the solid and disks. In order to determine the thickness of the

* Communicated by the Authors.

material used, two short pegs of brass, one vertically above the other, project from the disks, and their distance apart is measured by means of a wire gauge or calipers, when the disks are in contact and when the plate to be tested is between, the difference is the thickness of the plate of the material tested. The under surface of the lower disk may be kept polished, or it may be painted, in order that the heat radiated from it may remain constant during a series of experiments.

When steam is passed through the upper cylinder the temperature of the upper disk is raised to nearly 100° C.; heat flows through the plate of material experimented on to the lower disk, the temperature of which is therefore raised above that of the surrounding air. It begins in consequence to lose heat by radiation and conduction to the air, and eventually a stage is reached when this loss of heat is equal to the heat received from the material experimented on.

Hence if the amount of this loss is found by a separate experiment, a determination of the temperature gradient in the material experimented on at its surface of contact with the lower plate, will enable the thermal conductivity of the material to be found.

Theory.

If θ is the temperature at a point x above the under surface of the lower plate, θ_0 the temperature of the air, k_1 the internal conductivity, h_1 the external conductivity or "emissivity," Newton's law being supposed to hold for the limits of temperature used, p the perimeter, q the area, of cross section of the plate, the differential equation for the motion of heat in the plate, the isothermals being assumed plane, is satisfied if

$$\theta-\theta_0=A_1\cosh\sqrt{\frac{ph_1}{qk_1}}.x+B_1\sinh\sqrt{\frac{ph_1}{qk_1}}.x,$$

where A_1 and B_1 are constants.

The condition for continuity of flow at the under surface is

$$k_1\frac{d\theta}{dx}=h_1\theta \text{ for } x=0,$$

i. e.

$$k_1\sqrt{\frac{ph_1}{qk_1}}.B_1=h_1A_1;$$

$$\therefore\ \theta-\theta_0=A_1\left\{\cosh\sqrt{\frac{ph_1}{qk_1}}.x+\frac{h_1}{k_1\sqrt{\frac{ph_1}{qk_1}}}\sinh\sqrt{\frac{ph_1}{qk_1}}.x\right\}$$

Now for the lower brass disk

$$p/q = 2\pi r/\pi r^2 = 2/5{\cdot}7 = {\cdot}351,$$

while k_1 for brass $= {\cdot}25$, and it will be shown presently that $h_1 = {\cdot}0003$ about. Hence $\sqrt{\frac{ph_1}{qk_1}} = {\cdot}02$, and since the thickness of the plate is 1·3 cm. the maximum value of the expression in brackets is

$$\cosh {\cdot}026 + \frac{{\cdot}0003}{{\cdot}25 \times {\cdot}02} \sinh {\cdot}026 = 1{\cdot}0018;$$

or the temperature of the lower plate varies less than $\frac{1}{5}$ per cent., and for the purpose of the present experiment may be taken to be uniform throughout and equal to the indication θ_1 of the thermometer in the centre of the disk. The temperature of the lower disk is therefore given completely by the equation

$$\theta - \theta_0 = \{\theta_1 - \theta_0\} \left\{ \cosh \sqrt{\frac{ph_1}{qk_1}} . x + \frac{h_1}{k_1 \sqrt{\frac{ph_1}{qk_1}}} . \sinh \sqrt{\frac{ph_1}{qk_1}} . x \right\}.$$

The rate of flow of heat into the lower disk, the thickness of which is 1·3 centim., is given by $k_1 \frac{d\theta}{dx}$ for $x = 1{\cdot}3$. It is therefore

$$= k_1 \sqrt{\frac{ph_1}{qk_1}} \{\theta_1 - \theta_0\} \left\{ \sinh {\cdot}026 + \frac{h_1}{k_1 \sqrt{\frac{ph_1}{qk_1}}} . \cosh {\cdot}026 \right\}$$

$$= (\theta_1 - \theta_0) \left(1{\cdot}00026\, h_1 + {\cdot}026\, k_1 \sqrt{\frac{ph_1}{qk_1}} \right)$$

$$= (\theta_1 - \theta_0)(h_1 + {\cdot}00013)$$

$$= (\theta_1 - \theta_0)(h_1 + h_1'), \text{ say, where } h_1' = {\cdot}00013.$$

If k is the internal, h the external conductivity of the medium under test, the temperature θ at a point x above its under surface is given by:—

$$\theta - \theta_0 = (\theta_1 - \theta_0) \left\{ \cosh \sqrt{\frac{ph}{qk}} . x + \mathrm{B} \sinh \sqrt{\frac{ph}{qk}} . x \right\},$$

where B is a constant, the value of which is fixed by the

condition for continuity of flow at the surface of contact of the material and the lower plate, *i. e.*

$$k\sqrt{\frac{ph}{qk}}.\,\mathrm{B}=(\theta_1-\theta_0)(h_1+h_1').$$

Hence in the material under test

$$\theta-\theta_0=\{\theta_1-\theta_0\}\left\{\cosh\sqrt{\frac{ph}{qk}}.x+\frac{h_1+h_1'}{k\sqrt{\frac{ph}{qk}}}.\sinh\sqrt{\frac{ph}{qk}}.x\right\}.$$

If b is the thickness of the material, at $x=b$ we must have $\theta=\theta_2$, the temperature indicated by the thermometer in the upper brass plate, *i. e.*

$$\theta_2-\theta_0=(\theta_1-\theta_0)\left(\cosh\sqrt{\frac{ph}{qk}}.b+\frac{h_1+h_1'}{k\sqrt{\frac{ph}{qk}}}.\sinh\sqrt{\frac{ph}{qk}}.b\right).$$

The maximum value of $\sqrt{\frac{ph}{qk}}.b$ for the materials experimented on is ·2; hence the hyperbolic functions can be expanded in ascending powers of $\sqrt{\frac{ph}{qk}}.b$, and terms involving cubes can be neglected. Thus:—

$$\theta_2-\theta_0=(\theta_1-\theta_0)\left(1+\frac{ph}{2qk}b^2+\frac{h_1+h_1'}{k\sqrt{\frac{ph}{qk}}}\sqrt{\frac{ph}{qk}}.b\right),$$

or

$$\frac{\theta_2-\theta_1}{\theta_1-\theta_0}=\frac{b}{k}\left(h_1+h_1'+\frac{ph}{2q}.b\right).$$

The last term in the bracket never exceeds 4 per cent. of the first two terms, and the final result will be affected only to a very small extent if for h, h_1 is substituted.

Hence we have

$$k=\frac{\theta_1-\theta_0}{\theta_2-\theta_1}b\left(h_1+h_1'+\frac{ph_1}{2q}.b\right),$$

which enables k to be calculated from the result of the experiment if h_1 has been previously determined.

Determination of h_1.—To determine the value of this quantity the two plates were separated by a layer of air 3 mm. thick, enclosed by a ring of badly conducting material upon

which the upper plate rested; steam was passed through the box at the top, and the temperature of the upper plate raised to nearly 100° C. The lower plate was next heated a few degrees above 100° C. by means of a gas-flame and then allowed to cool, and as soon as its temperature fell to 100° C. the supply of steam to the upper can was cut off and the hot water in the can allowed to run out.

Both plates were then found to cool at almost the same rate; the difference in the temperatures of the two plates never exceeded 1° C., the upper plate being always the warmer. Since the conductivity of air is only about ·00005 C.G.S. units, the amount of heat conducted from the upper plate to the lower is negligible, so that each plate cools independently of the other.

The thermometer in the lower plate was watched as the temperature fell, and the times at which the mercury passed every alternate degree were observed; the results are given in the table below.

The temperature of the air was steady at 18° C.

Temperature of lower plate.		Time.	Fall of temp. per second.	Loss of heat, gram-degrees per second.	Loss of heat per sq. cm., gram-degrees per second.	h_1.
θ_1.	$\theta_1-\theta_0$.					
° C.	° C.	h m s				
96	78	3.17.42				
94	76	18.37	·0370	3·92	·0255	·000336
92	74	19.32	354	75	43	28
90	72	20.30	340	60	34	25
88	70	21.30	325	44	23	19
86	68	22.33	308	26	12	12
84	66	23.40	301	14	04	09
82	64	24.46	290	07	·0199	11
80	62	25.58	274	2·90	86	03
78	60	27.12	263	79	81	02
76	58	28.30	256	71	76	03
74	56	29.48	250	65	72	07
72	54	31.10	244	56	68	11
70	52	32.32	235	49	62	12
68	50	34.0	220	33	51	02
66	48	35.34	208	20	43	·000298
64	46	27.12	197	08	35	93
62	44	38.57	187	1·98	29	93
60	42	40.46	180	91	24	95
58	40	42.40	169	79	16	90
56	38	44.42	156	65	07	82
54	36	46.57	146	55	·0101	81
52	34	49.15	140	48	·0096	82
50	32	51.43				

The numbers in the 4th column are found by subtracting alternate times in the 3rd column from each other and dividing 4° C. by the differences.

The numbers in the 5th column are obtained from those in the 4th by multiplying by 1140 grams, the mass of the lower disk, and by ·093, the specific heat of brass, *i. e.* by 106.

The numbers in the 6th column are obtained by dividing by the total radiating surface of the lower disk, plus half the radiating surface of the nonconducting ring $= 148{\cdot}5 + 5{\cdot}5 = 154$ sq. centims.

The values of h are obtained from these numbers by dividing by the excess of the temperature of the lower disk over that of the air.

Experiments.

The following account of an experiment on a plate of glass will show the method of treatment in each case:—

A disk of plate-glass of diameter equal to that of the brass plates—11·4 centims.—was placed between them, and contact made by means of thin mercury films. Good contact was easily secured between the glass and the lower plate; but with much more difficulty in the case of the upper plate. Contact was eventually obtained by covering the surface of the upper plate with mercury, so that it adhered in pendent drops, at the same time placing a few drops of mercury on the surface of the glass, and then carefully lowering the upper plate on to the glass and allowing the excess of mercury to run out at the edges.

Steam was then passed through the box attached to the upper brass plate, and thermometers inserted in the upper and lower plates, the temperature of the air being indicated by a third thermometer placed as described above.

After a time, the duration of which depended on the conductivity and thickness of the substance experimented upon, the thermometers became steady and the temperatures were observed.

$\theta_2 =$ temperature of upper plate $= 96{\cdot}1°$ C.

$\theta_1 =$ „ „ lower „ $= 92{\cdot}4°$ C.

$\theta_0 =$ „ „ air $= 20{\cdot}0°$ C.

Thickness = ·292 centim.

The thickness was obtained by measuring with a micrometer-gauge the distance between the two pegs, first, when the plates were in contact with each other, and, secondly, when the glass was between them.

The following table gives the observations taken during each experiment, and shows the method of determining the conductivities from the observations :—

Remarks on Specimens Experimented on.

(*a*) The plates of Portland cement and plaster of Paris, &c., were made by pouring the liquid cement on to a glass plate, on which three equal beads strung together were laid, a second glass plate was pressed down upon these beads, after a few hours the glass plates could be removed and a plate of cement of uniform thickness was obtained.

The plate of plaster of Paris and sand consisted of two parts by weight of plaster of Paris to one part by weight of sand.

(*b*) When the substance experimented upon was a powder, the brass plates were kept apart by three stops of wood, and the powder was prevented from falling out at the edges by a very narrow circular ring of fibre.

(*c*) The three values given for the conductivity of garden-soil refer to three specimens of soil of the same kind but containing different amounts of moisture; the first value refers to dry soil, the second to slightly moist, and the third to damp soil. The results show how greatly a small amount of moisture affects the conductivity.

(*d*) The calico in the first experiment was dry, and weighed 8·91 grams. In the second experiment it was exposed for some time to a damp atmosphere, and its weight increased to 9·06 grams, so that the conductivity of calico increases 20 per cent. for 1·6 per cent. increase in weight due to moisture absorbed.

(*e*) In the three experiments on old flannel, the flannel contained different amounts of moisture. In the first experiment the flannel was quite dry and weighed 8·21 grams, in the second it was exposed to a damp atmosphere, and its weight was 8·29 grams, and in the third experiment was damped with water and its weight increased to 9·05 grams. Thus the conductivity of flannel increased about 11 per cent. for 1 per cent. increase in weight due to absorbed moisture, and afterwards increased 25 per cent. more for a further increase of 10 per cent. in weight.

The flannel in the third experiment is almost as good a conductor as dry calico.

The new flannel, flanelette, silk, and linen were dry.

Substance.	Thickness in cm.	Temperatures. Upper plate θ_2.	Lower plate θ_1.	Air θ_0.	$\frac{ph_1}{2q}b$ $=\cdot00005\,b$.	$h_1+h_1'+\frac{ph_1}{2q}b$ $=\cdot00044\,45\,46\}+$	$\theta_2-\theta_1$.	$\theta_1-\theta_0$.	$\frac{\theta_1-\theta_0}{\theta_2-\theta_1}$.	$\frac{\theta_1-\theta_0}{\theta_2-\theta_1}b$.	k.
							° C.	° C.			
Slate	·388	96·1	92·4	20·0	·00002	·00047	3·7	72·4	19·6	7·60	·00357
Glass	·292	94·6	89·8	18·5	2	47	4·8	71·3	14·8	4·32	·00203
Porcelain	·418	98·0	92·2	19·0	2	47	5·8	73·2	12·6	5·27	·00248
Fibre	·157	96·8	83·8	19·0	1	45	13·0	64·8	4·98	·6782	·00035
Leather, Cowhide	·403	97·0	72·3	19·6	2	46	24·7	52·7	2·13	·916	·00042
Leather, Chamois	·183	97·2	69·3	19·0	1	45	27·9	50·3	2·81	·514	·00015
Indiarubber	·183	97·8	88·0	17·5	1	46	9·8	70·5	7·19	1·316	·00061
Blotting-paper	·104	97·1	78·4	20·3	1	45	18·7	58·1	3·11	·323	·00015
Asbestos-paper	·285	96·3	78·2	17·8	1	45	18·1	60·4	3·34	·952	·00043
(*a*) Portland cement	·316	96·3	83·2	19·2	2	46	13·1	64·0	4·89	1·54	·00071
Plaster of Paris	·336	97·2	83·0	18·8	2	46	14·2	64·2	4·52	1·152	·00070
Plaster of Paris with sand	·321	96·6	86·7	19 3	2	46	9·9	67·4	6·81	2·19	·00101
(*b*) Plaster of Paris powder	·600	98·0	57·0	17·3	3	45	41·0	39·7	9·68	5·81	·00261
Sand	·540	96·9	71·8	17·7	3	47	25·1	54·1	2·16	1·17	·00055

Brickdust, yellow	·540	97·7	67·0	19·8	·00003	·00046	30·7	47·2	1·54	·832	·00038
Brickdust, red	·550	97·1	69·7	20·0	3	46	27·4	49·7	1·83	1·01	·00046
Sawdust	·540	98·0	52·5	20·8	3	44	45·5	31·7	·697	·376	·000165
Salt	·540	98·4	75·0	19·2	3	46	23·4	55·8	2·39	1·29	·00059
(*c*) Garden soil 1	·550	97·5	63·7	19·0	3	46	33·8	44·7	1·32	·716	·00033
,, ,, 2	·550	96·7	81·9	18·5	3	47	14·8	63·4	4·28	2·35	·00110
,, ,, 3	·630	95·6	83·7	19·3	3	47	11·9	64·4	5·41	3·41	·00160
(*d*) Calico 1	·221	97·5	69·4	17·2	1	44	28·1	52·2	1·86	·411	·00018
,, 2	·221	96·8	71·8	16·2	1	44	25·0	55·8	2·23	·493	·00022
(*e*) Old flannel 1	·316	97·0	56·4	18·0	2	44	40·6	38·4	·947	·299	·00013
,, ,, 2	·316	97·8	59·9	19·6	2	44	37·4	40·3	1·07	·338	·00015
,, ,, 3	·316	97·2	62·9	18·0	2	44	34·3	44·9	1·31	·414	·00018
New flannel	·397	97·8	50·1	16·4	2	43	47·7	33·7	·706	·280	·00012
Flanelette	·298	98·4	61·6	18·8	1	43	36·8	42·8	1·16	·346	·00015
Baize	·356	97·0	53·8	18·1	2	43	43·2	35·7	·827	·278	·00012
Linen	·196	96·7	73·3	16·5	1	44	23·4	56·8	2·43	·476	·00021
Silk	·080	96·2	74·9	17·2	0	44	21·3	57·7	2·71	·217	·000095

LV. *On the Relation between the Brightness of an Object and that of its Image.* By W. T. A. Emtage, *M.A.**

If an object situated in a medium of refractive index μ forms an image by means of rays refracted through any number of surfaces and finally into a medium of index μ', the luminosities, I, I′, of object and image satisfy the relation

$$I : I' = \mu^2 : \mu'^2,$$

supposing no light to be lost by reflexion at the surfaces or by absorption in the media.

This is generally proved, in the case where the pencils are not all very narrow and along an axis of revolution of the refracting surfaces, by showing that if this relation did not obtain we could get an image brighter than the object by viewing this through a suitable combination of lenses; a result which would be contrary to experience. The following direct proof may, however, be given for pencils of any sort:—

We notice first that if a small luminous surface sends a certain quantity of light to a second small surface, for the second to send the same quantity of light to the first it must be equally bright with it. For if s_1, s_2 are the areas of the two surfaces, I_1, I_2 their luminosities, d the distance between them, and θ_1, θ_2 the inclinations of their normals to d, the lights which they send to each other are

$$\frac{I_1\, s_1\, s_2 \cos\theta_1 \cos\theta_2}{d^2}, \text{ and } \frac{I_2\, s_1\, s_2 \cos\theta_1 \cos\theta_2}{d^2}.$$

And for these to be equal $I_1 = I_2$.

Now let s be a small area of an object in a medium of refractive index μ_1, and let it send out a sheaf of rays which is refracted across the surface RT into a second medium of index μ_2. The cone of rays from s converging to the point R is refracted into a cone of different solid angle. If i and r are the angles of incidence and refraction at R, the angular breadths of these solid angles in the plane of refraction are di, dr; and their angular breadths at right angles to this plane are proportional to $\sin i$, $\sin r$.

Now

$$\mu_1 \sin i = \mu_2 \sin r,$$

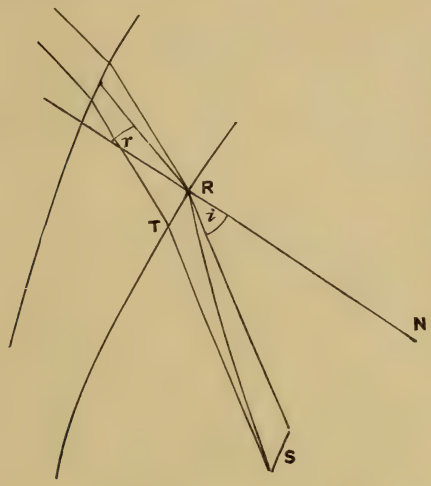
$$\therefore\ \mu_1 \cos i\, di = \mu_2 \cos r\, dr.$$

* Communicated by the Author.

And if the solid angles are ω_1, ω_2

$$\frac{\omega_1}{\omega_2} = \frac{di \sin i}{dr \sin r} = \frac{\mu_2^2 \cos r}{\mu_1^2 \cos i}.$$

Now let I_1 be the luminosity of s, and I_2 that of the surface R T as seen in the medium μ_2 by the light coming originally



from s. I_1 is the luminosity RT must have to send back to s the same amount of light as s sends to it, that is, as R T sends on into μ_2. If s_1 is the area of the surface RT, the quantities of light received and sent on by it are

$$I_1 s_1 \omega_1 \cos i, \text{ and } I_2 s_1 \omega_2 \cos r.$$

And these are equal. Thus from above

$$I_1 : I_2 = \mu_1^2 : \mu_2^2.$$

In the same manner the luminosity of the second surface across which the rays pass, as seen in the third medium of index μ_3, is proportional to μ_3^2; and so on.

If the rays on passing across the last surface form a true image of s, so that the combination of surfaces is aplanatic for s, the brightness of this image is the same as that of the last surface; for it subtends the same solid angle at the point of view as the portion of the surface which sends the same quantity of light to the point of view. Thus, if μ, μ' are the refractive indices of the medium in which the object is and that in which the eye is which sees the image, and I, I′ the luminosities of the object and image;

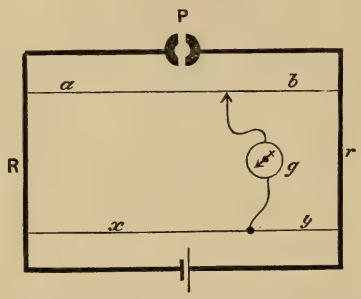
$$I : I' = \mu^2 : \mu'^2.$$

LVI. *Adjustment of the Kelvin Bridge.* By Rollo Appleyard*.

In a recent paper by Mr. J. H. Reeves† an alternative method of adjusting the ratios of the resistances in the Kelvin bridge was described; the following remarks may be regarded as a note upon that paper.

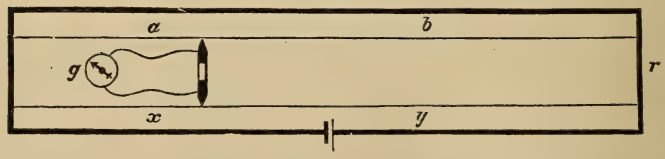
The measurement of a resistance by the arrangement adopted by Mr. Reeves involves two operations. It may be shown, however, that for certain purposes the two adjustments can be combined mechanically, and balance effected by a single test.

Fig. 1.



Consider the conductors R, r, a, b, x, and y in fig. 1. It is required to compare R with r. In the accepted form of the Kelvin bridge a, b, x, and y have fixed values, such that $ay=bx$, and balance is obtained by the one operation of varying R or r. Mr. Reeves prefers to keep R and r fixed; and, consequently, he has first to find provisional values of a, b, x, and y such that $ay \triangleq bx$; and then to balance again, using these values, to find $R:r$; with the plug, this time, inserted at P.

Fig. 2.



But fig. 1 suggests that $a+b$, and $x+y$, or convenient parts of them, may each be formed into a slide-wire, and that these two slide-wires may be placed parallel to one

* Communicated by the Physical Society: read April 24, 1896.

† *Supra*, p. 414.

another, with a double sliding-contact between them, as shown in fig. 2. Then, in whatever position the slider may be, the fundamental ratio $ay=bx$ is always maintained, and the first condition of the Kelvin bridge is mechanically fulfilled. The one adjustment consists in moving the double slider along the bridge until there is no deflexion of the galvanometer at g; in which case

$$\frac{a}{b}=\frac{x}{y}=\frac{R}{r}.$$

Since writing the above, I have referred to the original paper of Lord Kelvin (Proc. Roy. Soc. vol. xi. p. 313, 1861), and find that he proposes the use of parallel slide-wires for his auxiliary conductors; I have no doubt he had in view some such apparatus as that which I have here suggested. A Kelvin bridge with a single slide-wire was used by Matthiessen and Hockin in their differential method; it is described by Clerk-Maxwell in 'Elec. and Mag.' vol. i. p. 406 (1873).

LVII. *The Effect of Wave Form on the Alternate Current Arc.* *By* Julius Frith, 1851 *Exhibition Scholar**.

In the paper by Dr. Fleming and Mr. Petavel, recently read before the Physical Society, on the Alternate Current Arc†, I think too little attention was paid to the wave form of the alternate current used.

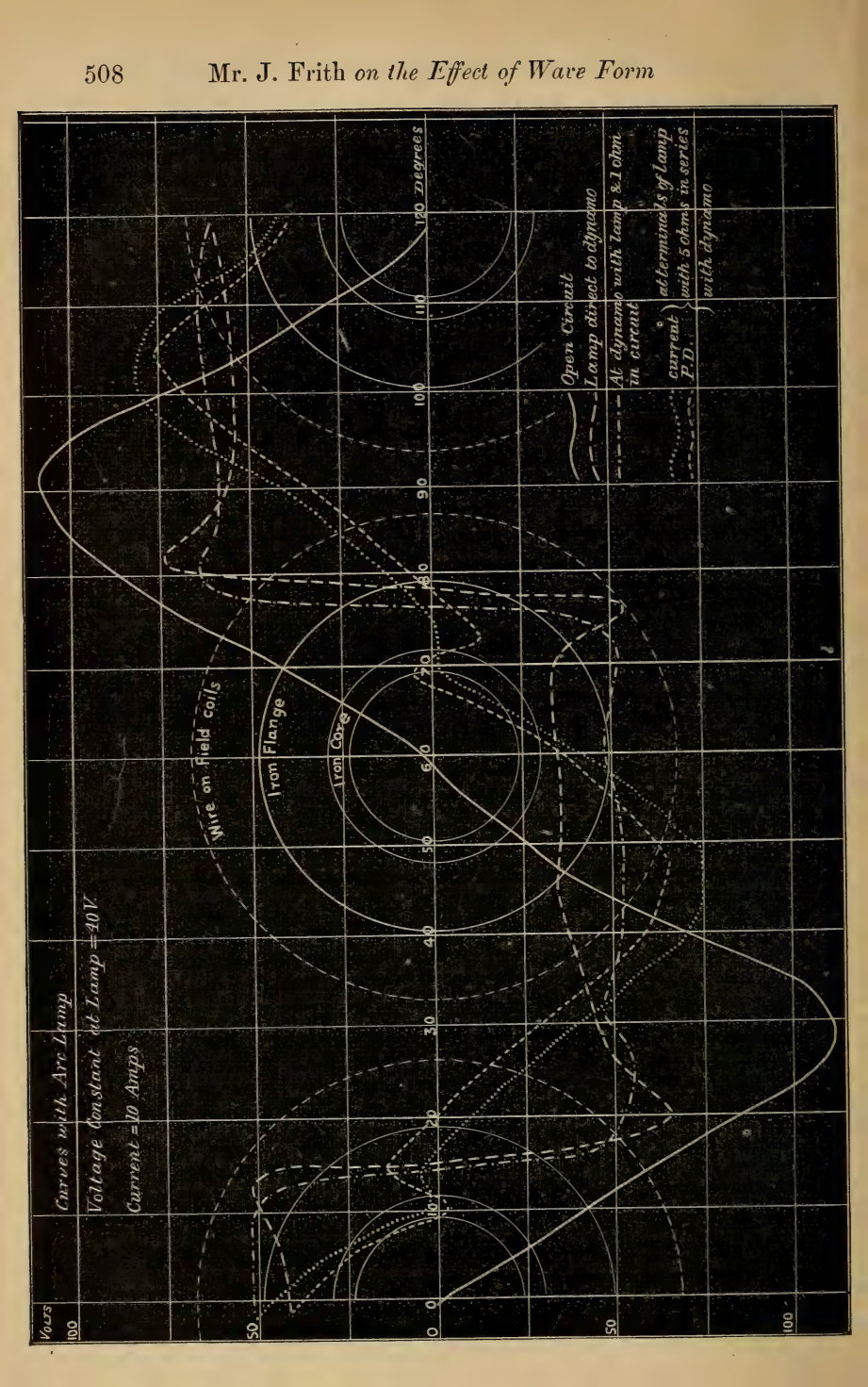
It is known that if the arc is allowed to exert a preponderating influence at all on the alternate current circuit, it alters the wave form of both the current and the P. D. in a very marked degree.

As an illustration of the change produced in the wave form by the character of the external circuit, I give some curves for alternate current arcs taken from a paper which I read before the Manchester Literary and Philosophical Society in March 1894.

Here is shown, first the E.M.F. curve of the machine on open circuit, which is rather more peaked than a sine curve and involves the third harmonic largely. Next is shown the curve obtained under the same conditions, but with an arc lamp taking 10 amperes at 40 volts joined direct to the machine.

The lag recorded is due to the self-induction of the machine,

* Communicated by the Physical Society: read April 24, 1896.
† *Supra*, p. 315.


Volts
100
50
0
50
100
Curves with Arc Lamp
Voltage constant at Lamp = 40 V.
Current = 10 Amps
Wire on field coils
Iron Flange
Iron Core
10
20
30
40
50
60
70
80
90
100
110
120 degrees
Open Circuit
Lamp direct to dynamo
At dynamo with lamp & 1 ohm in circuit
at terminals of lamp with 5 ohms in series with dynamo
current
P.D.

which was a "Wilde" alternator, the armature of which contained iron.

It will also be noticed that the first ordinates of the curves are not quite equal to the last ordinates; this is due to the curve being slightly asymmetrical. The whole period is only one third of the revolution of the alternator (shown as 120 degrees in the figure). In one complete revolution this lack of symmetry would of course disappear.

It is seen that the arc alters the wave form from a peaked curve to a very flat topped curve, changing sign almost instantaneously, and with two small maxima, which occur respectively just before and just after each reversal. The next curve shows the effect of adding a resistance of 1 ohm in series with the arc. This smooths out the two maxima, but otherwise does not affect the shape materially.

The current and P.D. curves with 5 ohms in series with the arc are next shown. Both waves now assume much more the form of the E.M.F. curve on open circuit, except at the zero line. Here the P.D. curve crosses the line twice in each direction, and the current curve runs parallel to the zero line for some way before crossing it.

These curves show, I think, the great effect that the arc has on the wave form, and also how this effect can be destroyed by placing resistance in series with the arc.

In a paper by Rössler and Wedding, which appeared in the 'Electrician' for August 31st, 1894, it was proved that an alternate current arc is more efficient, that is, gives a higher candle-power for the same electric power consumed, when the alternating current feeding it has a flat-topped than when it has a peaked wave form.

Rössler was, however, mistaken, I think, in assuming that the machine was making that curve. Indeed, this mistake runs through the whole of this otherwise most valuable paper. Rössler took three machines giving, as he thought, wave forms from the extremely peaked to the extremely flat wave, and determined the efficiencies of the same arc lamp for each of them. Whereas exactly the same results might have been obtained from one machine alone on causing it to give a higher voltage by increasing the field excitation and then absorbing the excess of voltage in resistance, exactly as in the case of the Wilde alternator above referred to.

I understand that in the experiments described by Dr. Fleming and Mr. Petavel there was always a resistance amounting to 7 ohms outside the arc, and hence a wave form was forced upon the arc which, as Rössler has proved, is not the most efficient one, and which the arc would convert into a

form better suited to itself if it had been allowed to do so, as it is in commerce. This consideration must affect, not only the efficiency, but also the curves which Fleming and Petavel obtained for the variations in the luminous intensity.

Further, this action of the arc in modifying the wave form may throw some light on the discrepancy between the efficiency of alternate current arcs as determined in the laboratory and that stated to be obtained in practice.

When an arc is run in the laboratory a large resistance is almost certain to be put in series with it to ensure that degree of steadiness which is essential to exact measurement, and hence the arc cannot alter the wave form. In the commercial use of arcs, on the contrary, the circumstances are widely different. In this case, for economic reasons, the arc must form a large percentage of the total "reactance" of the circuit, and therefore can easily alter the wave to the form required for the greatest efficiency.

It is interesting to note that the wave form giving the best result for the arc is almost exactly the opposite to that giving the best efficiency for transformers. In the former case a flat-topped wave is best, while for the maximum efficiency of transformers an exceedingly peaked wave is best, as lately found by Dr. Rössler.

This points to the building of alternators for use with transformers in such a way as to give peaked wave-forms.

In the case of the arc the building of machines to give the most efficient wave form is not so necessary, since, generally, the arc itself has the power of automatically converting *any* wave form into the one best suited to its requirements. Nevertheless, when the arc has to run in series with a large resistance it is of the utmost importance for obtaining the best efficiency that the machine should give a flat-topped wave.

City and Guilds of London Central Technical College,
April 2, 1896.

LVIII. *On the Calculation of the Conductivity of Mixtures of Electrolytes having a common Ion.* By Douglas McIntosh, *Physical Laboratory, Dalhousie College, Halifax, N.S.**

In a paper published in the April number of this Magazine (*supra*, p. 276) Prof. MacGregor showed how to obtain, by a graphical process, from observations of the electrical conductivity of a sufficient number of simple solutions

* Abstract of a paper read before the Nova Scotia Institute of Science on the 13th of April, 1896. Communicated by the Secretary of the Institute.

of two electrolytes having a common ion, the data necessary for the calculation of the conductivity of a solution containing both electrolytes, according to the dissociation theory of electrolytic conduction; and in order to test this theory he calculated the conductivities of a series of mixtures of solutions of sodium chloride and potassium chloride, which had been measured by Bender. He found that for dilute solutions his calculations agreed with Bender's observations within the limits of experimental error; but that as the strength of the solutions increased the differences became larger, until with a mixture of solutions containing each four gramme-molecules per litre of salt (the strongest solutions with which Bender worked) a difference of 3·6 per cent. was found. The method of calculation assumed that the ionic velocities of the constituent electrolytes were not changed by the mixing; and Prof. MacGregor attributed the differences between the calculated and observed values to the change which, as he pointed out, would probably be produced in these velocities by mixture.

At his suggestion I have made the observations and calculations described in this paper with the object of determining (1) what the differences between the observed and calculated values are, in the case of mixtures of sodium and potassium chloride solutions of greater strength than those examined by Bender; and (2) how the calculated and observed values are related in the case of solutions containing sodium chloride and hydrochloric acid—electrolytes whose ionic velocities differ from one another much more than those of sodium and potassium chlorides.

Preparation of Solutions and Determination of Conductivities.

The paper of which this is an abstract contains a full statement of the experimental methods employed and the precautions taken to secure accuracy. It will be sufficient here to make a general statement merely.

The salts and acid used were purchased as chemically pure, and the former re-purified by crystallization. They were found by the usual tests to be free from probable impurities. The water was doubly distilled and was also tested.

Simple solutions of salt or acid having been prepared their concentration was determined by volumetric analysis, the pipettes and burettes used having been tested by weighing the water they delivered. The volumetric analyses were found to be accurate to 0·1 per cent. when applied to solutions specially prepared by the mixture of exactly determined quantities of salt and water.

The mixtures in all cases consisted of equal volumes of the constituent solutions. In preparing them the same pipette was used for both solutions in exactly the same way, having been washed out before use in each case with a portion of the solution with which it was to be filled. In all cases solutions were prepared and analysed at 18° C.

The method employed in the observations of conductivity was that of Kohlrausch with alternating current and telephone, the apparatus consisting of a bridge-wire of german-silver about 3 metres long, wound on a marble drum, a set of resistance-coils (of which I needed to use but one), a small induction-coil with a very rapid vibrator, and an ordinary Bell telephone. I calibrated the bridge-wire by the method of Strouhal and Barus, and was able to determine resistances to within from 0·2 to 0·3 per cent.

The electrolytic cell was of a U-shape, and was placed in a water-bath kept at or near 18° C. by a thermostat and stirred by a current of air. Its temperature was read by a thermometer graduated to 0·1 degree Centigrade, and capable of being read easily to 0·05 degree. Its errors had recently been determined at the Physikalisch-technische Reichsanstalt, Berlin. When the bath could not be kept at 18° C., the temperature coefficient was determined. In the platinizing of the electrodes, and in the whole procedure of the determination of the conductivity, I followed carefully the recommendations of Kohlrausch's recent papers.

As I was to employ Kohlrausch's observations as data in the calculations, I reduced all my observations of conductivity to his standard (the conductivity of mercury at 0° C.). The factor requisite for this purpose was determined by comparing my own observations for certain solutions with the values given by him for the same solutions. This factor I found (as Bender had also) to vary somewhat with the conductivity of the solution used in finding it, but not with the nature of the solution. I therefore found its values as given by using a series of solutions of different conductivities; and in reducing the observed conductivity of a mixture to Kohlrausch's standard I used the value of the factor corresponding to the conductivity of the mixture. All observations given below are expressed in terms of Kohlrausch's standard.

Results of Observations on Mixtures.

The following tables give the concentrations of the constituent solutions of the mixtures examined and the observed conductivities of the mixtures: —

(A).—*Sodium and Potassium Chlorides.*

Constituent Solutions (gramme-molecules per litre).		Conductivity $\times 10^3$.
KCl.	NaCl.	
3·88	5·12	2494
3·20	"	2326
2·49	"	2187
1·93	"	2029
3·88	5·12	2494
"	4·28	2404
"	3·37	2316
"	2·56	2196
"	2·06	2124

(B).—*Sodium Chloride and Hydrochloric Acid.*

Constituent Solutions (gramme-molecules per litre).		Conductivity $\times 10^3$.
NaCl.	HCl.	
2·02	4·55	4932
"	3·89	4492
"	3·29	4089
"	3·19	4073
"	3·06	3958
"	2·66	3623
"	2·56	3489
"	2·34	3323
1·04	4·55	5069
"	3·97	4682
"	3·80	4315
"	3·10	3989
"	2·86	3696
"	2·18	3112
"	2·11	3025
"	1·93	2824
"	1·58	2427
"	1·15	1928
0·607	1·120	1813
"	0·970	1620
"	0·815	1412
"	0·730	1296·5
"	0·603	1114
"	0·485	952

The Data for the Calculations.

The method of calculating the conductivity of the mixtures was that described by Prof. MacGregor in the paper referred to above. It requires as data the change of volume on mixing (if not negligible), the conductivities of sufficiently extended series of the simple solutions of about the same dilution as the solutions mixed, and the molecular conductivities of the simple solutions at infinite dilution.

Specific gravity determinations showed that the change of volume on mixing was in all cases so small as to produce no practical effect on the calculated value of the conductivity.

Kohlrausch's tables* of the conductivity of solutions of sodium and potassium chlorides furnished sufficient data for calculating the conductivity of mixtures of these salts; but I found it necessary to make additional observations on solutions of hydrochloric acid. They are as follows:—

Concentration (gramme-molecules per litre).	Molecular Conductivity $\times 10^3$.	Concentration (gramme-molecules per litre).	Molecular Conductivity $\times 10^3$.
1·58	2550	2·80	2065
1·93	2403	2·88	2052
2·11	2347	3·15	1960
2·18	2305	3·29	1914
2·24	2290	3·39	1890
2·46	2245	3·60	1789
2·51	2192	3·83	1726
2·56	2164	4·13	1636
2·66	2141	4·55	1534
2·78	2090	4·87	1456

The values of the specific molecular conductivity at infinite dilution for potassium chloride, sodium chloride, and hydrochloric acid respectively, were taken to be 1220×10^{-8}, 1030×10^{-8}, and 3500×10^{-8} according to Kohlrausch's determination†.

Results of the Calculations.

(A) *Sodium and Potassium Chlorides.*

The following table contains in columns 1 and 2 the concentrations of the constituent solutions in the mixtures. Column 3 gives the calculated values of the conductivity;

* 'British Association Reports' (1893), p. 148.
† Wiedemann's *Annalen*, xxvi. p. 204.

column 4 the observed values obtained by graphical interpolation from the observations given above; and column 5 the excesses of the calculated over the observed values expressed as percentages:—

Constituent Solutions (gramme-molecules per litre).		Conductivity $\times 10^3$.		Difference per cent.
KCl.	NaCl.	Calculated.	Measured.	
3·75	5·12	2312	2469	−6·4
3·50	"	2276	2420	−6
3·00	"	2202	2313	−4·8
2·50	"	2109	2190	−3·7
2·00	"	2013	2049	−1·7
3·88	5·00	2323	2481	−6·4
"	4·50	2295	2429	−5·5
"	4·00	2292	2377	−3·6
"	3·50	2261	2324	−2·7
"	3·00	2227	2260	−1·4
"	2·50	2174	2189	−0·7
"	2·00	2096	2116	−1·0

The differences of the above table agree very well with those found by Prof. MacGregor* in the case of the mixtures examined by Bender, being of the same sign and, in general, for mixtures of about the same mean concentration, of approximately the same magnitude. The results seem, therefore, to be worthy of confidence, and they show clearly that the differences between the calculated and the observed values increase rapidly as the constituent solutions become more and more nearly saturated, reaching in the case of practically saturated solutions 6·4 per cent.

(B) *Sodium Chloride and Hydrochloric Acid.*

The following table gives the results in the case of mixtures of solutions of sodium chloride and hydrochloric acid:—

* *Supra*, p. 285.

Constituent Solutions (gramme-molecules per litre).		Conductivity of Mixture $\times 10^8$.		Difference per cent.
HCl.	NaCl.	Calculated.	Measured.	
2	2·02	3020	3008	+0·4
2·5	,,	3489·5	3456	+1·0
3·0	,,	3885	3888	−0·08
3·5	,,	4233·5	4260	−0·6
4·0	,,	4622·3	4580	+1·0
4·5	,,	4944	4880	+1·3
1	1·04	1751	1752	−0·005
1·5	,,	2373	2332	+1·7
2·0	,,	2928·3	2900	+0·9
2·5	,,	3428·5	3398	+0·9
3·0	,,	3906	3872	+0·9
3·5	,,	4340·7	4316	+0·6
4·0	,,	4715	4700	+0·3
4·5	,,	5055	5036	+0·4
·4	·607	829·8	838	−1·0
·5	,,	983·4	976	+0·8
·6	,,	1125·5	1116	+0·8
·7	,,	1255	1250	+0·4
·8	,,	1384·7	1388	−0·2
·9	,,	1524·6	1525	−0·025
1·0	,,	1658·9	1656	+0·16
1·1	,,	1787·6	1784	+0·2
1·2	,,	1917·1	1913	+0·2

It will be seen that in the series of weakest solutions, the differences between calculated and observed values are of such small magnitude and show such alternation of sign as to warrant the conclusion that they are due chiefly to accidental errors. In the two series of stronger solutions the differences are more irregular in magnitude and the alternation of sign is much less marked, the most of the differences being positive. The above results, therefore, seem to show that even in the case of two electrolytes with a common ion which differ so markedly in ionic velocity from one another as sodium chloride and hydrochloric acid, the dissociation theory enables us to calculate the conductivity of solutions containing both, within the limits of experimental error, up to a mean concentration of about one gramme-molecule per litre, and that in the case of solutions of greater mean concentration the calculated value is greater than the observed.

LIX. *Notices respecting New Books.*

The Magnetic Circuit. By Dr. H. du Bois. *Translated by* Dr. Atkinson. Pp. xviii+362. (London: Longmans, Green, & Co. 1896.)

In view of the ever increasing employment of magnetic circuits, in some form or other, in experimental and applied electricity, the importance of this work (so far the only one in which the subject has been treated with anything like completeness) will be recognized by both physicists and practical electricians.

The book is divided into two parts and eleven chapters, the first part (Chapters I.–V.) being devoted to the mathematical development of the theory, and the second chiefly to applications.

Chapters I.–IV. contain a short account of the electromagnetic field, the magnetization of bars and ellipsoids (considered as imperfect magnetic circuits), and the general theory of perfectly "rigid" and of temporary magnetization. The mode of treatment of this part in general follows that of Maxwell, but the definition of both "magnetic intensity" and magnetic induction by the currents induced in an exploring coil is an improvement on the usual method in which the forces acting on a magnetic pole in narrow cavities of different forms are considered at the outset. The term "magnetic intensity" is thus used throughout the book (instead of "magnetic force"), and magnetization is defined as the difference between magnetic induction and magnetic intensity, divided by 4π.

In Chapter V. are explained the author's theory of the magnetization of an iron toroid with a radial slit (the typical case of an imperfect magnetic circuit), and the verification of the results by the experiments of Lehmann.

With Part II. begins the more special part of the work—the properties of magnetic circuits as such. The discussion of the stress existing in a circuit follows Maxwell in assuming a longitudinal tension $BH/4\pi - H^2/8\pi$ to exist at each point in the direction of the lines of induction. Whether this is the magnitude of the stress existing (even in a rigidly magnetized part of the circuit where also B and H have the same direction) is, of course, open to question: the balance of evidence tends to show that the stress has a different value.

The remaining Chapters VII.–XI. are highly interesting, containing a discussion of the analogy between the magnetic and other circuits and the applicability of a so-called "Ohm's law," of the magnetic circuits of dynamos and electromagnets and principles of their design as realized in the du Bois electromagnet for producing very strong fields, and, finally, of methods and instruments for measuring field-intensity, magnetization, and induction. Of the instruments described, one of the most interesting is the author's magnetic balance, by means of which a complete magnetic cycle (H, I) can be obtained in a few minutes.

The translation, though good on the whole, is not altogether

free from inaccuracies. Thus, on page 340, § 221, the word "circuitous" (which does not express "*umständlich*") does not convey the author's opinion of a method necessarily full of detail and one which is required in certain special circumstances. A few other errors such as "normal" for "tangential" (page 83, theorem iv.), and the occasional use of "magnitude" for "quantity" are the most serious defects.

The author has made several additions containing results obtained chiefly during the past year.

The printing and general appearance of the book are excellent, and Dr. Atkinson deserves the thanks of English physicists for introducing to them so important an addition to the fascinating subject of Magnetism.

"*Die Ausdehnungslehre, vollständig und in strenger Form bearbeitet von* HERMANN GRASSMANN (Berlin, 1862)."

THIS book is the second part of the first volume of the miscellaneous mathematical and physical works of their illustrious author, which are in course of publication under the editorship of Dr. Fr. Engel, in cooperation with a strong body of five other eminent mathematicians. The volumes are being brought out at Leipzig, and the one before us gives a reprint of the edition brought out in Grassmann's lifetime in 1862. Clifford would have been delighted with this edition. In 1878 he contributed to the first volume of the 'American Journal of Mathematics' an article on "Applications of Grassmann's Extensive Algebra" (Mathematical Papers, pp. 266–276), which opens with the statement:—"Until recently I was unacquainted with the *Ausdehnungslehre*, and knew only so much of it as is contained in the author's geometrical papers in *Crelle's Journal* and in 'Hankel's Lectures on Complex Numbers.' I may, perhaps, therefore be permitted to express my profound admiration of that extraordinary work, and my conviction that its principles will exercise a vast influence upon the future of mathematical science." Prof. Tait writes:—"Hamilton and Grassmann, while their earlier work had much in common, had very different objects in view. Hamilton had geometrical application as his main object; when he realized the quaternion system, he felt that his object was gained, and thenceforth confined himself to the development of his method. Grassmann's object seems to have been, all along, of a much more ambitious character, viz. to discover, if possible, a system or systems in which every conceivable mode of dealing with sets should be included. That he made very great advances towards the attainment of this object all will allow; that his method, even as completed in 1862, fully attains it is not so certain" ("Quaternions" in *Encycl. Brit.* 9th edit., where more remarks on the same writers will be found). After these quotations as to the value of Grassmann's work, we need only state the following partciulars. There are a few words, by way of preface, by the editor, from which we learn that the text of the "1862" edition (the *first* edition was published in 1842) has been carefully

revised by the author's son. The text itself here occupies 379 pages and is followed by an alphabetical index of terms explained and by a Table of Contents. Then there is a collection of the most important passages in which the present work differs from the "1862" text. This occupies pp. 384–396. Then the important "Notes" take up pp. 397–495. A full index completes the volume. What is now wanted—as has been pointed out elsewhere more than once we think—is a translation into English which should render Grassmann's original views easily accessible to a much larger circle of readers than exists at present.

Graphical Calculus. By A. H. Barker, B.A., B.Sc. (Longmans, 1896. Pp. viii + 188.)

Prof. Goodman, who furnishes a brief introduction, writes:—"I have frequently had students come under my notice who, although fairly good mathematicians as far as bookwork is concerned, yet through not having had the advantage of a *practical* mathematical training were utterly at sea when they came to apply their mathematics to a simple engineering problem." It is Mr. Barker's object in this book to help such a one to acquire an intelligent *working* knowledge of the Calculus, and after a perusal of the major portion of it we can endorse Prof. Goodman's commendations. The student who carefully goes through all the constructions and keeps in mind the author's injunction to aim at grasping the *meaning* underlying all the symbols employed should be no mean proficient in their use, and not one "who performs algebraical operations in a haphazard fashion without making himself acquainted with the principle involved." Besides a careful discussion of the principles and of the methods of differentiation and integration there are numerous applications to practical questions. In an appendix there is a figure and a description (given here for the first time) of a Planimeter devised by Mr. Barker. We have detected very few slips: on p. 5 it is not *stated* that the curve is a circle; on pp. 61, 133 + is given where the sign should be − ; on p. 96 the value of e is slightly incorrect; and on p. 156, ex. (1), there would seem to be some slip. The type and the figures are good, but the punctuation is defective. One little point the author might correct in subsequent editions, viz. the frequent statement "as will be presently explained," and the like references to future chapters.

LX. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 147.]

January 22nd, 1896.—Dr. Henry Woodward, F.R.S., President, in the Chair.

THE following communications were read:—

1. 'On the Speeton Series in Yorkshire and Lincolnshire.' By G. W. Lamplugh, Esq., F.G.S.

Further work on the Speeton section, while extending our

knowledge of the palæontological details, has fully sustained the results of the author's previous investigations. The rapid attenuation and final disappearance of the Speeton Series in a westerly direction in Yorkshire is discussed, and though the available evidence is held to be insufficient to demonstrate the exact conditions, it is stated that, contrary to the accepted view, the lower zones are probably the first to die out and are overstepped or overlapped by the higher divisions, since at Knapton, 14 miles inland, only the upper zones of the coast-section can be proved to occur, as shown by the presence of marls with *Bel. minimus* passing upward into the Red Chalk, and by the fossils preserved in the old collections including *Hoplites Deshayesii* under the name of *Amm. knaptonensis*, Bean MS., and a few others of the same zone.

The ferruginous sands locally occurring beneath the Red Chalk on the western edge of the Yorkshire Wolds are recognized as agreeing in all respects with the Lincolnshire Carstone; and Mr. A. Strahan's conclusions as to the relations of this division to the Red Chalk are confirmed both in Yorkshire and in Lincolnshire.

In Mid-Lincolnshire all the palæontological zones of Speeton are identified and traced, the presence of the leading zonal types of the cephalopoda readily establishing the general correlation proposed by Prof. A. Pavlow and the author, in spite of the greatly modified lithological aspect of the deposits and the corresponding modification of their fauna. The chief features of this correlation, which differs in many respects from that adopted by Prof. Judd and the Geological Survey, are as follows:—

Speeton.		*Lincolnshire.*
	RED CHALK.	
Zone A. Passage Marls	=	Carstone (? in part or wholly).
B. Zone of *Bel. brunsvicensis*	=	Tealby Limestone and its southerly equivalents (also ? the lower part of the Carstone).
C. Zone of *Bel. jaculum*	=	Tealby Clay of the Wold escarpment (the lower portion of the zone doubtfully represented).
D. Zone of *Bel. lateralis*	=	'Tealby Clay' of the outliers west of Spilsby (Hundleby Clay), Claxby Ironstone, and Spilsby Sandstone.
E. 'Coprolite-bed'	=	Nodule-bed at the base of the Spilsby Sandstone.
	Kimeridge Clay.	

In Lincolnshire, in at least one instance, the synchronal boundary as indicated by the limits of a palæontological zone is shown not to pursue the same stratigraphical horizon throughout its course, proving that sediments of different character were accumulated simultaneously in comparative proximity to each other. The inherent divergence between the stratigraphical and palæontological methods in geology is thus once more illustrated.

The derivative character of the band of phosphatic nodules at

the base of the Spilsby Sandstone is stated to be very doubtful and the fossils of the so-called 'pebbles' are considered as probably representing an original fauna, poorly preserved in nodules formed during a temporary pause in the sedimentation.

The 'Zone of *Bel. lateralis*' (*Série Speetono-russe* of Pavlow) is shown to bridge over the space between undoubtedly Jurassic and undoubtedly Lower Cretaceous strata; but according to the recent results of Prof. A. Pavlow, if the accepted classification of other areas is to be upheld, the division between the two systems must be placed high enough to include this zone in the Jurassic, in spite of the local inconvenience of this arrangement.

2. 'On some Podophthalmous Crustaceans from the Cretaceous Formation of Vancouver and Queen Charlotte Islands.' By Henry Woodward, LL.D., F.R.S., P.G.S.

3. 'On a Fossil Octopus, *Calais Newboldi* (J. de C. Sby., MS.), from the Cretaceous of the Lebanon.' By Henry Woodward, LL.D., F.R.S., P.G.S.

4. 'On Transported Boulder Clay.' By the Rev. Edwin Hill, M.A., F.G.S.

The 'mid-Glacial' sands of the cliffs between Yarmouth and Lowestoft are overlain at Corton by Chalky Boulder Clay. But farther north than Corton some masses of the same clay occur in the interior of the cliffs, surrounded by the sands in undisturbed stratification, but passing into them by strings and patches such as suggest the melting off of enveloping ice. They have probably been floated and dropped there.

Again, gravels lying in a valley of Chalky Boulder Clay in West Suffolk (Cockfield &c.), and indicating considerable denudation of the Clay, yet have some patches and sheets of that Clay overlying them as if carried down or slipped down from higher ground.

This may explain some anomalous positions of Boulder Clay noted by writers. The Lowestoft observations suggest that Chalky Boulder Clay was being manufactured in one locality simultaneously with 'mid-Glacial' sands in another.

February 5th.—Dr. Henry Woodward, F.R.S., President, in the Chair.

The following communications were read:—

1. 'On the Morte Slates and Associated Beds in North Devon and West Somerset.—Part I.' By Henry Hicks, M.D., F.R.S., F.G.S.

In a paper read before the Society in 1890 the author stated that he had found the Morte Slates to be fossiliferous, and had come to the conclusion that they were the oldest rocks in the North Devon area and had been thrust over much newer rocks, producing a deceptive appearance of conformity; and that there was not a continuous upward succession in the rocks from the Bristol Channel to the neighbourhood of Barnstaple. Since that paper was read, the

author has obtained much additional evidence bearing on the succession, which is given in the present paper so far as the position and age of the Morte Slates in the Ilfracombe area is concerned.

The author describes the lithological characters and fossil contents of the Morte Slates, and their relationships to the Pickwell Down Sandstones and Ilfracombe Beds, treating of the development of the rocks in four areas, viz.:—Morthoe and Woolacombe to Bittadon; Rockham Bay, Bull Point, Lee, and Slade; Mullacott, Shelfin, and Ilfracombe; and Woolscott Barton, Smithson, and Berry Down. The great abundance of detrital mica in the Ilfracombe Beds and Pickwell Down group, and its almost entire absence from the Morte Slates is noticed. The faulted junction of the Pickwell Down Sandstones and Morte Slates is described, and it is held that the faulting has not thrust older beds over newer, as maintained by Jukes, but that the order of succession is the original one, whereas the overthrust fault separates the Morte Slates from the now underlying though newer Ilfracombe Beds, which often dip away from the Morte Slates, while different members of the Ilfracombe series abut against the Morte Slates, so that the latter form a complex group of older rocks bounded on either side by newer strata. The author states that the fossils found in the Morte Slates belong to several horizons, some probably as low in position as the base of the Silurian (Upper Silurian of the Geological Surveyors), while none of the beds of the series appear to be newer than the older Devonian. In some places newer rocks may occur amongst them as the result of faulting or unconformity, but not in order of succession.

A description of the species found in the Slates is appended to the paper.

2. 'Evidences of Glacial Action in Australia in Permo-Carboniferous Time.' By Prof. T. W. Edgeworth David, B.A., F.G.S.

The author, after summarizing the work of previous observers, gives an account of recent observations made by himself.

In Hallett's Cove, near Adelaide, the pre-Cambrian rocks are strongly glaciated, striæ being seen when the overlying glacial beds are removed, as sharply cut as though caused by recent glacial action, and trending nearly north and south, the ice having come from the south. The overlying glacial beds are in places fairly stratified, while parts contain abundance of well-striated boulders: these beds are from 23 to over 100 feet thick. Proofs were obtained that in this case the glaciation occurred in an age intermediate between Miocene and pre-Cambrian, and probably did not antedate the close of the Palæozoic period.

In Wild Duck Creek near Heathcote, Lower Silurian (Ordovician) beds exhibit strongly-grooved, polished surfaces, the grooves being from S. 5° E. to N. 5° W., the ice probably having come from the south. They are succeeded by Permo-Carboniferous glacial beds, consisting chiefly of mudstones with well-glaciated boulders.

At Bacchus Marsh Ordovician beds are also well-striated and

polished, and more or less *moutonnée.* Here also the ice came from a southerly point. These beds are succeeded by Permo-Carboniferous glacial beds having an approximate thickness of at least 2000 feet, consisting of mudstones with well-glaciated boulders.

It is extremely probable that the glacial beds of Bacchus Marsh, Wild Duck Creek, and Springhurst in Victoria were of homotaxial if not contemporaneous origin, and they may probably be correlated with the glacial conglomerates at Mount Reid in Tasmania, these correlations being mainly based on lithological evidence. The evidence for the correlation of the Bacchus Marsh glacial beds with the erratic-bearing Permo-Carboniferous mudstones of Maria Island, One Tree Point, and Bruni Island in Tasmania, of Maitland, Branxton, and Grasstree in N.S. Wales, and of the Bower River coalfields in Queensland, is that the genus *Gangamopteris* is distributed somewhat abundantly throughout the formations in all these localities.

This glaciation was probably homotaxial with that of the period of the Dwyka Conglomerate and Ecca Beds of Southern Africa, and of the Talchir Group of the Salt Range of India, the Boulder Beds in Western Rajputana, and the Panjáh Conglomerates of Kashmir. In the case of Southern Africa and India, as in that of Australia, the general direction in which the ice moved appears to have been from south to north. In the Bacchus Marsh beds there are at least 9 or 10 distinct boulder-beds separated by sandstones and conglomerates; this may possibly indicate a sequence of glacial periods separated by milder interglacial periods. The glacial conditions in Australia may have been prolonged into early Mesozoic times, as indicated by the Mesozoic facies of certain plants in the uppermost glacial beds of Bacchus Marsh.

February 26th.—Dr. Henry Hicks, F.R.S., President, in the Chair.

The following communications were read:—

1. 'On the Structure of the Plesiosaurian Skull.' By Charles W. Andrews, Esq., B.Sc., F.G.S.

2. 'On certain Granophyres, modified by the Incorporation of Gabbro Fragments, in Strath (Skye).' By Alfred Harker, Esq., M.A., F.G.S.

The rocks described form a group of irregular intrusions, the largest less than a mile in length, situated in the tract of volcanic agglomerate north and west of Loch Kilchrist. They differ from the normal granophyres, abundantly developed in the neighbourhood, in being darker, denser, and manifestly richer in the iron-bearing minerals, while in places are seen numerous small rock-fragments evidently of extraneous origin.

The fragments are mainly of gabbro. Closer examination shows that they have been abundantly distributed through the granophyre, but most of them have been more or less completely dissolved. The clearest evidence of this is afforded by the augite of the gabbro, which has been less readily attacked by the magma than the other

minerals, and is seen in isolated crystals in various stages of conversion to hornblende. The material dissolved has rendered the acid magma less acid, and has influenced accordingly the products of final consolidation. The granophyre is roughly estimated to have taken up about one-fourth of its mass of gabbro, and this material has been derived, not from the rocks seen in contact with the intrusions, but from some subterranean source.

3. 'Observations on the Geology of the Nile Valley, and on the Evidence of the greater Volume of that River at a former Period.' By Prof. E. Hull, M.A., LL.D., F.R.S., F.G.S.

The author draws attention to the two great periods of erosion of the Nile Valley, the first during the Miocene period, after the elevation of the Libyan region at the close of Eocene times, and the second during a 'pluvial' period extending from late Pliocene times into and including the Pleistocene. He notes the course of the river through escarpments of the granitic and schistose rocks of Assuan, the Nubian Sandstone, the Cretaceous limestone, and the Eocene limestone, and observes that in places the line of erosion of the primæval Nile was directed by dislocations of the strata. Evidence of the unconformity of the Nubian Sandstone upon the granites and schists of Assuan is given, and some observations made upon the age of the different parts of the Nubian Sandstone.

In the second part of the paper the terraces of the Nile Valley are described, and full details given of the characters of a second terrace, at a height varying from 50 to 100 feet above the lower one, which is flooded at the present day. The second terrace is devoid of vegetation, and its deposits have frequently furnished river-shells such as *Cyrena fluminalis*, *Ætheria semilunata*, *Unio*, *Paludina*, &c. The second terrace is traceable at intervals for a distance of between 600 and 700 miles above Cairo. Two old river-channels are also described, one at Koru Ombo, and the other at Assuan itself. The author discusses the mode of origin of the second terrace and the old river-valleys, and believes them to be due to the former greater volume of the river, and not to subsequent erosion of the valley. He gives further evidence of the existence of meteorological conditions sufficient to give rise to a 'pluvial' period, and points out that other authors have also considered that the volume of the Nile has been greater in former times.

4. 'The Fauna of the Keisley Limestone.—Part I.' By F. R. Cowper Reed, Esq., M.A., F.R.S.

LXI. *Intelligence and Miscellaneous Articles.*

ON THE DIFFUSION OF METALS. BY W. C. ROBERTS-AUSTEN, C.B., F.R.S., PROFESSOR OF METALLURGY, ROYAL COLLEGE OF SCIENCE.

PART I.—*Diffusion of Molten Metals.*

IN the first part of the paper the author alludes to some earlier experiments he made in 1883 on the diffusion of gold, silver,

and platinum in molten lead. He points out that although the action of osmotic pressure in lowering the freezing-point of metals has been carefully examined, very little attention has been devoted to the measurement, or even to the consideration, of the molecular movements which enable two or more metals to form a truly homogeneous fluid mass. The absence of direct experiments on the diffusion of molten metals is probably explained by the want of a sufficiently accurate method. Ostwald had stated, moreover, with reference to the diffusion of salts, that "to make accurate experiments in diffusion is one of the most difficult problems in practical physics," and the difficulties are obviously increased when molten metals diffusing into each other take the place of salts diffusing into water.

The continuation of the research was mainly due to the interest Lord Kelvin had always taken in these experiments. The want of a ready method for the measurement of comparatively high temperatures, which led to the abandonment of the earlier work, was overcome when the author arranged his recording pyrometer, and the use of thermo-junctions in connexion with this instrument rendered it possible to measure and record the temperature at which diffusion occurred. Thermo-junctions were placed in three or more positions in either a bath of fluid metal or an oven carefully kept hotter at the top than at the bottom. In the bath or oven, tubes filled with lead were placed, and in this lead, gold, or a rich alloy of gold, or of the metal under examination, was allowed to diffuse upwards against gravity. The amount of metal diffusing in a given time was ascertained by allowing the lead in the tubes to solidify; the solid metal was then cut into sections, and the amount of metal in the respective sections determined by analysis.

The movement in linear diffusion is expressed, in accordance with Fick's law, by the differential equation

$$\frac{dv}{dt}=k\frac{d^2v}{dx^2}.$$

In this equation x represents distance in the direction in which diffusion takes place, v is the degree of concentration of the diffusing metal, and t is the time; k is the diffusion constant, that is, the number which expresses the quantity of the metal in grams diffusing through unit area (1 square centim.) in unit time (one day) when unit difference of concentration (in grams per cubic centim.) is maintained between the two sides of a layer 1 centim. thick. The author's experiments have shown that metals diffuse in one another just as salts do in water, and the results were ultimately calculated by the aid of tables prepared by Stefan for the calculation of Graham's experiments on the diffusion of salts.

The necessary precautions to be observed and the corrections to be made are described at length, and the values of the diffusivity of various metals in lead are then given.

The values for k, the diffusivity, given in square centimetres per day, are as follows:—

	k.
Gold in lead.....	3·19 at 500°.
„ bismuth	4·52 „
„ tin	4·65 „
Silver in tin.....	4·14 „
Lead in tin	3·18 „
Rhodium in lead	3·04 „
Platinum in lead	1·69 at 490°.
Gold in lead.....	3·03 „
Gold in mercury	0·71 at 11°.

In order to afford a term of comparison, it may be stated that the diffusivity of chloride of sodium in water at 18° is 1·04.

The author at present refrains from drawing any conclusion as to the evidence which the results afford respecting the molecular constitution of metals. It is, however, evident that they will be of value in this connexion, because, with the exception of the gases, they present the simplest possible case of diffusion which can occur—the diffusion of one element into another.

Thus the relatively slow rate of diffusion of platinum as compared with gold points to its having a more complex molecule than the latter.

Part II.—*Diffusion of Solid Metals.*

The second part of the paper is devoted to the consideration of the diffusion of solid metals. Much of the evidence is historical, for there has long been a prevalent belief that diffusion can take place in solids, and the practice in conducting important industrial operations supports this view. In this connexion the author cites two truly venerable "cementation" processes. The object in the first of these is the removal of silver from a solid gold-silver alloy, while the second is employed in steel-making by the carburization of solid iron. In both of these processes, however, a gas may intervene, though the carburization of iron by the diamond, which had been effected *in vacuo* by the author, suggests that if a gas does intervene in the latter case, its quantity must be very minute. In connexion with the mobility of various elements in iron the work of Colson, of Osmond, and of Moissan is specially referred to.

The author points out that in 1820 Faraday and Stodart showed that platinum will alloy with steel at a temperature at which even the steel is not melted, and they express their interest in the formation of alloys by cementation, that is by the union of solid metals.

The remarkable view expressed by Graham, in 1863, that the "three conditions of matter (liquid, solid, and gaseous) probably always exist in every liquid or solid substance, but that one predominates over the other," is shown to have afforded ground for

the anticipation that metals would diffuse into each other at temperatures far below their melting-points. Reference is then made to the important work by Spring in 1886 on the lead-tin alloys, which retained a certain amount of molecular activity after they had become solid; and special importance is attached to the proof afforded by Spring, that alloys may be formed either by the strong compression of the finely divided constituent metals at the ordinary temperature (1882) or (1894) by the union of solid masses of metal compressed together at temperatures which varied from 180° in the case of lead and tin, to 400° in the case of copper and zinc; tin melting at 227° and zinc at 415°.

The evidence as to the volatilization of solid metals is then traced, and allusion is made to the expression of Robert Boyle's belief, that even such solid bodies as glass and gold might respectively "have their little atmospheres, and might in time lose their weight."

Merget's experiment on the evaporation of frozen mercury is quoted in relation to Gay-Lussac's well-known discovery that the vapours emitted by ice and water both at 0° C. are of exactly equal tension.

Demarçay's experiments on the volatilization of metals *in vacuo* at comparatively low temperatures is connected with the evidence afforded by Spring (1894), that the interpenetration of two metals at a temperature below the melting-point of the more fusible of the two is preceded by volatilization.

The author then points out that, interesting as the results of the earlier experiments are, as affording evidence of molecular interpenetration, they do not, for the purpose of measuring diffusivity, come within the prevailing conditions in the ordinary diffusion of liquids, in which the diffusing substance is usually in the presence of a large excess of the solvent, a condition which has been fully maintained in the experiments on the diffusion of liquid metals described in the first part of the paper. Van't Hoff has made it highly probable that the osmotic pressure of substances existing in a *solid solution* is analogous to that in liquid solutions, and obeys the same laws: and it is probable that the behaviour of a solid mixture, like that of a liquid mixture, would be greatly simplified if the solid solution were very dilute.

The author proceeds to describe his own experiments on the diffusion of solid metals. They are of the same nature as in the case of fluid metals, except that the gold, which is the metal chosen for examination, was placed at the bottom of a solid cylinder of lead instead of a fluid one.

In the first series of experiments, cylinders of lead, 70 millims. long, with either gold, or a rich alloy of gold and lead at their base, were maintained at a temperature of 251° (which is 75° below the melting-point of lead) for thirty-one days. At the end of this period the solid lead was cut into sections, and the amount of gold which had diffused into each of them was determined in the usual

way. Other experiments follow, in which the lead was maintained at 200°, and at various lower temperatures down to that of the laboratory. The following are the results:—

						k.
Diffusivity of gold in fluid lead at				550°	....	3·19
,,	,,	solid	,,	251	....	0·03
,,	,,	,,	,,	200	....	0·007
,,	,,	,,	,,	165	....	0·004
,,	,,	,,	,,	100	....	0·00002

The experiments at the ordinary temperature are still in progress, but there is evidence that slow diffusion of gold in lead occurs at the ordinary temperature. The author points out that if clean surfaces of lead and gold are held together *in vacuo* at a temperature of only 40° for four days, they will unite firmly, and can only be separated by the application of a load equal to one-third of the breaking strain of lead itself.

The author thinks it will be considered remarkable that gold placed at the bottom of a cylinder of lead, 70 millims. long (which is to all appearance solid), will have diffused to the top in notable quantities at the end of three days. He points out that at 100° the diffusivity of gold in solid lead can readily be measured, though its diffusivity is only 1/100,000 of that in fluid lead at a temperature of 500°. He also states that experiments which are still in progress show that the diffusivity of solid gold in solid silver, or copper, at 800° is of the same order as that of gold in solid lead at 100°.

He concludes by warmly thanking Mr. A. Stansfield, B.Sc., who assisted him in all but the earlier portion of the work, and by expressing the hope that the experiments described in the paper will show that the diffusion can readily be measured in solid metals, and that they will carry one step further the work of Graham.—*Proceedings of the Royal Society*, February 20, 1896, being an abstract of the Bakerian Lecture.

RÖNTGEN RAYS NOT PRESENT IN SUNLIGHT. BY M. CAREY LEA.

If Prof. Röntgen's views as to the nature of the x-rays are correct, it would seem that they ought to be found amongst the many forms of radiant energy received from the sun, and various observers have thought that they so found them. Some experiments, the most important of which will be here briefly stated, do not seem to support this opinion.

1. A very sensitive dry plate (S. 27) was placed between the leaves of a book so that 100 leaves and the red paper cover should be between the sensitive film and the sunlight. The book was then packed in a box-frame to exclude all light from the sides. A large and thick lead star was then fastened on the outside of the book and the arrangement was exposed to exceptionally bright sunshine from 11 A.M. to sunset, March 7. The plate when put

into a developing-bath behaved as if unexposed. A prolonged development did not bring out a trace of an image of the lead star.

It will be remembered that Prof. Röntgen found that the x-rays penetrated easily through a book of 1000 printed pages. Indeed G. Moreau has recently stated that in his hands the x-rays had penetrated through "several metres" of cardboard*. So that the above experiment seems to be very significant.

2. A piece of sheet aluminium 1·2 millim. thick was accurately fitted into a frame. A very sensitive plate was placed behind it and a lead star in front. With three hours' exposure not a trace of an image could be obtained. This experiment was varied by substituting thin aluminium-foil for the plate, also by using bromide-paper as the sensitive surface. No images in any case were obtained.

3. The sun's rays or some portion of its radiation passes readily through wood if the latter is not too thick. Thus, through a piece of white pine $\frac{3}{16}$ of an inch thick, images that could readily be developed were obtained by three minutes' exposure to afternoon sunlight. With half an hour's exposure the images were brilliant.

A panel about 12 inches square was removed from an inside shutter and replaced with a piece of white pine $\frac{1}{4}$ inch thick. When the room was thoroughly darkened, reddish light could be seen to pass through the board. So that wood of this thickness is plainly translucent to the sight.

The sun's light may be examined for x-rays also by fluorescence.

4. The panel just described was replaced by one of stout book-board. With the sun shining on this book-board directly and not through glass, paper marked with a saturated solution of barium platinocyanide exhibited no indications of fluorescence when placed behind the board.

5. Three thicknesses of Bristol-board were pasted together, a circle was cut out, to one side of which barium platinocyanide was applied. The circle was then placed in a pasteboard tube (an arrangement, I believe, proposed by Prof. Magie). When the sun was looked at through this tube the barium salt exhibited fluorescence. But the interposition between the card and the sun of very thin aluminium-foil sufficed to cut off the fluorescence.

These concurrent results seem to indicate the absence of x-rays from sunlight.

Charles Henry* quotes an opinion of H. Poincaré that all bodies whose phosphorescence is sufficiently intense emit in addition to luminous rays the x-rays of Röntgen, whatever may be the cause of their fluorescence. Henry quotes confirmatory experiments of his own made with zinc sulphide.

It seemed worth while to ascertain if this principle is of general application. A dilute solution of uranin was exposed to sunlight,

* *C. R.* cxxii. p. 238: quoted Chem. News, Feb. 21, 1896, p. 85 (No. 1891).

† *C. R.* cxxii. p. 312; Chem. News, Feb. 28, 1896, p. 98.

using a large surface of solution so as to get the best effect. A short distance over the surface was placed a sensitive film protected by aluminium-foil $\frac{1}{10}$ of a millimetre in thickness and with a lead star interposed. Two hours' exposure gave no result. The experiment was repeated with acid solution of quinine, with which five hours' exposure gave no result.

I have also examined the Welsbach light for x-rays. This light is usually burned under a chimney which increases the brightness but interposes glass between the source of light and the sensitive film. Even without a chimney the light is bright. The experiment was therefore made both ways. No x-rays could be detected. Nothing capable of passing through aluminium-foil $\frac{1}{10}$ of a millimetre in thickness by five hours' exposure to the uncovered flame. —*American Journal of Science*, May 1896.

ON A NEW AREOMETER. BY L.-N. VANDEVYVER.

The instrument is of glass, and consists of a cylinder divided in two parts by a horizontal watertight diaphragm C. The part B forms a small reservoir, closed at D by a ground stopper. To the cylinder is attached a stem carrying a scale which may be graduated in different ways. We shall assume that this scale is graduated for liquids heavier than water, with densities between 1·00 (water) and 1·6, for instance.

The liquid whose specific gravity is to be determined serves as ballast.

Suppose it desired to verify the initial point of the graduation. For this purpose the instrument is inverted, and the part B is filled with distilled water at a temperature t; the stopper is inserted without allowing air to enter; the apparatus can then be held upright without any fear of the stopper falling out; it is wiped and placed in a cylinder containing distilled water at the same *temperature t*. The point to which it sinks should correspond to the point 1·00 of graduation on the stem.

The same process is repeated for the liquid whose density is to be determined, care being taken to clean the apparatus each time it is used; distilled water being always used for the immersion. The liquid introduced into B being denser than water the total weight of the apparatus is greater, and it sinks to a greater depth, giving a new level which represents the density.

By taking for extreme limits of the scale densities which are sufficiently near together, the densimeter may be made very sensitive.

In order to prove this I constructed one specially for determining the densities of the worts of Belgian beers, which vary only between 1·00 and 1·06. The scale, extending over a length of about 24 centimetres, is graduated 1·00, 1·005, 1·01, 1·015, ... 1·06. Between each two of these divisions are four large and five intermediate ones, making nine equidistant divisions of about 2 millims. The third decimal can thus easily be read with accuracy; and the fourth can be obtained with approximation amply sufficient for practical purposes, as is proved by determinations made by the aid of the apparatus and rigorously controlled by the method of the specific-gravity flask.

This densimeter has many advantages which we may point out.

I. All ballast is suppressed, whether shot or mercury, and thus the apparatus is less heavy and less subject to fracture.

II. Correction for capillarity is dispensed with, seeing that we always work in distilled water, which can always be taken in the same conditions.

III. The volume of the liquid with which we work need not be measured, being constant. (In this respect the instrument has the advantage over other densimeters which depend on the same principle, such as those of Rousseau, Pàquet, Laska.)

IV. The density of even a small quantity of liquid may be determined.

V. It is convenient for determining viscous liquids.

VI. The temperatures of the liquid and of the distilled water do not differ during the operation.

VII. Calculation shows that the results obtained are independent of the temperature worked at, at least for considerable differences. Allowing for the expansion of the liquid, that of the apparatus, and that of the water of immersion, it will be seen that these effects almost exactly compensate one another, especially if the liquid and the water are taken at the same temperature. Experiment shows that even with the instrument mentioned above, which has been graduated at 15° C., we may take the density indifferently from 8° to 9° C. to 19° or 20° C., without the results differing by more than 1 to 2 units in the fourth decimal place.

VIII. The instrument can, lastly, be easily cleaned.

The apparatus is also constructed for liquids less dense than water. The division 1·00 is then at the top of the scale; and putting the division 1·00 at the middle of the scale, we can with one and the same instrument, within certain limits, determine densities above and below that of water, especially if we are satisfied with a result correct to the second decimal and approximately so to the third.—From a separate impression from the *Arch. Sci. Phys. et Nat.* t. xxxiv. p. 409, communicated by the Author.

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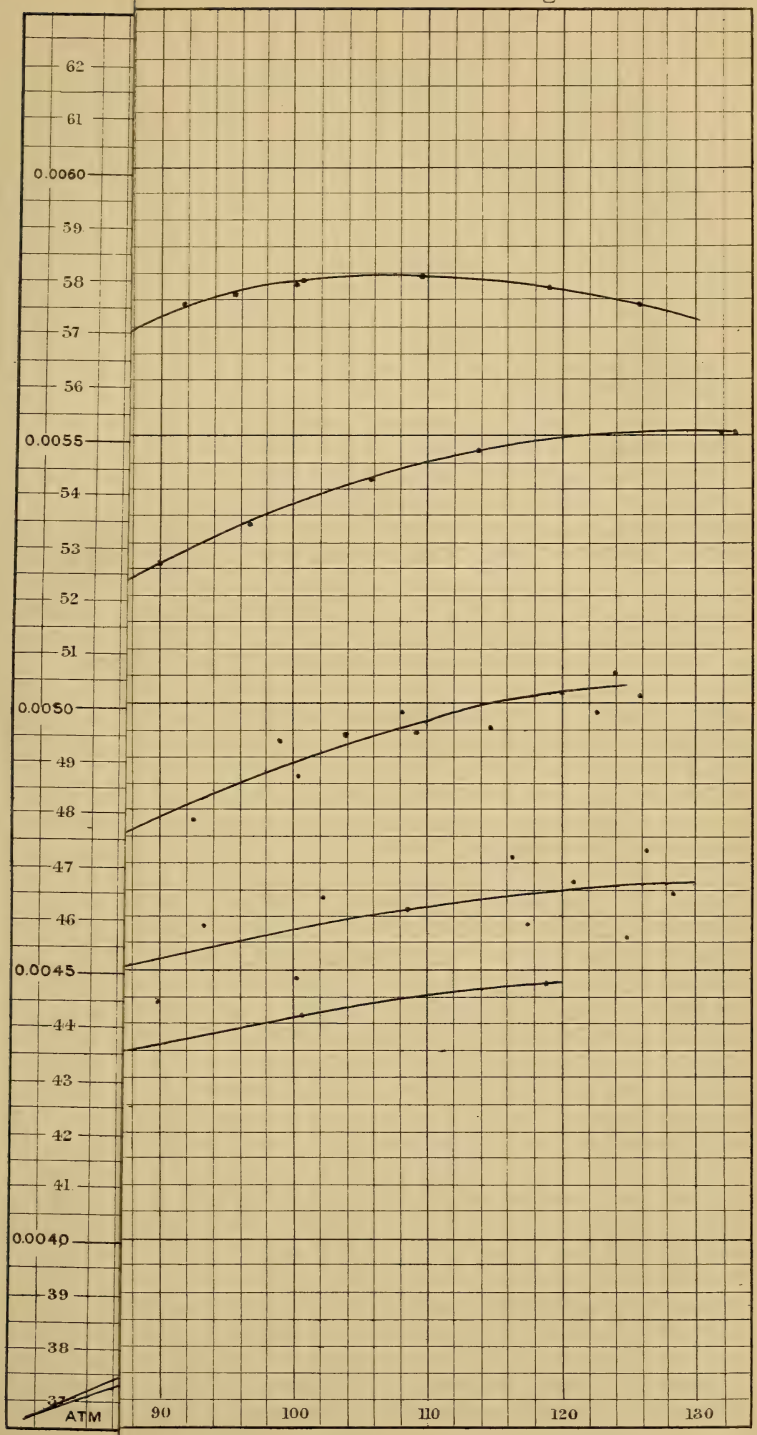
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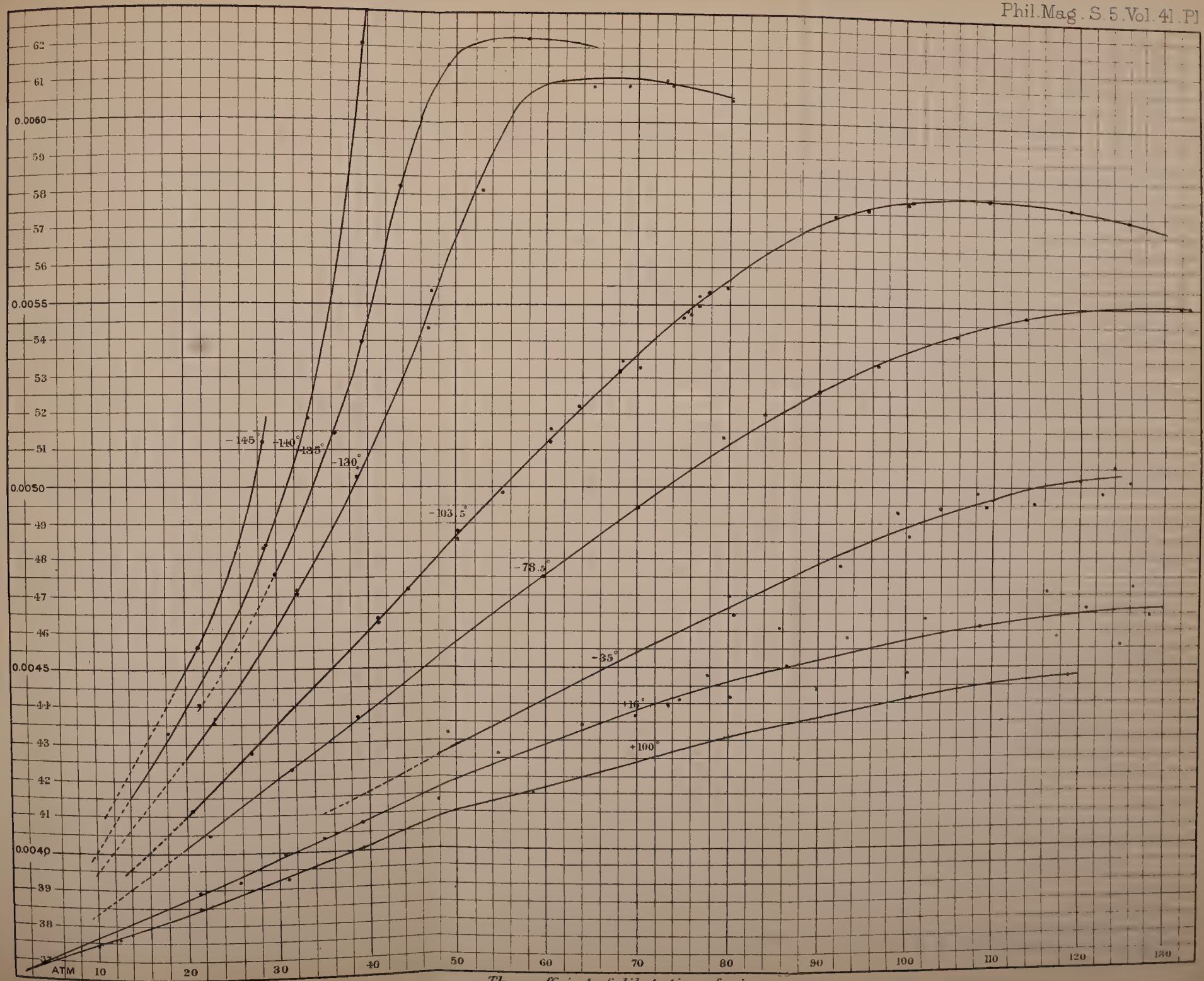

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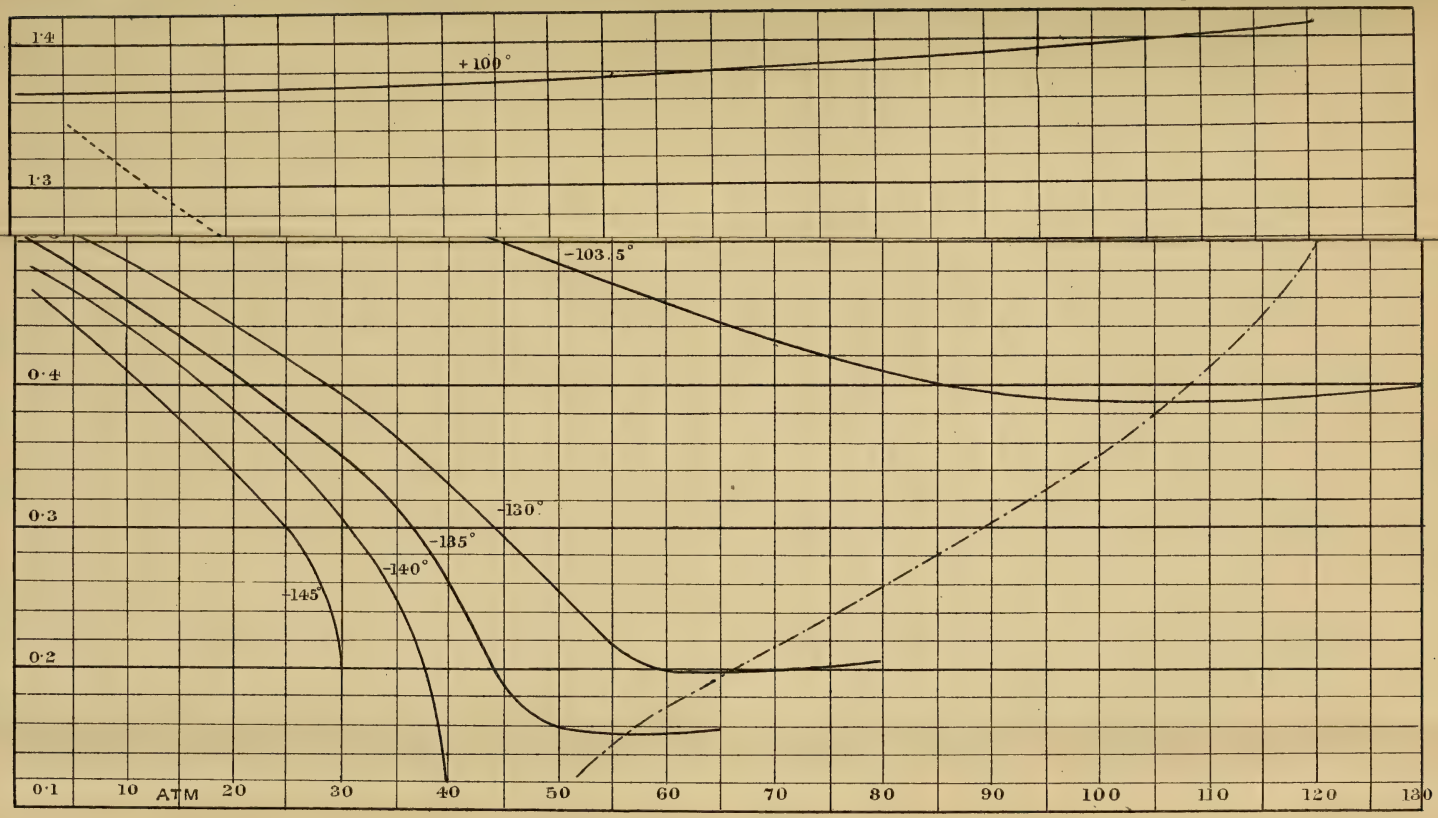

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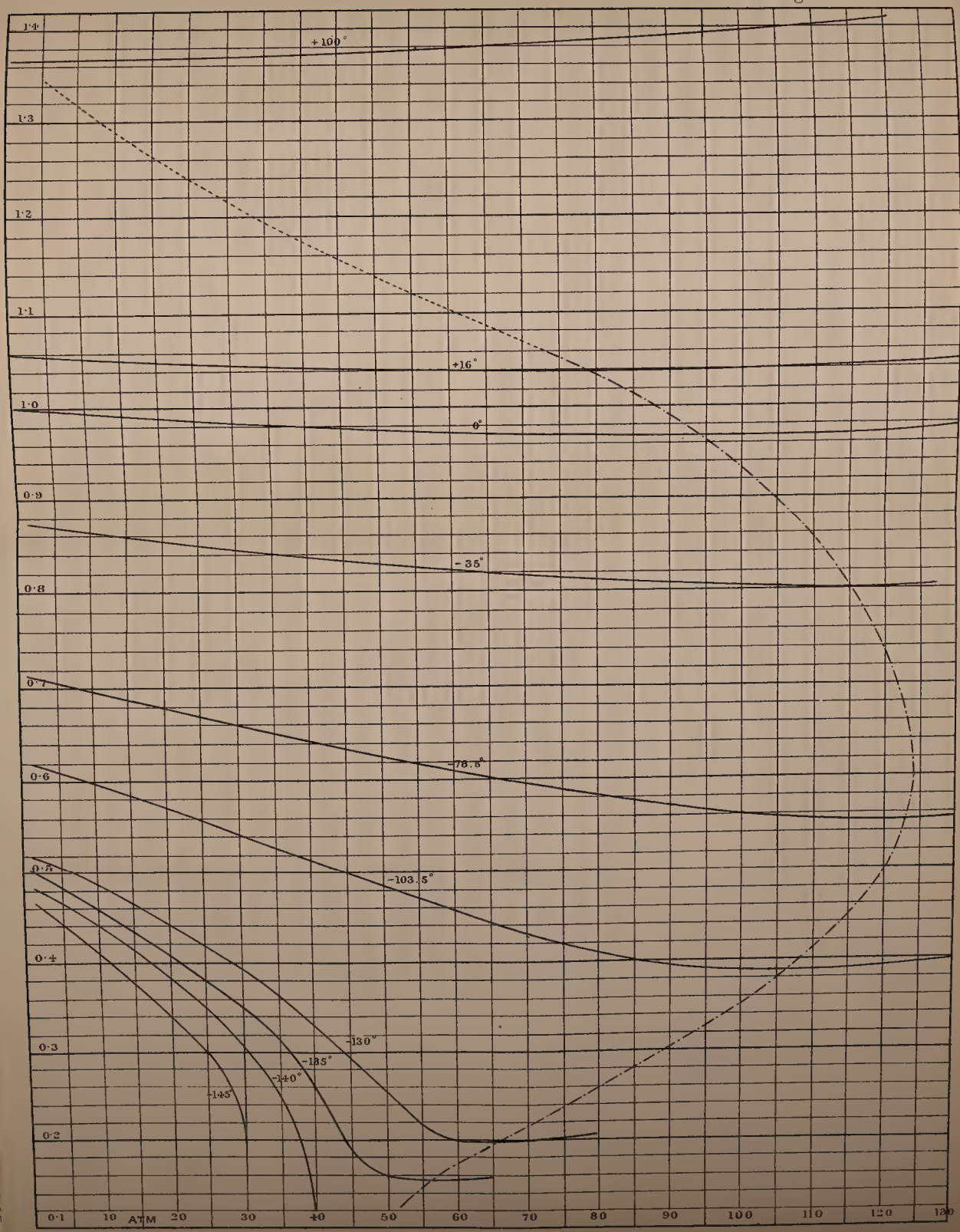

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Page 170, second line of table,
for Bell-metal at 0° vibrates during 55 secs.; at 100° it vibrates during 15 secs.
read " " " 55 " " " 40 "
Page 171, line 27 from top, *for* air of glass tubes of different diameters, renders
read air in glass tubes of different diameters, render
" 173, line 11 from top, *dele* The frequency of the vibrations of
" 195, " 12 from bottom, *for* from the globular *read* having the globular

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Page 367, Mr. W. H. Everett's paper, in first equation, for $rd^2\theta$ read $r^2d\theta$

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